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ON SOME APPROXIMATIVE DIRICHLET-POLYNOMIALS IN THE THEORY OF THE ZETA-FUNCTION OF RIEMANN

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1. The zeta-function of Riemann is defined in the complex $s = \sigma + it$ plane for the half-plane $\sigma > 1$ by

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \cdots + \frac{1}{n^s} + \cdots. \tag{1.1}$$

Here is valid also the product-representation of Euler

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}},$$

where p runs through the consecutive primes. From this representation it follows clearly that

$$\zeta(s) \neq 0 \qquad \text{for } \sigma > 1. \tag{1.2}$$

As is well-known, the function $\zeta(s)$ is regular in the whole plane except at $s = 1$, where there is a pole of the first order. It is also well-known that the distribution of its roots is of fundamental importance in the theory of numbers. We know from the functional equation

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \tag{1.3}$$

that in the half-plane $\sigma < 0$ the only zeros are $s = -2, -4, -6, \ldots$ and that there are an infinite number of roots $\varrho = \sigma_\varrho + i t_\varrho$, the so called "non trivial roots", such that

$$0 < \sigma_\varrho < 1. \tag{1.4}$$

The famous hypothesis of Riemann, unproved so far, states that these all lie on the line $\sigma = \frac{1}{2}$. Using the fact obvious from the functional-equation (1.3) that they are symmetrical with respect to $s = \frac{1}{2}$ we can express the content of this hypothesis in the form that

$$\zeta(s) \neq 0 \qquad \text{for} \quad \sigma > \frac{1}{2}. \tag{1.5}$$

No one has yet been able to prove even the existence of a ϑ with $\frac{1}{2} \leqq \vartheta < 1$ such that

$$\zeta(s) \neq 0 \qquad \text{for} \quad \sigma > \vartheta. \tag{1.6}$$

2. Next we consider the partial-sums

$$U_n(s) = \frac{1}{1^s} + \frac{1}{2^s} + \cdots + \frac{1}{n^s} \tag{2.1}$$

of the series (1.1). They obviously converge to $\zeta(s)$ for $\sigma > 1$. We ask whether these partial-sums share with $\zeta(s)$ the property of being non-vanishing in the half-plane $\sigma > 1$. We have found the somewhat striking

Theorem I. If there is an n_0 such that for $n > n_0$ the partial-sums $U_n(s)$ do not vanish in the half-plane $\sigma > 1$, then Riemann's conjecture (1.5) is true.[1)]

More generally

Theorem II. If there are positive numbers n_0 and K such that for $n > n_0$ the partial-sum $U_n(s)$ does not vanish in the half-plane

$$\sigma \geqq 1 + \frac{K}{\sqrt{n}}, \tag{2.2}$$

then Riemann's hypothesis (1.5) is true.[2)]

Still more generally

Theorem III. If there are positive numbers n_0, K and ϑ satisfying

1) This elegant form of the theorem is due to Prof. B. Jessen; my original form was more awkward.

2) This theorem is due to my pupil Mr. P. Ungár who observed that the method of proof of theorem I furnishes at the same time the proof of theorem II.

$$\frac{1}{2} \leqq \vartheta < 1 \tag{2.3}$$

such that for $n > n_0$ the sum $U_n(s)$ does not vanish in the half-plane

$$\sigma \geqq 1 + \frac{K}{n^{1-\vartheta}}, \tag{2.4}$$

then $\zeta(s) \neq 0$ in the half-plane $\sigma > \vartheta$.

A further not uninteresting generalisation is given by

Theorem IV. If there are positive n_0, K, K_1 and ϑ satisfying (2.3) such that for $n > n_0$ the polynomial $U_n(s)$ omits in the half-plane (2.4) a real value c_n with[1]

$$-\frac{K_1}{n^{1-\vartheta}} \leqq c_n \leqq \frac{K_1}{n^{1-\vartheta}},$$

then $\zeta(s) \neq 0$ for $\sigma > \vartheta$.

3. All these theorems admit a further generalisation which asserts that these theorems remain true even if there is an infinity of exceptional n's provided that there are "not too many". We state explicitly only the analogue of theorem II.

Theorem V. If there is a positive K such that—denoting by $a(x)$ the number of n-values not exceeding x for which $U_n(s)$ has zeros in the half-plane $\sigma \geqq 1 + \frac{K}{\sqrt{n}}$—we have

$$\lim_{x \to \infty} \frac{a(x)}{\log x} = 0, \tag{3.1}$$

then Riemann's hypothesis (1.5) is true.

Such connection between Riemann's hypothesis and the roots of the partial-sums seems not to have been observed so far. The very interesting question whether, supposing Riemann's hypothesis to be true, we can deduce consequences on the roots of the sections, remains open.

On the basis of theorem III we have an interesting situation for the roots of the partial-sums $U_n(s)$. If Riemann's hypothesis

[1] The stronger statement that the omitted value c_n must satisfy only $|c_n| \leqq K_1 n^{\vartheta - 1}$ we cannot prove.

(1.5) is not true, or more exactly $\sup \sigma_\varrho = \Theta > \frac{1}{2}$, then there is an infinity of n's such that $U_n(s)$ vanishes in the half-plane $\sigma > 1$ and even in the half-plane $\sigma > 1 + n^{\Theta - 1 - \varepsilon}$, where ε is an arbitrarily small preassigned number. But if Riemann's hypothesis (1.5) is true, then, curiously, the method fails and nothing can be said about the roots of $U_n(s)$ this way.

4. What can actually be said about the roots of $U_n(s)$? According to a theorem of K. Knopp[1)] every point of the line $\sigma = 1$ is a condensation-point for the zeros of $U_n(s)$. But in an interesting way this condensation happens at least for $|t| \geqq \tau_0$, where τ_0 is a sufficiently large numerical constant[2)] only from the left.

More exactly we can prove

Theorem VI. There exist numerical τ_0 and K_2 such that $U_n(s)$ does not vanish for

$$\tau_0 \leqq |t| \leqq e^{K_2 \log n \log\log n}, \qquad \sigma \geqq 1, \qquad n > n_0. \tag{4.1}$$

Further $U_n(s)$ does not vanish in the half-plane

$$\sigma \geqq 1 + 2\frac{\log\log n}{\log n}, \qquad n > n_0. \tag{4.2}$$

In the estimation (4.1) of the domain of non-vanishing we could replace $\log n \log\log n$ by $\log^k n$ with a suitable $k > 1$, using estimations of Vinogradoff instead of estimations of Weyl.

The first part of theorem VI shows the indicated behaviour of the roots of $U_n(s)$; but to prove only this for all sufficiently large t we could use a more elementary reasoning. We write $U_n(s)$ in the form

$$U_n(s) = \zeta(s) - r_n(s), \qquad r_n(s) = \sum_{\nu > n} \nu^{-s}. \tag{4.3}$$

In what follows we denote by $K_3, \ldots$ positive quantities, whose dependence upon eventual parameters will be indicated explicitly; if no such dependence is mentioned they denote numerical constants. If

1) See the paper of R. Jentzsch: Untersuchungen zur Theorie der Folgen analytischer Functionen. Acta Math. 41 (1918), p. 219—251, in particular p. 236.

2) This probably also holds with $\tau_0 = 0$.

$$1 < \sigma \leqq 2, \qquad |t| \geqq 4, \tag{4.4}$$

then

$$\left|(\nu+1)^{1-s} - \nu^{1-s} - \frac{1-s}{\nu^s}\right| \leqq \left|\left(1+\frac{1}{\nu}\right)^{1-s} - 1 - \frac{1-s}{\nu}\right| < \frac{K_3 t^2}{\nu^2},$$

and summing over $\nu > n$

$$|r_n(s)| < 2\frac{n^{1-\sigma}}{|t|} + \frac{2K_3|t|}{n}.$$

This is true for any s in the domain (4.4) and obviously for $\sigma \geqq 1$, $|t| \geqq 4$; hence for $n \geqq t^2$

$$|r_n(s)| \leqq \frac{K_4}{|t|}. \tag{4.5}$$

Since for a suitable positive K_5 we have[1] for $\sigma \geqq 1$, $|t| \geqq 4$

$$\frac{1}{|\zeta(s)|} < K_5 \log|t|, \tag{4.6}$$

it follows from this, (4.5) and (4.3), that for $\sigma \geqq 1$, $|t| \geqq K_6$, $n \geqq t^2$

$$\left|U_n(s)\right| > \frac{1}{K_5 \log|t|} - \frac{K_4}{|t|} > 0. \qquad \textit{Q. e. d.}$$

