On The Frobenius Problem

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Given positive integers $a_1 < a_2 < ... < a_n$ with $\gcd(a_1, a_2, ..., a_n) = 1$, the linear Diophantine of Frobenius asks for the largest integer m for which we cannot find nonnegative integers $x_1, x_2, ..., x_n$ such that $m = a_1x_1 + a_2x_2 + ... + a_nx_n$. We call this largest integer the Frobenius number $g(a_1, a_2, ..., a_n)$.
It is known that this number always exists and for $n = 2$ we have $g(a_1, a_2) = a_1a_2 - a_1 - a_2$. No explicit formula exists for $n > 2$.

Some Facts About Frobenius Numbers:

Curran Sharp (1884): $g(a_1, a_2) = a_1 a_2 - a_1 - a_2$

Erdos, Graham (1972):

$$g(a_1, a_2, ..., a_n) \leq 2a_n \left[\frac{a_1}{n}\right] - a_1$$

Vitek (1975):

$$g(a_1, a_2, ..., a_n) \leq \left[\frac{(a_2 - 1)(a_n - 2)}{2}\right] - 1$$

Selmer (1977):

$$g(a_1, a_2, ..., a_n) \leq 2a_{n-1} \left[\frac{a_n}{n}\right] - a_n$$

Beck, Diaz, Robins (2005):

$$g(a_1, a_2, ..., a_n) \leq \frac{\sqrt{a_1 a_2 a_3 (a_1 + a_2 + a_3)} - a_1 - a_2 - a_3}{2}$$

Rodseth (1978):

$$g(a_1, a_2, ..., a_n) \geq \sqrt[n-1]{(n-1)! a_1 a_2 ... a_n} - a_1 - a_2 - ... - a_n$$

A Natural Extension of The Frobenius Problem:

Given positive integers $a_1 < a_2 < ... < a_n$ with $\gcd(a_1, a_2, ..., a_n) = 1$, we say that m is $k-$representable if m can be represented in the form $m = a_1x_1 + a_2x_2 + ... + a_nx_n$, in exactly k ways. It can be shown that for any k, eventually every integer can be represented in more than k ways. We define $g_k(a_1, a_2, ..., a_n)$ to be the smallest integer beyond which every integer is represented in more than k ways. Obviously $g(a_1, a_2, ..., a_n) = g_0(a_1, a_2, ..., a_n)$.

Let $A = \{a_1, a_2, ..., a_n\}$ and:
$p_A(m) = \#\{(x_1, x_2, ..., x_n) \in \mathbb{N}_0^n : a_1x_1 + a_2x_2 + ... + a_nx_n = m\}$. In view of this function, $g_k(a_1, a_2, ..., a_n)$ is the smallest integer such that for every $m > g_k(a_1, a_2, ..., a_n)$ we have $p_A(m) > k$. Clearly:

$$\prod_{j=1}^{n} \frac{1}{1 - x^{a_j}} = \sum_{m=0}^{\infty} p_A(m) x^m.$$

We are going to find an explicit formula for $g_k(a_1, a_2)$. But first we need the following:

Theorem (Popoviciu, 1953): If $\gcd(a, b) = 1$ and $A = \{a, b\}$, then:

$$p_A(n) = \frac{n}{ab} - \left\{\frac{b^{-1}n}{a}\right\} - \left\{\frac{a^{-1}n}{b}\right\} + 1.$$

Here $\{x\} = x - [x]$, and $a^{-1}a \equiv 1 \mod b,\ b^{-1}b \equiv 1 \mod a$.

Proof: Let

$$f(z) = \frac{1}{(1-z^a)(1-z^b)z^{n+1}}.$$

Then $Res(f(z), z=0) = p_A(n)$, and

$$Rez(f(z), z=1) = \frac{-(a+b+2n)}{2ab}.$$

If $s^a = 1,\ s \neq 1$, then

$$Res(f(z), z=s) = \frac{-1}{a(1-s^b)s^n},$$

and if $s^b = 1,\ s \neq 1$, then

$$Res(f(z), z=s) = \frac{-1}{b(1-s^a)s^n}.$$

Now let $C_R = \{z : \ |z| = R\}$. It's easy to see that

$$\int_{C_R} f(z)dz \to 0,$$

as $R \to \infty$. So residue theorem gives us:

$$p_A(n) = \frac{a+b+2n}{2ab} + \sum_{s^a=1, s\neq 1} \frac{1}{a(1-s^b)s^n} + \sum_{s^b=1, s\neq 1} \frac{1}{b(1-s^a)s^n}.$$

If we let $b = 1$, we will have:

$$p_{\{a,1\}}(n) = \frac{a+2n+1}{2a} + \sum_{s^a=1, s\neq 1} \frac{1}{a(1-s)s^n}$$

On the other hand

$$p_{\{a,1\}}(n) = \#\{(x_1, x_2) \in {\mathbb{N}_0}^2 :\ x_1 a + x_2 = n\}$$

$$= \#\left(\left[0, \frac{n}{a}\right] \cap \mathbb{Z}\right) = \frac{n}{a} - \left\{\frac{n}{a}\right\} + 1.$$

Thus

$$\sum_{s^a=1, s\neq 1} \frac{1}{a(1-s)s^n} = \frac{1}{2} - \frac{1}{2a} - \left\{\frac{n}{a}\right\}.$$

Now since

$$\sum_{s^a=1, s\neq 1} \frac{1}{a(1-s^b)s^n} = \sum_{s^a=1, s\neq 1} \frac{1}{a(1-s)s^{b^{-1}n}},$$

the results follows easily. □

Theorem $g_k(a,b) = (k+1)ab - a - b$.

Proof: By Popviciu Theorem $p_{\{a,b\}}((k+1)ab - a - b) = k$. Let $n = (k+1)ab - a - b + m$, where $m \in \mathbb{N}$. Then since for any integer q, $\{\frac{q}{a}\} \le 1 - \frac{1}{a}$, we have:

$$p_{\{a,b\}}(n) = p_{\{a,b\}}((k+1)ab - a - b + m)$$

$$\ge \frac{(k+1)ab - a - b + m}{ab} - \left(1 - \frac{1}{a}\right) - \left(1 - \frac{1}{b}\right) + 1 = k + \frac{m}{ab} > k. \square$$

Theorem: Given $k \geq 2$, the smallest $k-$representable integer is $(k-1)ab$.

Proof: Let $n > 0$. Then $p_{\{a,b\}}((k-1)ab - n) \leq k - \frac{n}{ab} < k$. $\Box$.

The Continuous Version of The Problem:

Let $A \subseteq (0,1)$ be open and non-empty. Let $S(A)$ be the set of all real numbers representable as a finite sum of elements of A. Let $g(A) = \sup\{x \in \mathbb{R} : \ x \notin S(A)\}$.

Let $m(A) = r > 0$, where $m(A)$ is the Lebesgue measure of A.

Then it can be proved that:

if $0 < r \leq 0.1$, then $g(A) \leq (1-r)[\frac{1}{r}]$,

if $0.1 \leq r \leq 0.5$, then $g(A) \leq (1-r+r\{\frac{1}{r}\})[\frac{1}{r}]$,

if $0.5 \leq r \leq 1$, then $g(A) \leq 2(1-r)$.