MARKOV AND BERNSTEIN TYPE INEQUALITIES IN L_p FOR CLASSES OF POLYNOMIALS WITH CONSTRAINTS

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ABSTRACT

The Markov-type inequality

$$\int_{-1}^{1} |f'(x)|^p \, dx \leqslant c(p)\,(n(k+1))^p \int_{-1}^{1} |f(x)|^p \, dx$$

is proved for all real algebraic polynomials f of degree at most n having at most k, with $0 \leqslant k \leqslant n$, zeros (counting multiplicities) in the open unit disk of the complex plane, and for all $p > 0$, where $c(p) = c^{p+1}(1+p^{-2})$ with some absolute constant $c > 0$. This inequality has been conjectured since 1983 when the L_∞ case of the above result was proved. It improves and generalizes many earlier results. Up to the multiplicative constant $c(p) > 0$ the above inequality is sharp. A sharp Bernstein-type analogue for real trigonometric polynomials is also established, which is interesting on its own, and plays a key role in the proof of the Markov-type inequality.

1. *Introduction, notation*

Bernstein's inequality **[16**, pp. 39–41] asserts that

$$\max_{-\pi \leqslant t \leqslant \pi} |f'(t)| \leqslant n \max_{-\pi \leqslant t \leqslant \pi} |f(t)| \tag{0.1}$$

for every $f \in \mathcal{T}_n$, where $\mathcal{T}_n$ denotes the set of all trigonometric polynomials of degree at most n with real coefficients. The corresponding algebraic result **[16**, pp. 39–41], known as Markov's inequality, states that

$$\max_{-1 \leqslant x \leqslant 1} |f'(x)| \leqslant n^2 \max_{-1 \leqslant x \leqslant 1} |f(x)| \tag{0.2}$$

for all $f \in \mathcal{P}_n$, where $\mathcal{P}_n$ denotes the set of all algebraic polynomials of degree at most n with real coefficients. The Chebyshev polynomials $Q_n \in \mathcal{T}_n$ and $T_n \in \mathcal{P}_n$ defined by

$$Q_n(t) := \cos(nt + \alpha) \quad \text{for } \alpha \in \mathbb{R}, \tag{0.3}$$

$$T_n(x) := \cos(n \arccos x) \quad \text{for } -1 \leqslant x \leqslant 1 \tag{0.4}$$

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show that (0.1) and (0.2) are sharp. The substitution $x = \cos t$ in (0.1), together with (0.2), yields

$$|f'(y)| \leqslant \min\left\{n^2, \frac{n}{\sqrt{(1-y^2)}}\right\} \max_{-1 \leqslant x \leqslant 1} |f(x)| \quad \text{with } -1 \leqslant y \leqslant 1 \tag{0.5}$$

for every $f \in \mathscr{P}_n$. The sharp L_p version of Bernstein's inequality was first established by A. Zygmund [25, II (3.17) p. 11] for $p \geqslant 1$. It states that

$$\int_{-\pi}^{\pi} |f'(t)|^p\,dt \leqslant n^p \int_{-\pi}^{\pi} |f(t)|^p\,dt \tag{0.6}$$

for every $f \in \mathscr{T}_n$ and $1 \leqslant p < \infty$. For $0 < p < 1$, first G. Klein [14] and later P. Osval'd [21] proved (0.6) with a multiplicative constant $c(p)$. In [20] Nevai proved that $c(p) = 8/p$ is a possible choice. Subsequently, Máté and Nevai [19] showed the validity of (0.6) with a multiplicative absolute constant, and then V. V. Arestov [1] proved (0.6) (with the best constant 1) for every $0 < p < 1$. Recently M. von Golitschek and G. G. Lorentz [13] found a very elegant proof of Arestov's Theorem. Markov's inequality in L_p gives

$$\int_{-1}^{1} |f'(x)|^p\,dx \leqslant c^{p+1} n^{2p} \int_{-1}^{1} |f(x)|^p\,dx \tag{0.7}$$

for every $f \in \mathscr{P}_n$, where $c > 0$ is an absolute constant. This can be proved from the above L_p Bernstein-type inequalities by the substitution $x = \cos t$ and by using Nikolskii-type inequalities (cf. [19, 17]). Finding the best constant in (0.7) is still an open problem. Markov and Bernstein type inequalities in weighted spaces and in L_p norms play a key role in proving inverse theorems of approximation and of course have their own intrinsic interest.

