Game of Hex

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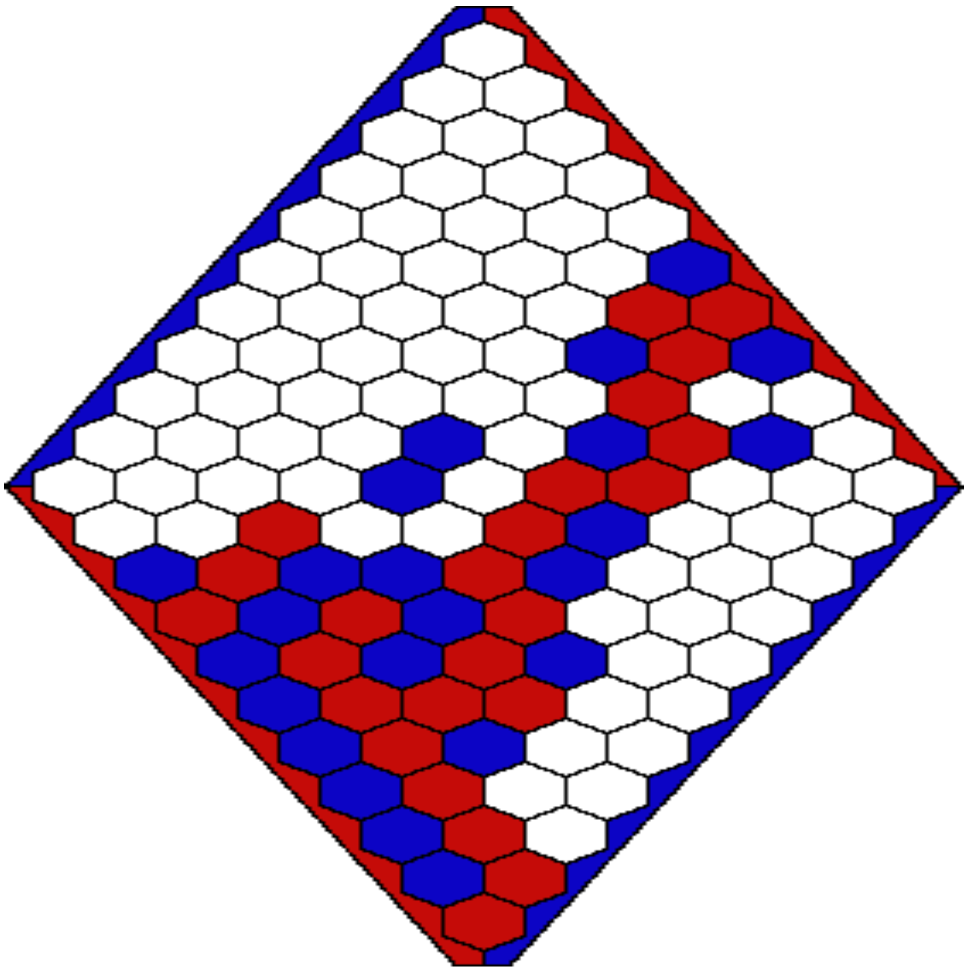
Hex is a board game played on a hexagonal grid, theoretically of any size and several possible shapes, but traditionally as a 11x11 rhombus. Each player has an allocated color, Red and Blue being conventional. Players take turns placing a stone of their color on a single cell within the overall playing board. The goal is to form a connected path of your stones linking the opposing sides of the board marked by your colours, before your opponent connects his sides in a similar fashion. The first player to complete his connection wins the game. The four corner hexagons each belong to two sides.