We do not know so far of a single $U_n(s)$ vanishing in the half-plane $\sigma > 1$. Beyond the obvious fact that $U_n(s) \neq 0$ there for $n \leqq 3$, we know only from a remark of Prof. B. Jessen that $U_4(s)$ as well as $U_5(s)$ does not vanish in the half-plane $\sigma \geqq 1$. For the set of values of $U_4(s)$ coincides "essentially" with that of the function

$$g_4(\varphi, \psi, \sigma) = 1 + \frac{1}{2^\sigma}e^{i\varphi} + \frac{1}{3^\sigma}e^{i\psi} + \frac{1}{4^\sigma}e^{2i\varphi}$$

and

$$\Re g_4(\varphi, \psi, \sigma) = 1 + \frac{1}{2^\sigma}\cos\varphi + \frac{1}{3^\sigma}\cos\psi + \frac{1}{4^\sigma}\cos 2\varphi,$$

so that for fixed $\sigma = \sigma_0 \geqq 1$

1) T. H. Gronwall: Sur la fonction $\zeta(s)$ de Riemann au voisinage de $\sigma = 1$. Rend. Circ. Mat. di Pal. T. XXXV, 1913, p. 95—102.

$$\Re g_4(\varphi, \psi, \sigma_0) \geqq 1 - \frac{1}{3^{\sigma_0}} + \min_{\varphi}\left(\frac{1}{2^{\sigma_0}}\cos\varphi + \frac{1}{4^{\sigma_0}}\cos 2\varphi\right) =$$

$$= 1 - \frac{1}{3^{\sigma_0}} - \frac{1}{8} - \frac{1}{4^{\sigma_0}}$$

and for $\sigma \geqq 1$

$$\Re U_4(s) \geqq \frac{7}{24}.$$

Similarly we have

$$\Re U_5(s) \geqq \frac{7}{24} - \frac{1}{5} = \frac{11}{120}.$$

5. We can prove all the corresponding theorems on replacing $U_n(s)$ with the Cesaro-means[1] of the series (1.1)

$$C_n(s) = \sum_{\nu \leqq n}' \frac{n-\nu+1}{n+1} \cdot \nu^{-s}. \tag{5.1}$$

To see what alterations are necessary in the proofs we shall treat explicitly only

Theorem VII. If there exist positive K and n_0 so that the polynomial $C_n(s)$ does not vanish in the half-plane

$$\sigma \geqq 1 + \frac{K}{\sqrt{n}},$$

then Riemann's hypothesis (1.5) is true.

Though the numerical evidences that the polynomials $C_n(s)$ do not vanish in the half-plane $\sigma \geqq 1$ are more numerous (e. g. the non-vanishing of $C_6(s)$ in this half-plane follows quite trivially), we cannot enlarge the domain of non-vanishing (4.1) of theorem VI for the Cesaro-means. For the Riesz-means

$$R_n(s) = \sum_{\nu \leqq n}' \left(1 - \frac{\log \nu}{\log n}\right) \nu^{-s},$$

for which the analogous theorems would have a somewhat enhanced interest because of the fact that they converge to $\zeta(s)$ in the closed half-plane $\sigma \geqq 1$, our method fails in principle.

1) Analogous theorems hold for Cesaro-means of higher order.

6. Another interesting series for the zeta-function is

$$\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}-\cdots+\frac{(-1)^{n+1}}{n^s}+\cdots, \tag{6.1}$$

which represents in the half-plane $\sigma > 0$ the function

$$\left(1-\frac{2}{2^s}\right)\zeta(s). \tag{6.2}$$

This function vanishes in the half-plane $\sigma > 0$ at the points $s = \varrho$ as well as at

$$s = w_k = 1 + \frac{2k\pi i}{\log 2} \tag{6.3}$$

$$k = \pm 1, \pm 2, \cdots.$$

Hence from the well-known theorem of Hurwitz it follows that the partial-sums

$$V_n(s) = \sum_{m \leqq n} (-1)^{m+1} m^{-s} \tag{6.4}$$

have roots "near" to w_k if n is sufficiently large, and we prove that these occur infinitely often in the half-plane $\sigma > 1$. Hence the analogue of theorem I is meaningless; but the analogues of theorems II, III, IV and V are true. We shall prove only

Theorem VIII. If there exist positive n_0 and K such that for $n > n_0$ the partial-sums $V_n(s)$ do not vanish in the half-plane $\sigma \geqq 1 + \frac{K}{\sqrt{n}}$, then Riemann's conjecture (1.5) is true,

and

Theorem IX. There is an infinity of values of n for which $V_n(s)$ vanishes in the half-plane $\sigma > 1$.

This will follow from the fact that those roots of $V_{2n}(s)$ which converge for a fixed k to w_k may be expressed asymptotically as

$$w_k + \frac{1}{4\log 2 \cdot \zeta'(w_k)} \cdot n^{-1+\frac{2k\pi i}{\log 2}}.$$

7. Returning to the partial-sums $U_n(s)$ we have mentioned the fact that every point of the line $\sigma = 1$ is a clustering point of the zeros of the polynomials $U_n(s)$. Are these all the clustering points? The answer, as we can prove easily, is affirmative. For

$$(\nu+1)^{1-s} - \nu^{1-s} - (1-s)\,\nu^{-s} = \nu^{1-s}\left(\left(1+\frac{1}{\nu}\right)^{1-s} - 1 - \frac{1-s}{\nu}\right) =$$

$$= \nu^{1-s}\int_0^{1/\nu}(1-s)(-s)(1+r)^{-s-1}\left(\frac{1}{\nu}-r\right)dr$$

and summing over ν

$$\left.\begin{aligned}\left|(n+1)^{1-s} - 1 - (1-s)\,U_n(s)\right| &\leq |s|\,|s-1| \sum_{1\leq\nu\leq n} \nu^{1-\sigma}\frac{1}{\nu^2}\max_{0\leq y\leq\frac{1}{\nu}}(1+y)^{-\sigma-1} \leq \\ &\leq |s|\,|s-1| \sum_{\nu=1}^{\infty} \nu^{-1-\sigma}.\end{aligned}\right\} \quad (7.1)$$

If a point $s^* = \sigma^* + it^*$ in the strip $\varepsilon \leq \sigma \leq 1-\varepsilon \left(0 < \varepsilon < \frac{1}{4}\right)$ could be an accumulation-point for the zeros of $U_n(s)$ then we have for an infinity of values of n that $U_n(s)$ vanishes in the domain

$$\frac{\varepsilon}{2} \leq \sigma \leq 1 - \frac{\varepsilon}{2}, \qquad |t| \leq (t^*+1). \tag{7.2}$$

But from (7.1) it follows that in the domain (7.2)

$$\left|(n+1)^{1-s} - 1 - (1-s)\,U_n(s)\right| \leq \frac{4\,|s|\,|s-1|}{\varepsilon}$$

$$|1-s|\,|U_n(s)| \geq (n+1)^{\frac{\varepsilon}{2}} - \frac{4\,|s|\,|s-1|}{\varepsilon} - 1 > 0$$

if $n > n_2 = n_2(t^*, \varepsilon)$, which is a contradiction. For the half-plane $\sigma < \varepsilon$ the proof runs similarly. Analogous statements hold for the Cesaro-means $C_n(s)$ and the Riesz-means $R_n(s)$. Similarly we can show that the complete set of accumulation points of zeros of $V_n(s)$ consists of the points of the line $\sigma = 0$, the non trivial roots ϱ of the zeta-function and the points w_k of (6.3).

8. These results suggest interesting further questions. Let there be given the series

$$e^{-\lambda_1 s} + a_2 e^{-\lambda_2 s} + \cdots + a_n e^{-\lambda_n s} + \cdots, \qquad 0 \leqq \lambda_1 < \lambda_2 < \cdots \to \infty, \quad (8.1)$$

which is convergent for $\sigma > 0$. We denote by H the clustering set of the zeros of its partial-sums. Is it always true that H consists of the zeros in $\sigma > 0$ of the function $f(s)$ defined by the series (8.1) and of the points of the line $\sigma = 0$? We can show that the set H can consist of the whole half-plane $\sigma \leqq 0$; by this we give the answer to a question raised by L. Fejér. Let $r_1, r_2, \cdots$ be the set of all positive rational numbers, arranged in such a way that every fixed one occurs infinitely often; we consider the product

$$g(s) = \prod_{\nu=1}^{\infty} \left[1 - \left(e^{-s-r_\nu}\right)^{2^\nu}\right]. \qquad (8.2)$$

Since the product

$$\prod_{\nu=1}^{\infty} \left[1 + e^{-2^\nu \sigma}\right]$$

is convergent for $\sigma > 0$ the product (8.2) can be expressed in the form (8.1) convergent for $\sigma > 0$. We observe that because of the rapid growth of the numbers 2^ν every partial-product is at the same time a partial-sum; hence all the numbers

$$s = -r_\nu + \frac{2 l \pi i}{2^\nu} \qquad \begin{array}{l} l = 0, \pm 1, \pm 2, \cdots \\ \nu = 1, 2, \cdots \end{array}$$

are roots of certain partial-sums. Since every fixed r_μ occurs infinitely often, every point of the line $\sigma = -r_\mu$ is a clustering point of such roots and so are all points of the half-plane[1] $\sigma \leqq 0$.