Denote by $\mathscr{P}(n,k)$ the set of all $p \in \mathscr{P}_n$ having at most k zeros (counting multiplicities) in the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. Markov and Bernstein type inequalities for constrained polynomials have been studied in many research papers where the classes $\mathscr{P}(n,k)$ for $0 \leqslant k \leqslant n$ are of special interest. One might correctly suspect that the restrictions on the zeros of a polynomial imply an improvement in inequalities (0.5), (0.6) and (0.7). In 1940 Erdös [12] proved that there is an absolute constant $c > 0$ such that

$$|f'(y)| \leqslant \min\left\{\frac{en}{2}, \frac{c\sqrt{n}}{(1-y^2)^2}\right\} \max_{-1 \leqslant x \leqslant 1} |f(x)| \quad \text{with } -1 \leqslant y \leqslant 1 \tag{0.8}$$

for every $f \in \mathscr{P}(n,0)$ having only real zeros. By taking the polynomials $p_n \in \mathscr{P}(n,0)$ defined by $p_n(x) := (1+x)^{n-1}(1-x)$, it is easy to see that the constant $\frac{1}{2}e$ in (0.8) is asymptotically sharp. In 1963 G. G. Lorentz [15] showed that there is an absolute constant $c > 0$ such that

$$|f'(y)| \leqslant c \min\left\{n, \frac{\sqrt{n}}{\sqrt{(1-y^2)}}\right\} \max_{-1 \leqslant x \leqslant 1} |f(x)| \quad \text{with } -1 \leqslant y \leqslant 1 \tag{0.9}$$

for every $f \in \mathscr{P}_n$ of the form

$$f(x) = \sum_{j=0}^{n} a_j(1+x)^j(1-x)^{n-j} \quad \text{with all } a_j \geqslant 0 \text{ or all } a_j \leqslant 0. \tag{0.10}$$

By an observation of G. G. Lorentz [22], every $f \in \mathscr{P}(n,0)$ is of the form (0.10), and therefore (0.9) holds for every $f \in \mathscr{P}(n,0)$. Inequality (0.9) is sharp up to the multiplicative absolute constant $c > 0$; namely it is shown in [8] that there is an absolute constant $c > 0$ such that

$$\sup_{f \in \mathscr{P}(n,0)} \frac{|f'(y)|}{\max_{-1 \leqslant x \leqslant 1} |f(x)|} \geqslant c \min\left\{n, \frac{\sqrt{n}}{\sqrt{(1-y^2)}}\right\} \tag{0.11}$$

for every $n \in \mathbb{N}$ and $y \in [-1,1]$. In 1972 Scheick [22] found the best possible constant in Lorentz's Markov-type inequality. Extending Erdös's Markov-type inequality, he proved that

$$\max_{-1 \leqslant x \leqslant 1} |f'(x)| \leqslant \tfrac{1}{2}en \max_{-1 \leqslant x \leqslant 1} |f(x)| \tag{0.12}$$

for every $f \in \mathscr{P}_n$ of the form (0.10), and hence for every $f \in \mathscr{P}(n,0)$. In 1980 Szabados and Varma [24] showed that there is a constant $c(k) > 0$ depending only on k so that

$$\max_{-1 \leqslant x \leqslant 1} |f'(x)| \leqslant c(k)\, n \max_{-1 \leqslant x \leqslant 1} |f(x)| \tag{0.13}$$

for every $f \in \mathscr{P}(n,k)$ with $0 \leqslant k \leqslant n$, having only real zeros. Subsequently Máté [18] proved that

$$\max_{-1 \leqslant x \leqslant 1} |f'(x)| \leqslant 6n \exp(\pi\sqrt{k}) \max_{-1 \leqslant x \leqslant 1} |f(x)| \tag{0.14}$$

for every $f \in \mathscr{P}(n,k)$ with $1 \leqslant k \leqslant n$, having $n-k$ zeros in $\mathbb{R}\backslash(-1,1)$. Szabados' conjecture, proved by P. Borwein [2] in 1985, establishes the Markov-type inequality

$$\max_{-1 \leqslant x \leqslant 1} |f'(x)| \leqslant 9n(k+1) \max_{-1 \leqslant x \leqslant 1} |f(x)| \tag{0.15}$$

