The Measure of Littlewood Polynomials

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We prove that the minimum value of the Mahler measure of a nonreciprocal polynomial with all odd coefficients is the golden ratio.

A Littlewood polynomial has all its coefficients equal to ± 1. We determine the smallest measures among reciprocal Littlewood polynomials with degree at most 72.

Introduction

The *Mahler measure* of a polynomial

$$f(x) = \sum_{i=0}^{n} a_i x^i = a_n \prod_{i=1}^{n} (x - \alpha_i)$$

is defined by

$$M(f) = |a_n| \prod_{i=1}^{n} \max\{1, |\alpha_i|\}. \qquad (1)$$

The measure is unchanged if coefficients are reversed. So if

$$f^*(x) = x^n f(1/x)$$

then $M(f^*) = M(f)$. If $f = \pm f^*$ then f is *reciprocal*.

For polynomials with integer coefficients Kronecker's Theorem states that $M(f) = 1$ if and only if $f(x)$ is a product of cyclotomic polynomials and the monomial x.

In 1933, D. H. Lehmer asked if for any $\epsilon > 0$ there exists $f(x) \in \mathbf{Z}[x]$ with

$$1 < M(f) < 1 + \epsilon$$

and this problem remains open.

Lehmer noted that

$$\ell(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

has measure

$$M(\ell) = 1.1762808\ldots$$

and this remains the smallest known measure greater than 1 of a polynomial with integer coefficients.

Smyth in 1971 answered Lehmer's question for the case of nonreciprocal polynomials, proving that if $f(x) \in \mathbf{Z}[x]$ is nonreciprocal and $f(0) \neq 0$ then

$$M(f) \geq M(x^3 - x - 1) = 1.324717\ldots$$

A *Littlewood polynomial* $f(x) = \sum_{i=0}^{n} a_i x^i$ has $a_i = \pm 1$ for each i.

We (PB and Stephen Choi) characterize the Littlewood polynomials of even degree with measure 1, providing a sharper version of Kronecker's theorem for this class of polynomials.

Recall that Φ_n is given by

$$\Phi_n(z) = \prod_{\substack{1 \le j \le n \\ \gcd(j,n)=1}} \Big(z - \exp(j 2\pi i/n) \Big).$$

Conjecture. A Littlewood polynomial $P(z)$ of degree $N-1$ has Mahler measure 1 if and only if P can be written in the form

$$P(z) = \pm \Phi_{p_1}(\pm z) \Phi_{p_2}\left(\pm z^{p_1}\right) \cdots \Phi_{p_r}\left(\pm z^{p_1 p_2 \cdots p_{r-1}}\right),$$

where $N = p_1 p_2 \cdots p_r$ and the p_i are primes, not necessarily distinct.

Our main result provides a lower bound on the measure for a larger class of nonreciprocal polynomials.

Theorem 1 *Suppose f is a monic, nonreciprocal polynomial with integer coefficients satisfying $f \equiv \pm f^*$ mod m for some integer $m \geq 2$. Then*

$$M(f) \geq \frac{m + \sqrt{m^2 + 16}}{4}, \tag{2}$$

and this bound is sharp when m is even.

Taking $m = 2$, we immediately obtain the golden ratio as a sharp lower bound for the measure of a nonreciprocal Littlewood polynomial.

Corollary 1 *If f is a monic, nonreciprocal polynomial whose coefficients are all odd integers, then $M(f) \geq M(x^2 - x - 1) = (1 + \sqrt{5})/2$. In particular, this bound holds for nonreciprocal Littlewood polynomials.*

A Pisot number is a real algebraic integer greater than 1, all of whose conjugates lie inside the open unit disk. Smyth's lower bound is the smallest Pisot number; the golden ratio is the smallest limit point of Pisot numbers.

Later we describes some computations for reciprocal Littlewood polynomials through degree 72 and list fifteen measures of Littlewood polynomials less than 1.6.

The smallest measure we find is $1.496711\ldots$, associated with the polynomial

$$x^{19}+x^{18}+x^{17}+x^{16}-x^{15}+x^{14}-x^{13}+x^{12}-x^{11}-x^{10}$$
$$-x^9-x^8+x^7-x^6+x^5-x^4+x^3+x^2+x+1$$

Proof of Theorem

Our proof follows Smyth 1971. We require the following inequality regarding coefficients of power series.