Somewhat more peculiar is the behaviour of the zeros of the sections of the series

$$1 + e^{-(s+10)} + e^{-2s} + e^{-3(s+10)} + \cdots + e^{-2ns} + e^{-(2n+1)(s+10)} + \cdots,$$

as P. Erdös remarked. As is easily shown, the set H consists

[1] Putting $e^{-s} = z$, we obtain a power series, regular for $|z| < 1$ with the property, that the roots of the partial-sums cluster to every point in $|z| \geqq 1$.

in this case of the lines $\sigma = 0$ and $\sigma = -10$. Probably one can prescribe in the half-plane $\sigma < 0$ the cluster set H of a Dirichlet-series regular in $\sigma > 0$.

9. The theorems I—V raise the question whether the zeros of the partial-sums of a series (8.1) convergent for $\sigma > 0$ can cluster to the points of the line $\sigma = 0$ *only from the left side.* That this is, indeed, possible is shown for example by series of the type

$$\sum_{\nu=1}^{\infty} a_\nu e^{-\nu s}, \tag{9.1}$$

where the a_ν's are positive and tend monotonously to 0 in such a manner that the line of convergence is the line $\sigma = 0$. That *all* the roots of *all* partial-sums of the series (9.1) lie in the half-plane $\sigma < 0$, follows from the theorem of Eneström-Kakeya. If the coefficients are chosen positive and increasing, then *all* the roots of *all* partial-sums lie in the half-plane $\sigma > 0$. For the sake of completeness we mention that the function

$$_1(s) = \prod_{\nu=1}^{\infty} \left[1 - \left(e^{-s-l_\nu}\right)^{2^\nu}\right],$$

where the sequence $l_\nu \to 0$ and contains an infinite number of both positive and negative terms, has an expansion of the form (8.1) (even (9.1)) with the property, that every point of the line $\sigma = 0$ is a condensation-point of zeros from the left half-plane and at the same time a condensation-point of zeros from the right half-plane.

10. Of course a direct approach to the investigation of the roots of partial-sums or arithmetical means in the half-plane $\sigma > 1$ seems to be very difficult; the stress of this paper is laid upon the *connection* between these questions and Riemann's hypothesis. In any case the results raise the question how the roots of the function given by a Dirichlet-series can influence the roots of its partial-sums or suitable means. In this direction no results are known so far which hold for the means of finite index. If the function is given by a Taylor-series the situation changes. If e. g.

$$f(s) = \sum_{\nu=0}^{\infty} a_\nu s^\nu$$

is an integral function of order 1, whose roots lie in the half-plane $\sigma < 0$, then all roots of all "Jensen-means" of $f(s)$

$$J_n(f) = a_0 + a_1 s + a_2\left(1-\frac{1}{n}\right)s^2 + a_3\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)s^3 + \cdots +$$
$$+ a_n\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)s^n$$

lie in the half-plane $\sigma \leqq 0$. The proof is very easy and runs on known lines.

I wish to thank Mrs. Helen K. Nickerson for linguistic assistance in the preparation of the manuscript.

11. Now we pass on to the proof of theorems I—IV. Obviously it is sufficient to prove theorem IV. We base the proof on an important theorem of H. Bohr[1], combining it with a classical reasoning of Landau[2]. First we recall that given a sequence

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to \infty, \tag{11.1}$$

we call the sequence B of linearly independent numbers

$$\beta_1, \beta_2, \ldots \tag{11.2}$$

a basis of (11.1), if

$$\lambda_n = r_{n_1}\beta_1 + r_{n_2}\beta_2 + \cdots + r_{n_{q_n}}\beta_{q_n}$$

with rational r_{n_ν}'s. If $f(s)$ is defined in the half-plane $\sigma > A$ by the absolutely convergent series

$$f(s) = \sum_{\nu=1}^{\infty} a_\nu e^{-\lambda_\nu s}, \tag{11.3}$$

1) H. Bohr, Zur Theorie der allgemeinen Dirichletschen Reihen. Math. Ann. B 79, 1919, p. 136—156. See in particular Satz. 4.

2) E. Landau: Über einen Satz von Tschebischeff. Math. Ann. 61, 1905, p. 527—550. The whole method of **14.** is due to him. He proved moreover that the integral (14.2) converges for $\sigma > \gamma$, but we do not use this fact.

Bohr calls every function

$$g(s) = \sum_{\nu=1}^{\infty} b_\nu e^{-\lambda_\nu s}$$

equivalent to $f(s)$, if for suitable $\varphi_1, \varphi_2, \cdots$ and $n = 1, 2, \cdots$

$$b_n = a_n e^{-i(r_{n_1}\varphi_1 + r_{n_2}\varphi_2 + \cdots + r_{n_{q_n}}\varphi_{q_n})} \tag{11.4}$$

Obviously $g(s)$ is also absolutely convergent for $\sigma > A$. Now the above-mentioned theorem of Bohr asserts that the sets of values assumed by $f(s)$ and $g(s)$ resp. in the half-plane $\sigma > A$ are identical.

We may apply this theorem to $f(s) = U_n(s)$ with

$$B \equiv (\log 2, \log 3, \cdots, \log p, \cdots), \quad \varphi_\nu = \pi \quad (\nu = 1, 2, \cdots). \tag{11.5}$$

If $\nu = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, then

$$\log \nu = \sum_{j=1}^{r} \alpha_j \log p_j;$$

hence

$$b_\nu = \exp\left(-i\pi \sum_{j=1}^{r} \alpha_j\right) = \lambda(\nu),$$

where, as usual, $\lambda(\nu)$ denotes Liouville's number theoretic function. Hence the set of values assumed by $U_n(s)$ in the half-plane

$$\sigma > 1 + Kn^{\vartheta-1} \tag{11.6}$$

is identical with that assumed here by the polynomial

$$W_n(s) = \sum_{\nu \leq n} \frac{\lambda(\nu)}{\nu^s}. \tag{11.7}$$

If $U_n(s)$ does not assume the value c_n in the half-plane (11.6) the same holds for $W_n(s)$. But then the function

$$W_n(s) - c_n, \tag{11.8}$$

which is real on the real axis and is positive at infinity, if $n > n_1$, where

$$K_1 n_1^{\vartheta-1} < 1, \tag{11.9}$$

is necessarily positive on the whole positive axis in the half-plane (11.6) and in particular non negative for $s = 1 + Kn^{\vartheta-1}$. This gives that if

$$n > K_7 = \max(n_0, n_1),$$

$$\sum_{\nu \leqq n}' \frac{\lambda(\nu)}{\nu^{1+Kn^{\vartheta-1}}} \geqq c_n. \tag{11.10}$$

12. Using the restrictions on c_n, we may write (11.10) for $n > K_7$ in the form

$$\sum_{\nu \leqq n} \lambda(\nu)\, \nu^{-1-Kn^{\vartheta-1}} \geqq -K_1 n^{\vartheta-1}. \tag{12.1}$$

Since

$$\left| e^{-Kn^{\vartheta-1}\log\nu} - 1 + Kn^{\vartheta-1}\log\nu \right| < \frac{1}{2}\cdot\frac{K^2}{n^{2(1-\vartheta)}}\log^2\nu,$$

the error made in replacing the left hand member of (12.1) by

$$\sum_{\nu \leqq n}' \frac{\lambda(\nu)}{\nu} - Kn^{\vartheta-1}\sum_{\nu \leqq n}' \frac{\lambda(\nu)\log\nu}{\nu}$$

is in absolute value

$$\leqq \frac{K^2}{2\,n^{2(1-\vartheta)}}\sum_{\nu \leqq n}' \frac{\log^2\nu}{\nu} < \frac{K^2}{6\,n^{2(1-\vartheta)}}\log^3 n < \frac{K^2}{6} n^{\vartheta-1}$$

if $n > n_2 = K_8(\vartheta)$. Hence for $n > \max(K_7, K_8(\vartheta))$

$$\sum_{\nu \leqq n}' \frac{\lambda(\nu)}{\nu} > Kn^{\vartheta-1}\sum_{\nu \leqq n}' \frac{\lambda(\nu)\log\nu}{\nu} - \left(K_1 + \frac{K^2}{6}\right) n^{\vartheta-1}. \tag{12.2}$$

Now since for $\sigma > 0$

$$\sum_{\nu=1}^{\infty}{}' \frac{\lambda(\nu)\log\nu}{\nu}\cdot\frac{1}{\nu^s} = \frac{\zeta(2s+2)\,\zeta'(s+1) - 2\,\zeta(s+1)\,\zeta'(2s+2)}{\zeta(s+1)^2},$$

the usual method of complex integration shows that

$$\sum_{\nu} \frac{\lambda(\nu) \log \nu}{\nu}$$

is convergent and that its sum is $-\frac{\pi^2}{6}$. Therefore for $n > K_9$ we have $\sum_{\nu \leqq n} \frac{\lambda(\nu) \log \nu}{\nu} \geqq -2$ and for $n > \max(K_7, K_8(\vartheta), K_9) = K_{10}(\vartheta)$

$$L(n) \equiv \sum_{\nu \leqq n} \frac{\lambda(\nu)}{\nu} > -\left(2 + K_1 + \frac{K^2}{6}\right) n^{\vartheta - 1} = -K_{11} n^{\vartheta - 1}. \quad (12.3)$$