for every $f \in \mathscr{P}(n,k)$ with $0 \leqslant k \leqslant n$, having $n-k$ zeros in $\mathbb{R}\backslash(-1,1)$. Inequality (0.15) was extended in [6] to all $f \in \mathscr{P}(n,k)$ with $0 \leqslant k \leqslant n$. Another proof of (0.15) for all $f \in \mathscr{P}(n,k)$ with $0 \leqslant k \leqslant n$, is obtained in [9] with the constant 11 instead of 9. The fact that (0.15) is sharp up to the multiplicative absolute constant was shown by Szabados [23, Example 1]. While (0.15) is essentially sharp, it is a good estimate only for $|f(y)|$ with $|y|$ close to 1.

It was proved in [11] that there is an absolute constant $c > 0$ such that

$$|f'(y)| \leqslant \frac{c\sqrt{(n)}\,(k+1)^2}{\sqrt{(1-y^2)}} \max_{-1 \leqslant x \leqslant 1} |f(x)| \quad \text{with } -1 < y < 1 \tag{0.16}$$

for every $f \in \mathscr{P}(n,k)$ with $0 \leqslant k \leqslant n$. Subsequently it was shown in [9] that there is an absolute constant $c > 0$ so that

$$|f'(y)| \leqslant \frac{c\sqrt{(n(k+1))}}{1-y^2} \max_{-1 \leqslant x \leqslant 1} |f(x)| \quad \text{with } -1 < y < 1 \tag{0.17}$$

for every $f \in \mathcal{P}(n,k)$ with $0 \leqslant k \leqslant n$. When $y = 0$, inequality (0.17) is sharp up to the multiplicative constant $c > 0$; so it is verified in **[6]** that there is an absolute constant $c > 0$ such that

$$\sup_{f \in \mathcal{P}(n,k)} \frac{|f'(0)|}{\max_{-1 \leqslant x \leqslant 1} |f(x)|} \geqslant c\sqrt{(n(k+1))} \tag{0.18}$$

for every $0 \leqslant k \leqslant n$.

The unpleasant thing about the Bernstein-type inequalities (0.16) and (0.17) is the fact that none of them matches the inequality (0.5) in the unrestricted case $k = n$ (note that $\mathcal{P}(n,n) = \mathcal{P}_n$). In **[4]** the authors established the 'right' Markov–Bernstein type inequality in L_∞ for $\mathcal{P}(n,k)$ with $0 \leqslant k \leqslant n$, which contains all of the earlier L_∞ results as special cases up to a multiplicative constant $c > 0$. Namely, there is an absolute constant $c > 0$ such that

$$|f'(y)| \leqslant c \min\left\{n(k+1), \left(\frac{n(k+1)}{1-y^2}\right)^{1/2}\right\} \max_{-1 \leqslant x \leqslant 1} |f(x)| \quad \text{with } -1 \leqslant y \leqslant 1 \tag{0.19}$$

for every $f \in \mathcal{P}(n,k)$ with $0 \leqslant k \leqslant n$.

The purpose of this paper is to establish the 'right' Markov and Bernstein type inequalities for $\mathcal{P}(n,k)$ with $0 \leqslant k \leqslant n$ in L_p for every $0 < p \leqslant \infty$. Up to a multiplicative constant $c(p)$ depending only on p, our results are sharp and contain all the earlier results as special cases. Our proofs are based on a nice combination of results and methods worked out in **[9, 4, 5, 19, 10]**; however we have to cope with a lot of technical details.

2. *Results and Proofs*

Throughout this paper $\mathbb{N}$ will denote the set of nonnegative integers.

THEOREM 1. *Let $\chi:[0,\infty) \to \mathbb{R}$ be a convex and nondecreasing function and let $0 < p \leqslant 1$. There is an absolute constant $c_1 > 0$ such that*

$$\int_{-\pi}^{\pi} \chi\left(c_1 p^2 \left|\frac{f'(t)}{\sqrt{(n(k+1))}}\right|^p\right) dt \leqslant \int_{-\pi}^{\pi} \chi(|f(t)|^p)\, dt$$

for every $f \in \mathcal{T}_n$ of the form

$$f(t) := h(\cos t)\, q(t), \tag{1}$$

where $h \in \mathcal{P}_{n-k}$ has no zeros in the open unit disk, and $q \in \mathcal{T}_k$, $n, k \in \mathbb{N}$ for $0 \leqslant k \leqslant n$.