Lemma 1 *(Schinzel p. 392.) Suppose* $\varphi(z) = \sum_{i \geq 0} \gamma_i z^i$ *with* $\gamma_i \in \mathbf{C}$ *is analytic in an open disk containing* $|z| \leq 1$ *and satisfies* $|\varphi(z)| \leq 1$ *on* $|z| = 1$*. Then for* $i \geq 1$

$$|\gamma_i| \leq 1 - |\gamma_0|^2$$

.

Proof of Theorem Suppose

$$f(z) = \sum_{i=0}^{n} a_i z^i = \prod_{i=1}^{n} (z - \alpha_i)$$

Write

$$f^*(z) = \sum_{i=0}^{n} d_i z^i$$

so $d_0 = 1$, and let

$$\sum_{i \geq 0} e_i z^i$$

be the power series for $1/f^*(z)$. Note that the e_i are integers.

Let

$$G(z) = f(z)/f^*(z) = \sum_{i \geq 0} q_i z^i,$$

so $q_i \in \mathbf{Z}$ for $i \geq 0$. Clearly $q_0 = a_0$.

The key observation is that since

$$f \equiv \pm f^* \mod m$$

we have

$$a_j \equiv q_0 d_j \mod m.$$

So by an easy induction, for $j \geq 1$,

$$m \mid q_j$$

and, if $q_j \neq 0$, then $m \leq |q_j|$

Now let $\epsilon = \pm 1$ and let

$$g(z) = \epsilon \prod_{|\alpha_i|<1} \frac{z - \alpha_i}{1 - \overline{\alpha_i} z} \quad \text{and} \quad h(z) = \prod_{|\alpha_i|>1} \frac{1 - \overline{\alpha_i} z}{z - \alpha_i},$$

so

$$\frac{g(z)}{h(z)} = \frac{\prod_{i=1}^{n} (z - \alpha_i)}{\prod_{i=1}^{n} (1 - \overline{\alpha_i} z)}$$

$$= \frac{\prod_{i=1}^{n} (z - \alpha_i)}{\prod_{i=1}^{n} (1 - \alpha_i z)} = \frac{f(z)}{f^*(z)} = G(z).$$

Clearly all poles of both $g(z)$ and $h(z)$ lie outside the unit disk, so both functions are analytic in a region containing $|z| \le 1$.

Further, if $|z| = 1$ and $\beta \in \mathbf{C}$ then

$$\left(\frac{z - \beta}{1 - \overline{\beta} z} \right) \overline{\left(\frac{z - \beta}{1 - \overline{\beta} z} \right)} = \left(\frac{z - \beta}{1 - \overline{\beta} z} \right) \left(\frac{1/z - \overline{\beta}}{1 - \beta / z} \right) = 1,$$

so $|g(z)| = |h(z)| = 1$ on $|z| = 1$.

Let

$$g(z) = \sum_{i \geq 0} b_i z^i \quad \text{and} \quad h(z) = \sum_{i \geq 0} c_i z^i.$$

Let k be the smallest integer for which $q_k \neq 0$, so $|q_k| \geq m$.

Since $g(z) = h(z)G(z)$,

$$b_i = c_i q_0$$

for $0 \leq i < k$ and

$$b_k = c_0 q_k + c_k q_0$$

Thus

$$|c_0 m| \leq |c_0 q_k| = |b_k - c_k q_0| \leq 2 \max\{|b_k|, |c_k|\}. \tag{3}$$

Assume without loss of generality that $|c_k| \geq |b_k|$. By Lemma 1,

$$|c_k| \leq 1 - c_0^2.$$

So with the observation that

$$|c_0| = |h(0)| = \prod_{|\alpha_i|>1} 1/|\alpha_i| = 1/M(f)$$

we have

$$M(f)m \leq 2(M(f)^2 - 1).$$

The theorem follows, and the bound is achieved when m is even by

$$f(z) = z^2 \pm mz/2 - 1.$$

Reciprocal Littlewood polynomials with small measure

We describe an algorithm for searching for reciprocal Littlewood polynomials with small Mahler measure.