13. The inequality (12.3) can be written in the form

$$L(n) + K_{11} n^{\vartheta - 1} > 0 \quad (13.1)$$

for integral $n > K_{10}(\vartheta)$. Now we consider for a continuously varying $x \geqq K_{10}(\vartheta)$ the function

$$L(x) + 2 K_{11} x^{\vartheta - 1},$$

and we assert that this is positive for all $x \geqq K_{10}(\vartheta)$. We consider the x-values belonging to the interval $m \leqq x < m + 1$, m an integer and $\geqq K_{10}(\vartheta)$. For $x = m$ our assertion is evident. To estimate this function for other x-values we remark that $L(x)$ being a step-function is constant for $m \leqq x < m + 1$ and hence

$$\begin{aligned} L(x) + 2 K_{11} x^{\vartheta - 1} &= L(m) + K_{11} m^{\vartheta - 1} + K_{11}\left(2 x^{\vartheta - 1} - m^{\vartheta - 1}\right) \geqq \\ &\geqq \left(L(m) + K_{11} m^{\vartheta - 1}\right) + K_{11}\left(2 (m+1)^{\vartheta - 1} - m^{\vartheta - 1}\right) \geqq \\ &\geqq K_{11} (m+1)^{\vartheta - 1}\left(2 - \left(1 + \frac{1}{m}\right)^{1 - \vartheta}\right) \geqq 0. \end{aligned}$$

14. Now for $\sigma > 0$ we have

$$\sum_{\nu = 1}^{\infty} \frac{\lambda(\nu)}{\nu} \cdot \frac{1}{\nu^s} = \frac{\zeta(2s + 2)}{\zeta(s + 1)} = s \int_1^{\infty} \frac{L(x)}{x^{s+1}} dx,$$

$$s \int_0^{\infty} \frac{x^{\vartheta - 1}}{x^{s+1}} dx = \frac{s}{s + 1 - \vartheta};$$

therefore for $\sigma > 0$

$$\int_1^{\infty} \frac{L(x) + 2K_{11}x^{\vartheta-1}}{x^{s+1}}\,dx = \frac{\zeta(2s+2)}{s\zeta(s+1)} + \frac{2K_{11}}{s+1-\vartheta};$$

or for $\sigma > 1$

$$\int_1^{\infty} \frac{L(x) + 2K_{11}x^{\vartheta-1}}{x^{s}}\,dx = \frac{\zeta(2s)}{(s-1)\zeta(s)} + \frac{2K_{11}}{s-\vartheta}. \qquad (14.1)$$

From **13.** it follows that the numerator of the integrand is positive for all sufficiently large x's; hence we may apply the following theorem of Landau[1]: if a function $\varphi(s)$ is defined for $\sigma > 1$ as

$$\varphi(s) = \int_1^{\infty} \frac{A(x)}{x^s}\,dx, \qquad (14.2)$$

where $A(x)$ does not change sign for $x > x_0$ and $\varphi(s)$ is regular on the real axis for $s > \gamma\ (< 1)$, then $\varphi(s)$ is regular in the entire half-plane $\sigma > \gamma$.

Hence we have only to consider the singularities of the right of (14.1) on the real axis. The first term is regular for $s > \frac{1}{2}$, the second for $s > \vartheta$, hence their sum is regular for $s > \vartheta$. Then Landau's theorem applied to (14.1) shows that the function on the right is regular in the half-plane $\sigma > \vartheta$. But then $\zeta(s)$ cannot vanish in this half-plane and theorem IV is proved.

The basis of the proof is the observation that for given arbitrarily small positive ε and η we can find $\tau_1 = \tau_1(\varepsilon, \eta)$ such that for $\sigma > 1 + \eta$ we have

$$\left| \zeta(s + i\tau_1) - \frac{\zeta(2s)}{\zeta(s)} \right| \leqq \varepsilon.$$

15. The proof of theorem V runs on the same line but instead of Landau's theorem we use the following theorem of Pólya.[2] Considering functions of type (14.2) let

1) See above p. 13, note 2.

2) G. Pólya: Über das Vorzeichen des Restgliedes im Primzahl-Satz. Gött. Nachr. 1930, p. 19—27. A special case of his theorem is given here in a slightly altered form.

$$1 = x_0 < x_1 < x_2 < \cdots < x_n < \cdots \tag{15.1}$$

be a sequence[1] which does not cluster to any finite positive value and with the property

$$(-1)^\nu A(x) \geqq 0 \qquad \text{for} \qquad x_\nu \leqq x < x_{\nu+1}. \tag{15.2}$$

The values x_ν are called sign-changing values. If $B(x)$ is defined by

$$B(\omega) = \sum_{\substack{\nu \\ x_\nu \leqq \omega}} 1 \tag{15.3}$$

he assumes that $A(x)$ has "not too many" sign-changing values, or more exactly

$$\overline{\lim_{\omega \to \infty}} \frac{B(\omega)}{\log \omega} = 0. \tag{15.4}$$

Then Pólya's theorem asserts that if Θ is the exact regularity-abscissa of $\varphi(s)$ and $\varphi(s)$ is meromorphic in a half-plane $\sigma > \Theta - b$ $(b > 0)$, then the statement of Landau's theorem holds i. e. the point $s = \Theta$ is a singular point of $\varphi(s)$. Applying this theorem to the function on the right in (14.1) we see that the condition of meromorphism is fulfilled. If we can deduce from (3.1) that the number of the sign-changing places of $L(x) + 2K_{11}x^{\vartheta-1}$ satisfies (15.4) the proof of theorem V will be completed.

We consider first the integral x-values. If n is a value sufficiently large such that $U_n(s) \neq 0$ for $\sigma \geqq 1 + \frac{K}{\sqrt{n}}$ — or briefly if n is a "good" value—the reasoning of **11.** and **12.** gives that $L(n) + K_{11}n^{\vartheta-1} > 0$. Then the reasoning of **13.** shows that for good n's

$$L(x) + 2K_{11}x^{\vartheta-1} > 0 \qquad n \leqq x < n+1. \tag{15.5}$$

If n is not a good value—or let us rather call it a "bad" value —then $L(n) + K_{11}n^{\vartheta-1}$ may be positive; and in this case (15.5) is true again. Finally if n is a bad value for which $L(n) + K_{11}n^{\vartheta-1}$ is $\leqq 0$, then since both of the functions[2]

$$L(x) + K_{11}x^{\vartheta-1}, \qquad L(x) + 2K_{11}x^{\vartheta-1}$$

1) This can consist of a finite number of terms or even of the single term x_0.
2) Using the fact that $L(x)$ is constant there, being a step-function.

are monotonously decreasing for $n \leqq x < n+1$, the second of them is either positive throughout $n \leqq x < n+1$ or negative throughout or finally positive for $x = n$ and decreasingly negative for $x \to (n+1)$ from the left. Hence the number of sign-changes $\leqq \omega$ for $L(x) + 2K_{11} x^{\vartheta - 1}$ cannot surpass three times the number of bad n's $\leqq \omega$; but then using (3.1) we see that Pólya's further condition (15.4) is fulfilled, indeed.

We can easily show that if Riemann's hypothesis (1.5) is not true, then the partial-sums $U_n(s)$ vanish infinitely often in the half-plane $\sigma > 1$ or more generally for every positive ε and for an infinity of n's $U_n(s)$ vanishes in the half-plane

$$\sigma > 1 + \frac{1}{n^{1-\Theta+\varepsilon}},$$

where $\sup \sigma_\varrho = \Theta > \frac{1}{2}$. For if we have an ε_1 with $\Theta - \varepsilon_1 > \frac{1}{2}$ such that $U_n(s) \neq 0$ in the half-plane

$$\sigma > 1 + \frac{1}{n^{1-\Theta+\varepsilon_1}}$$

for all sufficiently large n, then from theorem III we could conclude that $\zeta(s) \neq 0$ in the half-plane $\sigma > \Theta - \varepsilon_1$, a contradiction. This reasoning fails completely if $\Theta = \frac{1}{2}$; the identity (14.1) valid for $\sigma > 1$

$$\int_1^\infty \frac{L(x) + 2K_{11} x^{-\frac{1}{2}}}{x^s} dx = \frac{(2s-1)\zeta(2s)}{2(s-1)\zeta(s)} \cdot \frac{1}{s-\frac{1}{2}} + \frac{2K_{11}}{s-\frac{1}{2}},$$

whose right-side is continuable over the whole plane and behaves asymptotically[1)] if $s \to \frac{1}{2} + 0$ as

$$\frac{1}{s-\frac{1}{2}}\left(2K_{11} - \frac{1}{\zeta\left(\frac{1}{2}\right)}\right) = \left(2K_{11} + \frac{1}{\left|\zeta\left(\frac{1}{2}\right)\right|}\right)\frac{1}{s-\frac{1}{2}},$$

shows that the point $s = \frac{1}{2}$ is certainly a singular point.