Using $\chi(x) := x$ if $0 < p \leqslant 1$, and $\chi(x) := x^p$ if $p > 1$, immediately gives the following corollary.

COROLLARY 2. *There is a constant $c_2(p) > 0$ such that*

$$\int_{-\pi}^{\pi} |f'(t)|^p\, dt \leqslant c_2(p)\,(n(k+1))^{p/2} \int_{-\pi}^{\pi} |f(t)|^p\, dt$$

for every $f \in \mathcal{T}_n$ of the form (1) *and for every $p > 0$, where $c_2(p) := c_3^{p+1}(1+p^{-2})$ with some absolute constant $c_3 > 0$.*

THEOREM 3. *There is a constant $c_4(p) > 0$ such that*

$$\int_{-1}^{1} |f'(x)|^p \, dx \leqslant c_4(p)\,(n(k+1))^p \int_{-1}^{1} |f(x)|^p \, dx$$

for every $f \in \mathscr{P}_n$ of the form

$$f(x) = h(x)\,q(x), \tag{2}$$

where $h \in \mathscr{P}_{n-k}$ has no zeros in the open unit disk, $q \in \mathscr{P}_k$, $n, k \in \mathbb{N}$ for $0 \leqslant k \leqslant n$, and for every $p > 0$, where $c_4(p) := c_5^{p+1}(1+p^{-2})$ with some absolute constant $c_5 > 0$.

The following examples show the sharpness of Corollary 2 and Theorem 3 up to a multiplicative positive constant depending only on p.

EXAMPLE 4. There are $h_{n,k,p} \in \mathscr{T}_n$ of the form

$$h_{n,k,p}(t) = (1+\cos t)^s q_{n,k,p}(t) \quad \text{for } q_{n,k,p} \in \mathscr{T}_k \tag{3}$$

such that

$$\int_{-\pi}^{\pi} |h'_{n,k,p}(t)|^p \, dt \geqslant c_6^{p+1} p(p+1)^{-1} (n(k+1))^{p/2} \int_{-\pi}^{\pi} |h_{n,k,p}(t)|^p \, dt$$

for all integers $0 \leqslant k \leqslant n$ and for all $p > 0$, where $c_6 > 0$ is an absolute constant.

EXAMPLE 5. There are $H_{n,k,p} \in \mathscr{P}_n$ of the form

$$H_{n,k,p}(x) = (1+x)^s Q_{n,k,p}(x) \quad \text{for } Q_{n,k,p} \in \mathscr{P}_k \tag{4}$$

such that

$$\int_{-1}^{1} |H'_{n,k,p}(x)|^p \, dx \geqslant c_7^{p+1} p(p+1)^{-1} (n(k+1))^p \int_{-1}^{1} |H_{n,k,p}(x)|^p \, dx$$

for all integers $0 \leqslant k \leqslant n$ and for all $p > 0$, where $c_7 > 0$ is an absolute constant.

To prove Theorem 1 we need a series of lemmas.

LEMMA 1.1. *There is an absolute constant $c_8 > 0$ such that*

$$|g(z)| \leqslant c_8 \max_{t \in \mathbb{R}} |g(t)|$$

for every $g \in \mathscr{T}_{(l+1)n}$ of the form

$$g(t) = (1-\cos(t-\beta))^{l(n-k)} (1+\cos(t-\alpha))^m (1-\cos(t-\alpha))^{n-k-m} q(t), \tag{5}$$

where $\alpha, \beta \in \mathbb{R}$, $q \in \mathscr{T}_{(l+1)k}$, $n, k, m, l \in \mathbb{N}$ with $0 \leqslant k \leqslant n$, $0 \leqslant m \leqslant n-k$, and for every $z \in \mathbb{C}$ such that

$$|\operatorname{Im} z| \leqslant 32^{-1}(l+1)^{-2}(n(k+1))^{-1/2}.$$

The proof of the above lemma rests on the following result.