Algorithm Given a positive integer d, we wish to determine all reciprocal Littlewood polynomials $f(x) = \sum_{i=0}^{d} a_i x^i$ having $1 < M(f) < M$, where M is a fixed constant.

Following Boyd, we use the Graeffe root-squaring algorithm to screen out most polynomials f having $M(f) > M$ and all polynomials with $M(f) = 1$ in an efficient way.

Recall that the Graeffe operator G applied to a polynomial $f(x)$ written as

$$f(x) = g(x^2) + xh(x^2)$$

yields the polynomial

$$Gf(x) = g(x)^2 - xh(x)^2.$$

The roots of Gf are precisely the squares of the roots of f, and $M(Gf) = M(f)^2$.

Let $a_{k,m}$ denote the coefficient of x^k in $G^m f(x)$.

Boyd shows that

$$\left|a_{k,m}\right| \le \binom{d}{k} + \binom{d-2}{k-1}\left(M^{2^m} + M^{-2^m} - 2\right) \quad (4)$$

for all m

If in addition $a_{1,m} \ge d-4$ and $m \ge 1$, then

$$\left|a_{k,m}\right| \le \binom{d}{k} + \binom{d-4}{k-2}\left(M^{2^m} + M^{-2^m} - 2\right) + 2\left(M^{2^{m-1}} + M^{-2^{m-1}} - 2\right)\left(\binom{d-4}{k-3} + \binom{d-4}{k-1}\right). \quad (5)$$

We apply the Graeffe operator to each polynomial at most m_0 times, where m_0 is another fixed parameter of the algorithm.

A polynomial f is rejected at stage m if the appropriate inequality (4) or (5) is not satisfied for some k, or if $G^m f = G^{m-1} f$.

In the latter case, Kronecker's theorem implies that f is a product of cyclotomic polynomials.

Let Φ_n denote the nth cyclotomic polynomial. If $n = 2^r s$ with s odd, then

$$G^m \Phi_{2^r s} = \Phi_s^{2^{r-1}}$$

when $m \geq r$, so the Graeffe method is guaranteed to detect a product of cyclotomic polynomials with total degree d if $m \geq 1 + \log_2 d$.

Results and analysis

We ran our program at HPC@SFU, the high performance computing centre at Simon Fraser University, on The Bugaboos, a Beowulf cluster with 96 nodes, each with two AMD Athlon 1.2 GHz processors.

In two weeks we searched through degree 72, using as many as 64 processors at once and totaling 426 days of CPU time.

Our program finds 1643 Littlewood polynomials with degree at most 72 that survive ten iterations of root-squaring; of these, 1487 have measure less than $M = 5/3$.

Only 127 distinct measures less than 5/3 appear, since most measures occur several times.

15 known measures of Littlewood polynomials less than 1.6**.**

1.49671107561
1.50613567955
1.50646000575
1.53691794778
1.55107223951
1.55603019132
1.57930874185
1.58234718368
1.58501169305
1.59185616779
1.59287323067
1.59341317381
1.59504631133
1.59700500917
1.59918220880

The seventh polynomial above is the only one listed whose noncyclotomic part is reducible.

There are certainly an infinite number of polynomials having $\{-1, 0, 1\}$ coefficients with smaller measure. For example, the measure of

$$x^{2n+2} + x^{2n+1} + x^{n+2} + x^{n+1} + x^n + x + 1$$

approaches $1.255433\ldots$ as $n \to \infty$. This is the smallest known limit point of measures of integer polynomials.

There are in fact an infinite number of limit points of measures of polynomials with $\{-1, 0, 1\}$ coefficients less than 1.382.

There are also infinitely many integer polynomials with reducible noncyclotomic part having measure less than 1.4967, since two noncyclotomic polynomials are known with measure less than $1.4967/1.2554 \approx 1.1922$.

It therefore seems quite possible that Littlewood polynomials with Mahler measure smaller than $1.496711\ldots$ exist. It appears likely however that additional techniques would be required in further searches.