1) We use the fact—which plays here a decisive role—that $\zeta\left(\frac{1}{2}\right)$ is negative.

16. Now we turn to the proof of theorem VI. The second part is only mentioned for the sake of completeness; for in the half-plane

$$\sigma \geqq 1 + 2\frac{\log\log n}{\log n}$$

$$|U_n(s)| = |\zeta(s) - r_n(s)| \geqq |\zeta(s)| - |r_n(s)| \geqq \prod \frac{1}{1+\frac{1}{p^\sigma}} -$$

$$-\sum_{\nu > n} \nu^{-\sigma} = \frac{\zeta(2\sigma)}{\zeta(\sigma)} - \frac{n^{1-\sigma}}{\sigma - 1} > K_{12}\cdot(\sigma-1) - \frac{1}{(\sigma-1)\log^2 n} >$$

$$> \frac{K_{12}}{(\sigma-1)\log^2 n}\left(4(\log\log n)^2 - \frac{1}{K_{12}}\right) > 0$$

for $n > n_0$.

For the proof of (4.1) we use an inequality of Weyl[1)] according to which for $\sigma > 0$, r an integer, $t > 3$, $N \leqq N' < 2N$ we have

$$\left|\sum_{\nu=N}^{N'} \frac{1}{(\nu+1)^s}\right| < 2^{17}\log^{2^{1-r}} t\cdot$$

$$\cdot\left\{N^{1-2^{1-r}-\sigma}\, t^{\frac{1}{(r+1)2^{r-1}}} + N^{1-\sigma}\, t^{-\frac{1}{(r+1)2^{r-1}}} \log^{\frac{r-1}{2^{r-1}}} N\right\}.$$

For $\sigma \geqq 1$ we have

$$\left.\begin{aligned} &\left|\sum_{\nu=N}^{N'} \frac{1}{(\nu+1)^s}\right| < 2^{17}\log^{2^{1-r}} t\,\cdot \\ &\cdot\left\{N^{-2^{1-r}}\, t^{\frac{1}{(r+1)2^{r-1}}} + t^{-\frac{1}{(r+1)2^{r-1}}} \log^{\frac{r-1}{2^{r-1}}} N\right\}. \end{aligned}\right\} \qquad (16.1)$$

To estimate $r_n(s)$ for $\sigma > 1$ and $n > t^2$ we start from

$$\left|(\nu+1)^{1-s} - \nu^{1-s} - \frac{1-s}{\nu^s}\right| < \frac{K_3\, t^2}{\nu^2}, \qquad t \geqq 4;$$

1) See Landau: Vorl. über Zahlenth. II. Theorem 389.

summing over ν

$$|r_n(s)| = \left|\sum_{\nu > n} \nu^{-s}\right| \leqq \frac{K_4}{t}. \tag{16.2}$$

Obviously the same estimation holds for $\sigma \geqq 1$ as we have seen in (4.5).

For $\sqrt{t} \leqq n \leqq t^2$ we have

$$|r_n(s)| < \left|\sum_{n < \nu \leqq t^2} \nu^{-s}\right| + \frac{K_4}{t} \equiv |S_1| + \frac{K_4}{t}. \tag{16.3}$$

To estimate $|S_1|$ we split it into $O(\log t)$ sums of the form (16.1); applying (16.1) to each of them with

$$r = 2, \qquad \sqrt{t} \leqq N \leqq t^2,$$

we obtain that for $t > K_{13}$ each such sum is in absolute value

$$< K_{14}\sqrt{\log t}\left\{t^{-\frac{1}{12}} + t^{-\frac{1}{6}}\sqrt{\log t}\right\} < \frac{K_{15}}{\log^3 t},$$

i. e. $|S_1| < \frac{K_{16}}{\log^2 t}$, and from (16.3) for $t > K_{13}$ we have

$$|r_n(s)| < \frac{K_{17}}{\log^2 t}. \tag{16.4}$$

As appears from (16.2) and (16.4) this estimation is valid for all $n \geqq \sqrt{t}$.

Now we suppose only

$$n > t^{\frac{1}{\log\log t}}. \tag{16.5}$$

Then using (16.4) we have

$$\left.\begin{aligned} |r_n(s)| &\leqq \left|\sum_{n < \nu \leqq \sqrt{t}} \nu^{-s}\right| + \left|\sum_{\nu > \sqrt{t}} \nu^{-s}\right| < \frac{K_{17}}{\log^2 t} + \left|\sum_{n < \nu \leqq \sqrt{t}} \nu^{-s}\right| = \\ &= \frac{K_{17}}{\log^2 t} + |S_2|. \end{aligned}\right\} \tag{16.6}$$

Because of (16.5) we can split S_2 into $O(\log\log t)$ sums of the form

$$S_2^{(k)} = \sum_{t^{\frac{2}{k+2}} < \nu \leq t^{\frac{2}{k+1}}} \nu^{-s}, \qquad 3 \leq k \leq 2 \log\log t. \qquad (16.7)$$

$S_2^{(k)}$ can be split into $O(\log t)$ sums of the form (16.1); applying (16.1) to them with

$$r = k, \qquad t^{\frac{2}{k+2}} \leq N \leq t^{\frac{2}{k+1}}$$

we obtain that they are in absolute value

$$< K_{18} \log^{2^{1-k}} t \left\{ t^{-\frac{2}{k+2}\cdot\frac{1}{2^{k-1}}+\frac{1}{(k+1)2^{k-1}}} + t^{-\frac{1}{(k+1)2^{k-2}}} \log^{\frac{k-1}{2^{k-1}}} t \right\} < K_{19} \log^{-4} t,$$

and hence

$$|S_2| < \frac{K_{19}}{\log^2 t}.$$

Thus for $t > K_{20}$, $n > t^{\frac{1}{\log\log t}}$, $\sigma \geq 1$

$$|r_n(s)| < \frac{K_{20}}{\log^2 t}. \qquad (16.8)$$

The inequality $n > t^{\frac{1}{\log\log t}}$ can obviously be written in the form

$$|t| \leq e^{K_{21} \log n \log\log n}.$$

Finally using Gronwall's inequality (4.6) we obtain for $\sigma \geq 1$, $t > K_{22}$

$$|U_n(s)| \geq |\zeta(s)| - |r_n(s)| > \frac{1}{K_5 \log t} - \frac{K_{20}}{\log^2 t} > 0. \qquad \textit{Q. e. d.}$$

17. Next we sketch the proof of theorem VII. A reasoning similar to that of **11.** shows that from our hypothesis it follows that for $n > K_{23}$

$$\sum_{\nu=1}^{n}{}' (n-\nu+1) \frac{\lambda(\nu)}{\nu^{1+Kn^{-\frac{1}{2}}}} \geqq 0,$$

and by an argument similar to that of **12.** we obtain for these n's

$$C(n) \equiv \sum_{\nu \leqq n}{}' (n-\nu+1) \frac{\lambda(\nu)}{\nu} \geqq -K_{24} \sqrt{n}$$

and

$$C(x) + 2 K_{24} \sqrt{x} \geqq 0 \qquad \text{for } x > K_{25}. \tag{17.1}$$

Next we have to find the analogue of formula (14.1). Generally if for $\sigma > 0$

$$f(s) = \sum_{\nu=1}^{\infty}{}' \frac{d_\nu}{\nu^s}, \qquad \sum_{\nu \leqq n}{}' d_\nu = D_n,$$

$$\sum_{n \leqq m}{}' D_n = \sum_{\nu \leqq m}{}' (m-\nu+1)\, d_\nu = S_m,$$

then

$$\left.\begin{aligned} f(s) &= \sum_{\nu=1}^{\infty}{}' D_\nu \left(\nu^{-s} - (\nu+1)^{-s}\right) = \\ &= \sum_{\nu=1}^{\infty}{}' S_\nu \left(\nu^{-s} - 2(\nu+1)^{-s} + (\nu+2)^{-s}\right) = \\ &= \sum_{\nu=1}^{\infty}{}' S_\nu \left\{ s \int_{\nu}^{\nu+1} x^{-s-1}\, dx - s \int_{\nu+1}^{\nu+2} x^{-s-1}\, dx \right\} = \\ &= s \sum_{\nu=1}^{\infty}{}' S_\nu \int_{\nu}^{\nu+1} \left(x^{-s-1} - (x+1)^{-s-1}\right) dx = \\ &= s \int_{1}^{\infty} S(x) \left(x^{-s-1} - (x+1)^{-s-1}\right) dx = \\ &= s(s+1) \int_{1}^{\infty} S(x) \left(\int_{0}^{1} (x+y)^{-s-2}\, dy \right) dx. \end{aligned}\right\} \tag{17.2}$$