LEMMA 1.2. *There is an absolute constant* $c_9 > 0$ *such that*

$$|P(\alpha)| \leqslant c_9 \max_{-1 \leqslant x \leqslant 1} |P(x)|$$

for every $P \in \mathscr{P}_{2(l+1)n}$ *of the form*

$$P(x) := (x-a_1)^{2l(n-k)}(x-a_2)^{2m}(x-a_3)^{2(n-k-m)}Q(x), \tag{6}$$

where a_1, a_2, $a_3 \in [-1, 1]$, $Q \in \mathscr{P}_{2(l+1)k}$, $n, k, m, l \in \mathbb{N}$ *with* $0 \leqslant k \leqslant n$, $0 \leqslant m \leqslant n-k$, *and for every*

$$\alpha \in [1, 1+(1024(l+1)^4 n(k+1))^{-1}].$$

We reduce the proof of Lemma 1.2 to the following two results proved in [**4**, Lemma 13; **7**, Lemma 1, Corollary 1], respectively.

LEMMA 1.3. *There is an absolute constant* $c_{10} > 0$ *such that*

$$|P(\alpha)| \leqslant c_{10} \max_{-1 \leqslant x \leqslant 1} |P(x)|$$

for every $P \in \mathscr{P}_{2(l+1)n}$ *of the form*

$$P(x) := (x+1)^{2l(n-k)+2m}(x-a_4)^{2(n-k-m)}Q(x), \tag{7}$$

where $\alpha_4 \in [-1, 1]$, $Q \in \mathscr{P}_{2(l+1)k}$, $n, k, m, l \in \mathbb{N}$ *with* $0 \leqslant m \leqslant n-k$, *and for every*

$$\alpha \in [1, 1+(4(l+1)^2 n(k+1))^{-1}].$$

LEMMA 1.4. *Assume that* $P \in \mathscr{P}_{2(l+1)n}$ *and that*

$$|P(1)| = \max_{-1 \leqslant x \leqslant 1} |P(x)|. \tag{8}$$

Then P *has at most* $c_{11}(l+1)nr^{1/2}$ *zeros (counting multiplicities) in* $[1-r, 1]$ *for every* $r > 0$, *where* $c_{11} = 2\sqrt{2}$ *is a suitable choice.*

Proof of Lemma 1.2. If $\frac{1}{2}n \leqslant l \leqslant n$, then the conclusion of the lemma is a well-known consequence of the Chebyshev inequality [**16**, p. 43] for all $P \in \mathscr{P}_{2(l+1)n}$. Therefore, in what follows, let $0 \leqslant k < \frac{1}{2}n$. Without loss of generality we may assume that $\frac{1}{2}(n-k) \leqslant m \leqslant n-k$, otherwise $\frac{1}{2}(n-k) \leqslant n-k-m \leqslant n-k$. Note that $0 \leqslant k < \frac{1}{2}n$ and $\frac{1}{2}(n-k) \leqslant m$ imply that $\frac{1}{4}n < m$. A simple variational method yields that it is sufficient to prove the lemma under the assumption that $Q \in \mathscr{P}_{(l+1)k}$ has all its zeros in $[-1, 1]$. We may also assume that

$$\max_{-1 \leqslant x \leqslant 1} |P(x)| = 1; \tag{9}$$

the general case can easily be reduced to this by a linear transformation (note that $|P|$ is increasing on $[1, \infty)$, since all the zeros of Q are in $[-1, 1]$). Therefore, by Lemma 1.4, we can deduce that

$$\max\{a_1, a_2\} \leqslant 1 - c_{12} \tag{10}$$

with $c_{12} := (4(l+1)c_{11})^{-2} = (8.2^{1/2}(l+1))^{-2}$. For the sake of brevity let

$$\tilde{a}_1 := \min\{a_1, a_2\} \quad \text{and} \quad \tilde{a}_2 := \max\{a_1, a_2\}. \tag{11}$$