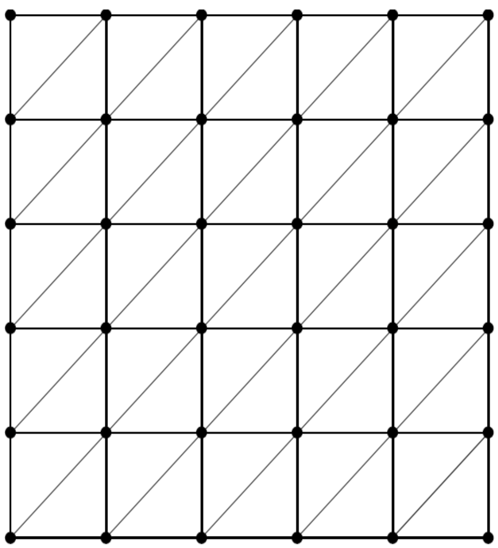
Hex was invented by a Danish poet and mathematician Piet Hein in 1942 at the Niels Bohr Institute for Theoretical Physics, and became popular under the name of Polygon. It was re-discovered in 1948 by John Nash, when he was a graduate student at Princeton. At that time, this game was commonly called John, referring mainly to the fact that it was often played on the hexagonal tiles of bathroom floors. Parker Brothers marketted a version of the game in 1952 under the name Hex.

Hex Theorem: The game Hex can never end in draw

This follows from the fact that if all cells of the board are occupied, then a winning chain for Red or Blue must necessarily exist. David Gale (1979) proved that this result is equivalent to the Brouwer fixed-point theorem for two dimensional squares. John Nash showed that a winning strategy always exists for the first player.

Hex Theorem implies Brouwer fixed-point Theorem

Proof: We can modify the hex board so that the game is played on the unit square. Subdivide the square as shown in the picture:



Each player claims a vertex and whenever a player has claimed two adjacent vertices, the edge joining them belongs to that player. Thus the number of vertices in the subdivision of the square is the number of hexagons in the original hex board. The aim is again to connect opposite sides. Now let $\mathbb{I}$ be the unit interval and $f : \mathbb{I}^2 \longrightarrow \mathbb{I}^2$ be any continuous function. Then we want to show that there exists $x \in \mathbb{I}^2$ such that $f(x) = x$.

Let $f(x) = (f_1(x), f_2(x))$. We just need to show that for every $\epsilon > 0$, there exists $x \in \mathbb{I}^2$ such that $|f(x) - x| < \epsilon$. So suppose $\epsilon > 0$ is given. Since f is uniformly continuos, there is $\delta > 0$ so that if $x, y \in \mathbb{I}^2$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. We can assume that $\delta \leq \epsilon$. Now choose an integer $n > \frac{2}{\delta}$ and subdivide $\mathbb{I}^2$ into a $n \times n$ hex board. Let V be the vertices of this hex board and $x = (x_1, x_2)$.

Define:
$H^+ = \{x \in V : \ f_1(x) - x_1 \geq \epsilon\}, \qquad H^- = \{x \in V : \ x_1 - f_1(x) \geq \epsilon\}$
$V^+ = \{x \in V : \ f_2(x) - x_2 \geq \epsilon\}, \qquad V^- = \{x \in V : \ x_2 - f_2(x) \geq \epsilon\}.$

1) No vertex of H^+ is adjacent to a vertex of H^-.
Because if $x \in H^+$ and $y \in H^-$ were adjacent, then $|x - y| < \delta$.
we would have: $x_1 - y_1 \geq -\epsilon, \qquad f_1(x) - x_1 \geq \epsilon, \qquad y_1 - f_1(y) \geq \epsilon.$
Hence $f_1(x) - f_1(y) \geq \epsilon$. So $|f(x) - f(y)| \geq \epsilon$, which is not true.

2) No vertex of H^+ has first coordinate equal to 1 and no vertex of H^- has first coordinate equal to 0.

Now 1) and 2) show that $H^+ \cup H^-$ cannot form a winning set for a hex player trying to join the sides $x_1 = 0$ and $x_1 = 1$. Similarly, $V^+ \cup V^-$ cannot form a winning set for a hex player wishing to join the sides $x_2 = 0$ and $x_2 = 1$. Thus by Hex Theorem $H^+ \cup H^- \cup V^+ \cup V^- \neq V$. Thus there exists $x^* \in V - H^+ \cup H^- \cup V^+ \cup V^-$. Clearly $|f(x^*) - x^*| < \epsilon$ and the proof is complete. $\square$