Hence for $\sigma > 0$

$$\frac{\zeta(2s+2)}{\zeta(s+1)} = s(s+1)\int_1^{\infty} C(x)\left(\int_0^1 (x+y)^{-s-2}\,dy\right)dx. \tag{17.3}$$

Let

$$J \equiv s(s+1)\int_1^{\infty}\sqrt{x}\left(\int_0^1 (x+y)^{-s-2}\,dy\right)dx =$$

$$= s(s+1)\int_1^{\infty} x^{-s-\frac{3}{2}}\left(\int_0^1\left(1+\frac{y}{x}\right)^{-s-2} dy\right)dx.$$

But then

$$\frac{J - \dfrac{s(s+1)}{s+\dfrac{1}{2}}}{s(s+1)(s+2)} = \frac{J - s(s+1)\displaystyle\int_1^{\infty} x^{-s-\frac{3}{2}}\,dx}{s(s+1)(s+2)} =$$

$$= -\int_1^{\infty} x^{-s-\frac{3}{2}}\int_0^1 dy\left(\int_1^{1+\frac{y}{x}} \zeta^{-s-3}\,d\zeta\right)dx,$$

and since for $\sigma > -\frac{3}{2}$ the last expression is in absolute value

$$< \frac{1}{2}\int_1^{\infty} x^{-\sigma-\frac{5}{2}}\,dx$$

it represents a function $\vartheta(s)$ regular and bounded in every half-plane $\sigma \geqq -\frac{3}{2} + \varepsilon$. Hence for $\sigma > 0$

$$\left.\begin{aligned} s(s+1)\int_1^{\infty}\sqrt{x}\left(\int_0^1 (x+y)^{-s-2}\,dy\right)dx &= \\ = \frac{s(s+1)}{s+\dfrac{1}{2}} + s(s+1)(s+2)\,\vartheta(s) \end{aligned}\right\} \tag{17.4}$$

and from (17.3) and (17.4)

$$s(s+1)\int_1^\infty \left(C(x)+2K_{24}\sqrt{x}\right)\left(\int_0^1 (x+y)^{-s-2}\,dy\right)dx =$$
$$= \frac{\zeta(2s+2)}{\zeta(s+1)} + 2K_{24}\frac{s(s+1)}{s+\frac{1}{2}} + 2K_{24}\,s(s+1)(s+2)\,\vartheta(s),$$

or for $\sigma > 1$

$$\left.\begin{aligned} &\int_1^\infty \left(C(x)+2K_{24}\sqrt{x}\right)\left(\int_0^1 (x+y)^{-s-1}\,dy\right)dx = \\ &= \frac{\zeta(2s)}{(s-1)\zeta(s)}\cdot\frac{1}{s} + \frac{2K_{24}}{s-\frac{1}{2}} + 2K_{24}(s+1)\,\vartheta(s-1). \end{aligned}\right\} \tag{17.5}$$

This is the required analogue of (17.1). Now Landau's theorem cannot be applied directly; but we can apply his method. Suppose we have proved the following Lemma. If

$$\varphi_1(s) = \int_1^\infty E(x)\left(\int_0^1 (x+y)^{-s-1}\,dy\right)dx \tag{17.6}$$

is convergent in the half-plane $\sigma > 1$ with $E(x)$ non negative for all sufficiently large x and is regular on the real axis for $s > \gamma\ (< 1)$, then $\varphi_1(s)$ is also regular in the whole half-plane $\sigma > \gamma$.

Then (17.1) and the representation (17.5) give that the requirements of the lemma are satisfied with $\gamma = \frac{1}{2}$; hence $(s-1)\,\zeta(s) \neq 0$ in the half-plane $\sigma > \frac{1}{2}$ and theorem VII is proved.

Now we prove the lemma following Landau's paradigma; we prove more, viz. that the representation (17.6) is convergent for $\sigma > \gamma$. Suppose the representation were convergent on the real axis only for $s > \delta$ where $\gamma < \delta \leqq 1$. Then $\varphi_1(s)$ is obviously absolutely convergent in the half-plane $\sigma > \delta$ and regular here. Hence the Taylor-expansion around $s_1 = 2$ is convergent at least in the circle $|s - s_1| < 2 - \delta$; since according to our hypothesis $\varphi_1(s)$ is also regular for $s = \delta$ the radius of the circle of convergence is greater than $2 - \delta$ and hence there is a $\delta_1 < \delta$ such that the Taylor-series is convergent at $s = \delta_1$. But this Taylor-series is

$$\varphi_1(\delta_1) = \sum_{\nu=0}^{\infty} \frac{(-\delta_1+2)^\nu}{\nu!} \int_1^\infty E(x)\left(\int_0^1 (x+y)^{-3} \log^\nu (x+y)\, dy\right) dx\,;$$

since the integrand and all terms are positive we can interchange the summation and integration and hence

$$\varphi_1(\delta_1) = \int_1^\infty E(x)\left(\int_0^1 (x+y)^{-\delta_1-1}\, dy\right) dx,$$

i. e. the integral is convergent for $s = \delta_1 < \delta$, a contradiction. Hence the lemma is proved and the proof of theorem VII is completed.

18. Now we sketch the proof of theorem VIII. The arguments of **11.** and **12.** show that for $n > K_{25}$

$$\sum_{\nu \leqq n} (-1)^{\nu+1} \lambda(\nu)\, \nu^{-1-Kn^{-\frac{1}{2}}} \geqq 0\,;$$

for these n's

$$L_1(n) \equiv \sum_{\nu \leqq n} (-1)^{\nu+1} \frac{\lambda(\nu)}{\nu} > -K_{26}\, n^{-\frac{1}{2}}$$

and for $x > K_{27}$

$$L_1(x) + 2K_{26}\, x^{-\frac{1}{2}} > 0. \tag{18.1}$$

Next we must find the generating functions of $\sum (-1)^{\nu+1} \frac{\lambda(\nu)}{\nu^s}$. We assert that for $\sigma > 1$

$$g(s) = \sum_{\nu=1}^{\infty} (-1)^{\nu+1} \frac{\lambda(\nu)}{\nu^s} = \left(1 + \frac{2}{2^s}\right) \frac{\zeta(2s)}{\zeta(s)}. \tag{18.2}$$

For

$$g(s) = 2 \sum_{\nu \text{ odd}} \frac{\lambda(\nu)}{\nu^s} - \sum_{\nu=1}^{\infty} \frac{\lambda(\nu)}{\nu^s} = 2 \prod_{p>2} \frac{1}{1+\frac{1}{p^s}} - \frac{\zeta(2s)}{\zeta(s)} =$$

$$= \left\{2\left(1+\frac{1}{2^s}\right) - 1\right\} \frac{\zeta(2s)}{\zeta(s)} = \left(1 + \frac{2}{2^s}\right) \frac{\zeta(2s)}{\zeta(s)}.$$

Hence for $\sigma > 0$

$$s \int_1^{\infty} \frac{L_1(x)}{x^{s+1}}\, dx = \left(1 + \frac{2}{2^{s+1}}\right) \frac{\zeta(2s+2)}{\zeta(s+1)}$$

$$\int_1^{\infty} \frac{L_1(x) + 2K_{26}\, x^{-\frac{1}{2}}}{x^{s+1}}\, dx = \left(1 + \frac{2}{2^{s+1}}\right) \frac{\zeta(2s+2)}{s\zeta(s+1)} + \frac{2K_{26}}{s+\frac{1}{2}},$$

or for $\sigma > 1$

$$\int_1^{\infty} \frac{L_1(x) + 2K_{26}\, x^{-\frac{1}{2}}}{x^{s}}\, dx = \left(1 + \frac{2}{2^{s}}\right) \frac{\zeta(2s)}{(s-1)\zeta(s)} + \frac{2K_{26}}{s-\frac{1}{2}}.$$

Now the remainder of the proof is similar to that of theorem IV.

19. Finally we show that for an infinity of n's the partial-sums[1)]

$$V_{2n}(s) = \sum_{\nu \leqq 2n} (-1)^{\nu+1} \nu^{-s}$$

vanish in the half-plane $\sigma > 1$. For this purpose we consider the values $V_{2n}(w_k)$, k fixed and $w_k = 1 + \dfrac{2k\pi i}{\log 2} \equiv 1 + iv_k$.

20. First we show that

$$\left| V_{2n}(w_k) + \frac{1}{4}(n+1)^{-1+iv_k} \right| < \frac{(2+|v_k|)^2}{n^2}. \tag{20.1}$$

Starting from the identity

$$\left(1 - \frac{2}{2^s}\right) U_n(s) = V_{2n}(s) - 2^{1-s}\left(\frac{1}{(n+1)^s} + \cdots + \frac{1}{(2n)^s}\right)$$

we obtain, on setting $s = w_k$,

$$V_{2n}(w_k) = \frac{1}{(n+1)^{w_k}} + \frac{1}{(n+2)^{w_k}} + \cdots + \frac{1}{(2n)^{w_k}}. \tag{20.2}$$

1) The restriction to even indices is only for the sake of convenience.