Let P be of the form (6) and

$$\tilde{P}(x) := (x-\tilde{a}_2)^{2l(n-k)+2m}(x-a_3)^{2(n-k-m)}Q(x). \tag{12}$$

Since $(x-\tilde{a}_2)/(x-\tilde{a}_1)$ is increasing on $[\tilde{a}_2, \infty)$, we obtain

$$\frac{|P(\alpha)|}{\max\limits_{-1\leqslant x\leqslant 1}|P(x)|}\leqslant\frac{|P(\alpha)|}{\max\limits_{\tilde{a}_2\leqslant x\leqslant 1}|P(x)|}\leqslant\frac{|\tilde{P}(\alpha)|}{\max\limits_{\tilde{a}_2\leqslant x\leqslant 1}|\tilde{P}(x)|}\tag{13}$$

for every $\alpha\in[1,\infty)$. From Lemma 1.3, by a linear transformation, we can easily deduce that

$$\frac{|\tilde{P}(\alpha)|}{\max\limits_{\tilde{a}_2\leqslant x\leqslant 1}|\tilde{P}(x)|}\leqslant c_{10},\tag{14}$$

where

$$1\leqslant\alpha\leqslant 1+(1-\tilde{a}_2)(8(l+1)^2n(k+1))^{-1}\leqslant 1+(1024(l+1)^4n(k+1))^{-1}.$$

Combining (13) and (14), we get the lemma.

Now we obtain Lemma 1.1 from Lemma 1.2 as follows.

Proof of Lemma 1.1. If $f\in\mathcal{T}_{(l+1)n}$ is of the form (5), then there is a $Q\in\mathcal{P}_{2(l+1)k}$ such that

$$\begin{aligned}g(t)g(-t) &= (\cos t-\cos\beta)^{2l(n-k)}(\cos t+\cos\alpha)^{2m}(\cos t-\cos\alpha)^{2(n-k-m)}q(t)q(-t)\\ &= (\cos t-\cos\beta)^{2l(n-k)}(\cos t+\cos\alpha)^{2m}(\cos t-\cos\alpha)^{2(n-k-m)}Q(\cos t).\end{aligned}\tag{15}$$

For the sake of brevity let

$$P(x):=(x-\cos\beta)^{2l(n-k)}(x+\cos\alpha)^{2m}(x-\cos\alpha)^{2(n-k-m)}Q(x).\tag{16}$$

Then Lemma 1.2 yields

$$\begin{aligned}|g(\mathrm{i}\delta)|^2 &= |g(\mathrm{i}\delta)g(-\mathrm{i}\delta)| = |P(\cos(\mathrm{i}\delta))|\\ &\leqslant \max_{1\leqslant\alpha\leqslant 1+\delta^2}|P(\alpha)|\leqslant\max_{1\leqslant\alpha\leqslant 1+(1024(l+1)^4n(k+1))^{-1}}|P(\alpha)|\\ &\leqslant c_{10}\max_{-1\leqslant x\leqslant 1}|P(x)| = c_{10}\max_{t\in\mathbb{R}}|g(t)g(-t)|\leqslant c_{10}\max_{t\in\mathbb{R}}|g(t)|^2\end{aligned}$$

whenever

$$|\delta|\leqslant 32^{-1}(l+1)^{-2}(n(k+1))^{-1/2},\tag{17}$$

and the lemma follows for all $z\in\mathbb{C}$ with $\operatorname{Re} z=0$ and

$$|\operatorname{Im} z|\leqslant 32^{-1}(l+1)^{-2}(n(k+1))^{-1/2}.$$

If $\operatorname{Re} z\neq 0$, then we study $\tilde{g}(t):=g(t-\operatorname{Re} z)$; this is of the form (5) as well with $\tilde{\alpha}:=\alpha+\operatorname{Re} z$, $\tilde{\beta}:=\beta+\operatorname{Re} z$, and $\tilde{q}(t)=q(t-\operatorname{Re} z)\in\mathcal{T}_{(l+1)k}$, and the case already proved gives the lemma.

From Lemma 1.1, by Cauchy's Integral Formula, we immediately obtain the following.

COROLLARY 1.5. *There is an absolute constant* $c_{14}>0$ *such that*

$$\max_{t\in\mathbb{R}}|g'(t)|\leqslant c_{13}(l+1)^2(n(k+1))^{1/2}\max_{t\in\mathbb{R}}|g(t)|$$

for every $g\in\mathcal{T}_{(l+1)n}$ *of the form* (5).

From Corollary 1.5, by a variational method, we easily obtain the next corollary.

COROLLARY 1.6. *Let c_{13} be as in Corollary* 1.5. *Then*

$$|g'(t_0)| \leqslant c_{13}(l+1)^2(n(k+3))^{1/2} \max_{\tau \in \mathbb{R}} |g(\tau)| \quad \text{with } t_0 \in \mathbb{R}$$

for every $g \in \mathcal{T}_{(l+1)n}$ of the form

$$g(t) = (1-\cos(t-\beta))^{l(n-k)}\, h(\cos t)\, q(t), \tag{18}$$

where $\beta \in \mathbb{R}$, $q \in \mathcal{T}_{(l+1)k}$, $n, k, l \in \mathbb{N}$ with $0 \leqslant k \leqslant n$, and where $h \in \mathcal{P}_{n-k}$ has no zeros in the open unit disk.

Proof. Let $t_0 \in \mathbb{R}$, $n, k, l \in \mathbb{N}$ with $0 \leqslant k \leqslant n$, $\beta \in \mathbb{R}$, and $q \in \mathcal{T}_{(l+1)k}$ be fixed. By a simple compactness argument there is a function

$$g^*(t) := (1-\cos(t-\beta))^{l(n-k)} h^*(\cos t)\, q(t) \tag{19}$$

with $h^* \in \mathcal{P}_{n-k}$ having no zeros in the open unit disk such that

$$\sup_g \frac{|g'(t_0)|}{\max_{\tau \in \mathbb{R}} |g(\tau)|} = \frac{|g^{*\prime}(t_0)|}{\max_{\tau \in \mathbb{R}} |g^*(\tau)|}, \tag{20}$$

where the sup in (20) is taken for all g of the form (18). We show that $h^* \in \mathcal{P}_{n-k} \backslash \mathcal{P}_{n-k-2}$ has all but one of its zeros (counting multiplicities) at either -1 or $+1$. If $f^*(\alpha) = 0$ and $\alpha \in \mathbb{C} \backslash \mathbb{R}$, then for a sufficiently small $\varepsilon > 0$,

$$h_\varepsilon(x) := h^*(x)\left(1 - \frac{\varepsilon(x-\cos t_0)^2}{(x-\alpha)(x-\bar{\alpha})}\right) \in \mathcal{P}_{n-k} \tag{21}$$

has no zeros in the open unit disk, and

$$g_\varepsilon(t) := (1-\cos(t-\beta))^{l(n-k)}\, h_\varepsilon(\cos t)\, q(t) \tag{22}$$

contradicts the maximality of g^*.

Now assume that there are $\alpha, \beta \in \mathbb{R} \backslash [-1, 1]$ such that $(x-\alpha)(x-\beta)$ is a factor of $f(x)$. Then for a sufficiently small $\varepsilon > 0$ it follows that

$$h_\varepsilon(x) := h^*(x)\left(1 - \frac{\varepsilon \operatorname{sign}(\alpha\beta)(x-\cos t_0)^2}{(x-\alpha)(x-\beta)}\right) \in \mathcal{P}_{n-k} \tag{23}$$

has no zeros in the open unit disk, and (22) would contradict the maximality of g^*. Also, $\deg h^* \geqslant n-k-1$, otherwise for sufficiently small $\varepsilon > 0$ it would follow that

$$h_\varepsilon(x) := h^*(x)(1-\varepsilon(x-\cos t_0)^2) \in \mathcal{P}_{n-k} \tag{24}$$

has no zeros in the open unit disk, and g_ε defined by (22) would contradict the maximality of g^*. So now we get the desired conclusion by Corollary 1.5.

LEMMA 1.7. *There is an absolute constant $c_{14} > 0$ such that*

$$\max_{-\pi \leqslant \tau \leqslant \pi} |g(\tau)|^p \leqslant c_{14}(l+1)^2(n(k+3))^{1/2} \int_{-\pi}^{\pi} |g(\tau)|^p \, d\tau$$

for every $g \in \mathcal{T}_{(l+1)n}$ of the form (18) *and for every $0 < p \leqslant 2$.*