Now we have

$$\left(1+\frac{1}{\nu}\right)^{iv_k}-1-\binom{iv_k}{1}\frac{1}{\nu}-\binom{iv_k}{2}\frac{1}{\nu^2}=3\binom{iv_k}{3}\int_0^{1/\nu}\left(\frac{1}{\nu}-y\right)^2(1+y)^{iv_k-3}\,dy;$$

hence

$$\left|(\nu+1)^{iv_k}-\nu^{ia_k}-\binom{iv_k}{1}\nu^{-1+iv_k}-\binom{iv_k}{2}\nu^{-2+iv_k}\right|<|v_k|\frac{(2+|v_k|)^2}{6}\cdot\frac{1}{\nu^3}.$$

Putting $\nu=(n+1),(n+2),\cdots 2n$ and summing we obtain

$$\left.\begin{aligned}\left|(2n+1)^{iv_k}-(n+1)^{iv_k}-\binom{iv_k}{1}\sum_{\nu=n+1}^{2n}\nu^{-1+iv^k}-\binom{iv_k}{2}\sum_{\nu=n+1}^{2n}\nu^{-2+iv_k}\right|<\\ <|v_k|\frac{(2+|v_k|)^2}{12}\cdot\frac{1}{n^2}.\end{aligned}\right\}\tag{20.3}$$

The first two bracketed terms can be written in the form

$$\left.\begin{aligned}&\left((2n+1)^{iv_k}-(2n+2)^{iv_k}\right)+\left((2n+2)^{iv_k}-(n+1)^{iv_k}\right)=\\ &=-(2n+1)^{iv_k}\left(\left(1+\frac{1}{2n+1}\right)^{iv_k}-1\right)+(n+1)^{iv_k}\left(2^{iv_k}-1\right)=\\ &=-(2n+1)^{iv_k}\left(\left(1+\frac{1}{2n+1}\right)^{iv_k}-1\right);\end{aligned}\right\}\tag{20.4}$$

hence from (20.2), (20.3) and (20.4)

$$\left.\begin{aligned}\left|-(2n+1)^{iv_k}\left(\left(1+\frac{1}{2n+1}\right)^{iv_k}-1\right)-iv_k\,V_{2n}(w_k)-\binom{iv_k}{2}\sum_{\nu=n+1}^{2n}\nu^{-2+iv_k}\right|<\\ <|v_k|\frac{(2+|v_k|)^2}{12}\cdot\frac{1}{n^2}.\end{aligned}\right\}\tag{20.5}$$

Now

$$\left(1+\frac{1}{2n+1}\right)^{iv_k}-1-\frac{iv_k}{2n+1}=\int_0^{\frac{1}{2n+1}}\left(\frac{1}{2n+1}-y\right)(1+y)^{iv_k-2}\,dy\,(iv_k)\,(iv_k-1)$$

$$\left|\left(1+\frac{1}{2n+1}\right)^{iv_k}-1-\frac{iv_k}{2n+1}\right|<\frac{1}{2}\,\frac{|v_k|\,(1+|v_k|)}{(2n+1)^2};$$

hence from (20.5)

$$\left|-iv_k\,(2n+1)^{-1+iv_k}-iv_k\,V_{2n}(w_k)-\binom{iv_k}{2}\sum_{\nu=n+1}^{2n}\nu^{-2+iv_k}\right|<$$

$$<|v_k|\,\frac{(2+|v_k|)^2}{12}\cdot\frac{1}{n^2}+\frac{1}{8}\,\frac{|v_k|\,(1+|v_k|)}{n^2}<\frac{|v_k|\,(2+|v_k|)^2}{4}\cdot\frac{1}{n^2}$$

or

$$\left|V_{2n}(w_k)+(2n+1)^{-1+iv_k}-\frac{1-iv_k}{2}\sum_{\nu=n+1}^{2n}\nu^{-2+iv_k}\right|<\frac{(2+|v_k|)^2}{4}\cdot\frac{1}{n^2}. \qquad (20.6)$$

Further

$$\left(1+\frac{1}{\nu}\right)^{-1+iv_k}-1-(-1+iv_k)\,\frac{1}{\nu}=$$

$$=\int_0^{1/\nu}\left(\frac{1}{\nu}-y\right)(1+y)^{-3+iv_k}\,dy\,(-1+iv_k)\,(-2+iv_k)$$

$$\left|(\nu+1)^{-1+iv_k}-\nu^{-1+iv_k}+(1-iv_k)\,\nu^{-2+iv_k}\right|<\frac{1}{\nu^3}\cdot\frac{1}{2}\,(2+|v_k|)^2;$$

a summation over $\nu=(n+1),\cdots,2n$ gives

$$\left|-\frac{(2n+1)^{-1+iv_k}}{2}+\frac{(n+1)^{-1+iv_k}}{2}-\frac{1-iv_k}{2}\sum_{\nu=n+1}^{2n}\nu^{-2+iv_k}\right|<\frac{(2+|v_k|)^2}{4}\cdot\frac{1}{n^2}.$$

Putting this into (20.6) we obtain

$$\left|V_{2n}(w_k)+\left(\frac{3}{2}\,(2n+1)^{-1+iv_k}-\frac{1}{2}\,(n+1)^{-1+iv_k}\right)\right|<\frac{1}{2}\,\frac{(2+|v_k|)^2}{n^2}. \qquad (20.7)$$

If we use the transformation (20.4) again the sum of the two bracketed terms is

$$\left.\begin{aligned}&\frac{3}{2}\left((2n+1)^{-1+iv_k}-(2n+2)^{-1+iv_k}\right)+\\&+\left(\frac{3}{2}\cdot 2^{-1+iv_k}-\frac{1}{2}\right)(n+1)^{-1+iv_k}=\frac{1}{4}(n+1)^{-1+iv_k}-\\&-\frac{3}{2}(2n+1)^{-1+iv_k}\left(\left(1+\frac{1}{2n+1}\right)^{-1+iv_k}-1\right).\end{aligned}\right\}\quad(20.8)$$

Since the second term of the right-hand member of (20.8) is in absolute value

$$<\frac{3}{2}\cdot\frac{1}{2n+1}\left(1+|v_k|\right)\frac{1}{2n+1}<\frac{3}{8}\frac{\left(2+|v_k|\right)^2}{n^2}$$

we obtain from (20.7) and (20.8)

$$\left|V_{2n}(w_k)+\frac{1}{4}(n+1)^{-1+iv_k}\right|<\frac{\left(2+|v_k|\right)^2}{n^2},\qquad(20.9)$$

i. e. (20.1) is proved.

Now we consider $V'_{2n}(w_k)$. Obviously

$$V'_{2n}(w_k)=\left(\left(1-\frac{2}{2^s}\right)\zeta(s)\right)'_{s=w_k}+\sum_{\nu>2n}{}'_{\nu}\frac{(-1)^{\nu+1}\log\nu}{\nu^{w_k}}.\qquad(21.1)$$

The first term is obviously

$$\log 2\cdot\zeta(w_k).\qquad(21.2)$$

To estimate the second term of (21.1) we observe that

$$\left|\frac{\log(2l+1)}{(2l+1)^{w_k}}-\frac{\log(2l+2)}{(2l+2)^{w_k}}\right|\leq\frac{1}{2l+1}\log\frac{2l+2}{1l+1}+$$

$$+\log(2l+2)\left|(2l+1)^{-w_k}-(2l+2)^{-w_k}\right|<\frac{\left(2+|v_k|\right)\log(2l+2)}{(2l+1)^2};$$

hence for $n>10$ the second term of (21.1) is in absolute value

$$<\left(2+|v_k|\right)\sum_{\nu>n}{}'\frac{\log(2\nu+2)}{(2\nu+1)^2}<\left(2+|v_k|\right)\frac{\log n}{n}.$$

From this (21.2) and (21.1) we obtain

$$\left| V_{2n}'(w_k) - \log 2 \cdot \zeta(w_k) \right| < \left(2 + |w_k|\right) \frac{\log n}{n}. \qquad (21.3)$$

Now we consider the expression

$$F_{2n}(s) = V_{2n}(w_k) + V_{2n}'(w_k)(s - w_k), \qquad (21.4)$$

which is linear and has its zero at

$$w_{kn}' = w_k - \frac{V_{2n}(w_k)}{V_{2n}'(w_k)};$$

using (20.1) and (21.3) we get

$$w_{kn}' = w_k + \frac{1}{4 \log 2 \cdot \zeta(w_k)} (n+1)^{-1+iv_k} + \Theta_3 K_{27}(k) \frac{\log n}{n^2}. \quad (21.5)$$

$$|\Theta_3| \leqq 1.$$

22. We show that for fixed k and $n \to \infty$

$$w_{kn}^* = w_k + \frac{1}{4 \log 2 \cdot \zeta(w_k)} (n+1)^{-1+iv_k} \qquad (22.1)$$

is the asymptotical expression of that root of $V_{2n}(s)$ which $\to w_k$ if $n \to \infty$. To show this we consider the circle

$$|s - w_{kn}'| = K_{27}(k) \frac{\log^2 n}{n^2}; \qquad (22.2)$$

(21.5) and (22.1) show that w_{kn}^* lies in this circle. We have

$$V_{2n}(s) = F_{2n}(s) + \sum_{j=2}^{\infty} \frac{1}{j!} V_{2n}^{(j)}(w_k)(s - w_k)^j. \qquad (22.3)$$

For $F_{2n}(s)$ we have identically

$$F_{2n}(s) = V_{2n}'(w_k)(s - w_{kn}'),$$

so that on the whole circumference of the circle (22.2), if n is sufficiently large,

$$|F_{2n}(s)| > \frac{1}{2} |\zeta(w_k)| \frac{K_{27}(k)\log^2 n}{n^2} = K_{28}(k)\frac{\log^2 n}{n^2}. \quad (22.4)$$

Since on the circle $|s - w_k| \leqq \frac{1}{2}$

$$|V_{2n}(s)| \leqq K_{29}(k)$$

independently of n we have from Cauchy's estimation of coefficients

$$\left|\frac{V_{2n}^{(j)}(w_k)}{j!}\right| < K_{29}(k)\, 2^j$$

and on the circle (22.2)

$$|s - w_k| \leqq |s - w'_{kn}| + |w'_{kn} - w_k| \leqq K_{27}(k)\frac{\log^2 n}{n^2} + \frac{K_{30}(k)}{n} + K_{27}(k)\frac{\log n}{n^2} < K_{31}(k)\frac{1}{n};$$

hence on the circle (22.2) the second term of (22.3) is in absolute value

$$< \sum_{j=2}^{\infty} K_{29}(k)\left(\frac{2K_{31}(k)}{n}\right)^j < K_{32}(k)\frac{1}{n^2}. \quad (22.5)$$

From (22.4) and (22.3) we obtain for $n > K_{33}(k)$ on the circumference of (22.2)

$$|F_{2n}(s)| > K_{28}(k)\frac{\log^2 n}{n^2} > \frac{K_{32}(k)}{n^2} > \left|\sum_{j=2}^{\infty}\frac{1}{j!} V_{2n}^{(j)}(w_k)(s - w_k)^j\right|;$$

hence it follows from Rouché's theorem that the circle (22.2) contains as many zeros of $V_{2n}(s)$ as of $F_{2n}(s)$, i. e. exactly one. But this circle is contained in the circle

$$|s - w^*_{kn}| < |s - w'_{kn}| + |w'_{kn} - w^*_{kn}| < 2K_{27}\frac{\log^2 n}{n^2}. \quad (22.6)$$

23. To prove finally that for an infinity of n's the polynomials $V_{2n}(s)$ vanish in the half-plane $\sigma > 1$ we have only to observe that for fixed k

$$v_k \log(n+1) - v_k \log n \to 0.$$

Then for an infinity of n's we have

$$\Re w^*_{kn} > 1 + \frac{1}{5 \log 2 \,|\, \zeta\, w_k) \,|} \cdot \frac{1}{n+1};$$

hence from (22.6) the real part of the corresponding root of $V_{2k}(s)$ is

$$> 1 + \frac{1}{5 \log 2 \,|\, \zeta(w_k) \,|} \cdot \frac{1}{n+1} - \frac{2 K_{27}(k) \log^2 n}{n^2} > 1,$$

if n is sufficiently large.

We remark finally that for fixed k and $n \to \infty$ these roots in the half-plane $\sigma > 1$ lie in half-planes of the form

$$1 + \frac{K_{33}(k)}{n},$$

i. e. their location does not refute the hypothesis of theorem VIII. It would be interesting to study these roots if k is not fixed.

It is perhaps of some interest to note that for fixed k the behaviour of the corresponding roots of the Riesz-means is different. Denoting by w''_{kn} that root of the n^{th} Riesz-mean of the series (6.1) which for $n \to \infty$ tends to w_k, we have

$$\left| w''_{kn} - w_k + \frac{1}{\log n} \right| < \frac{K_{34}(k)}{\log^2 n}.$$

Hence these roots converge to w_k from the left in a particularly simple way. Thus there is some chance that the behaviour of the roots of the Riesz-means is more regular.

Added in proof.

24. An easy modification of the proofs gives also a more general theorem from which I mention only two special cases.

Theorem X. If for a modulus k there is a character $\chi(n)$

such that the partial-sums of the corresponding L-function of Dirichlet

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{24.1}$$

do not wanish in the half-plane $\sigma > 1$, then Riemann's hypothesis (1.5) is true.

I cannot prove that this property of the partial-sums of (24.1) implies the non-vanishing of $L(s,\chi)$ itself.

Of course theorem X admits all refinements similarly as theorems II–V refine theorem I.

The interest of theorem X compared to theorem VIII lies obviously in the fact that the function $L(s,\chi)$ has no roots on the line $\sigma = 1$, in contrast to $\left(1 - \frac{2}{2^s}\right)\zeta(s)$.

Theorem XI. If for real sequence $\beta_1, \beta_2, \ldots$ the partial-sums of the series

$$f_\beta(s) = \prod_{\nu=1}^{\infty} \frac{1}{1 - \frac{e^{i\beta_\nu}}{p_\nu^s}} = \sum_{n=1}^{\infty} \frac{d_n}{n^s} \tag{24.2}$$

do not vanish in the half-plane $\sigma > 1$, then Riemann's hypothesis (1.5) is true.

Prof. Jessen[1)] proved that for "almost all" β-sequences the functions $f_\beta(s)$ do not vanish in the half-plane $\sigma > \frac{1}{2}$. To obtain an explicit $f_\beta(s)$ which has this property, we may choose according to a remark of Prof. A. Selberg

$$e^{i\beta_\nu} = (-1)^\nu,$$

where $p_1 = 2$, $p_2 = 3$, ... denote the increasingly ordered sequence of primes.

Theorem XI admits the same refinements as theorem X.

25. We proved implicitly that if the partial-sums

1) B. Jessen: Some analytical problems relating to probability. Journ. of Math. and Physics. Mass. Inst. Techn. vol. XIV (1935), p. 24—27.

$$L(n) = \sum_{\nu \leq n} \frac{\lambda(\nu)}{\nu} \tag{25.1}$$

are of one sign for all sufficiently large n's or are for these n's

$$> -K_{11}\, n^{\vartheta - 1}, \tag{25.2}$$

then the hypothesis (1.6) is true. Pólya[1] remarked, that if

$$L_1(n) = \sum_{\nu \leq n} \lambda(\nu) \tag{25.3}$$

is non-positive for all sufficiently large n's, then Riemann's hypothesis (1.5) is true; with the same reasoning he could prove that from the inequality

$$L_1(n) < K_{35}\, n^{\vartheta}, \tag{25.4}$$

valid for all sufficiently large n's, the conjecture (1.6) follows. It seems to me that the condition (25.2) is somewhat less deep than (25.4), i. e. one can deduce (25.2) from (25.4). If we replace, however, (25.2) and (25.4) by twosided inequalities, the corresponding statement follows by partial summation.

Pólya showed by computation the validity of (25.3) for $2 \leq n \leq 1500$; this has been extended by H. Gupta[2] up to 20,000. The young danish mathematicians Erik Eilertsen, Poul Kristensen, Aage Petersen, Niels Ovè Roy Poulsen, and Aage Winther calculated the values of $L(n)$ for $n \leq 1000$. They found all of them to be positive; for $L(1000)$ they found the value

$$L(1000) = 0{,}028970560\,.$$

It is remarkable that in this range the minimal value is attained at $n = 293$ and that

$$L(293) = 0{,}005102273,$$

a much smaller value than $L(1000)$.

1) G. Pólya: Verschiedene Bemerkungen zur Zahlentheorie. Jahresb. der deutsch. Math. Ver. 28 (1919), p. 31—40.

2) H. Gupta: On a table of values of $L_1(n)$. Proc. Indian Acad. Sci. Sect. A. vol. 12 (1940), p. 407—409.

26. I conclude with two remarks. As Paul Erdös remarked, he can prove that to any given closed set H in the domain $|z| \geqq 1$ which contains the circumference of the unit-circle one can give a power-series, convergent in $|z| < 1$, such that the roots of its partial-sums cluster in $|z| \geqq 1$ to the points of H and only to those.

As Prof. Sherwood Sherman remarked, my statement on p. 13 about the Jensen-means is true only if we suppose in addition of the roots of the function $f(s)$, that the sum of their reciprocal values is convergent.

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