Proof. Let $t_0 \in [-\pi, \pi]$ be such that

$$|g(t_0)| = \max_{-\pi \leqslant \tau \leqslant \pi} |g(\tau)|. \tag{25}$$

By Corollary 1.6 and the Mean Value Theorem we obtain that there is a ξ between t_0 and t such that

$$\begin{aligned} |g(t)| &\geqslant |g(t_0)| - |g(t) - g(t_0)| = |g(t_0)| - |t - t_0|\,|g'(\xi)| \\ &= \max_{-\pi \leqslant \tau \leqslant \pi} |g(\tau)| - |t - t_0|\, c_{13}(l+1)^2(n(k+3))^{1/2} \max_{-\pi \leqslant \tau \leqslant \pi} |g(\tau)| \\ &\geqslant 2^{-1} \max_{-\pi \leqslant \tau \leqslant \pi} |g(\tau)| \end{aligned}$$

whenever

$$|t - t_0| \leqslant (2c_{13})^{-1}(l+1)^{-2}(n(k+3))^{-1/2}. \tag{26}$$

For the sake of brevity let

$$I := [t_0 - (2c_{13})^{-1}(l+1)^{-2}(n(k+2))^{-1/2}, t_0 + (2c_{13})^{-1}(l+1)^{-2}(n(k+2))^{-1/2}].$$

Then (26) and $0 < p \leqslant 2$ imply that

$$\int_{-\pi}^{\pi} |g(\tau)|^p \, d\tau \geqslant \int_I |g(\tau)|^p \, d\tau \geqslant c_{13}^{-1}(l+1)^{-2}(n(k+3))^{-1/2} 2^{-2} \max_{-\pi \leqslant \tau \leqslant \pi} |g(\tau)|^p,$$

and the lemma is proved.

LEMMA 1.8. *There are $f_{n,k} \in \mathcal{T}_n$ of the form*

(i) $f_{n,k}(t) := (1 + \cos t)^{n-k} q_{n,k}(t)$ *with* $q_{n,k} \in \mathcal{T}_k$

such that

(ii) $\int_{-\pi}^{\pi} (f_{n,k}(t))^2 \, dt = 1,$

(iii) $(f_{n,k}(0))^2 \geqslant c_{15}(n(k+3))^{1/2},$

(iv) $f'_{n,k}(0) = 0$

hold for all integers $0 \leqslant k \leqslant n$, where $c_{15} > 0$ is an absolute constant.

Proof. It follows from [5, Corollary 3.3] that there are $F_{n,k} \in \mathcal{P}_n$ of the form

$$F_{n,k}(x) := (1 + x)^{n-k} Q_{n,k}(x) \quad \text{with } Q_{n,k} \in \mathcal{P}_k \tag{27}$$

so that

$$\int_{-1}^{1} (F_{n,k}(x))^2 \, dx = 1, \tag{28}$$

$$|F_{n,k}(1)|^2 \geqslant c_{16}\, n(k+3) \tag{29}$$

hold for all integers $0 \leqslant k \leqslant n$, where $c_{16} > 0$ is an absolute constant. Now let

$$f_{n,k}(t) := \left(\int_{-1}^{1} (F_{n,k}(\cos t))^2 \, dt \right)^{-1/2} F_{n,k}(\cos t). \tag{30}$$

Obviously $f_{n,k} \in \mathcal{T}_n$ is of the form (i), and

$$\int_{-\pi}^{\pi} (f_{n,k}(t))^2\,dt = 1, \tag{31}$$

$$f'_{n,k}(0) = 0 \tag{32}$$

for all integers $0 \leqslant k \leqslant n$. To prove (iii), first note that Lemma 1.7 implies (with $g = f_{n,k}$, $l = 0$, $h = 1$, $\beta = \pi$, $p = 2$) that

$$\begin{aligned}\int_{-\pi}^{\pi} (f_{n,k}(t))^2\,dt &= \int_A (f_{n,k}(t))^2\,dt + \int_{[-\pi,\pi]\setminus A} (f_{n,k}(t))^2\,dt \\ &\leqslant m(A) \max_{t\in\mathbb{R}} (f_{n,k}(t))^2 + \int_{[-\pi,\pi]\setminus A} (f_{n,k}(t))^2\,dt \\ &\leqslant m(A)\, c_{14}(n(k+3))^{1/2} \int_{-\pi}^{\pi} (f_{n,k}(t))^2\,dt + \int_{[-\pi,\pi]\setminus A} (f_{n,k}(t))^2\,dt \end{aligned} \tag{33}$$

holds for any measurable set $A \subset [-\pi,\pi]$. Hence $m(A) \leqslant (2c_{14})^{-1}(n(k+3))^{-1/2}$ implies that

$$\int_{-\pi}^{\pi} (f_{n,k}(t))^2\,dt \leqslant 2 \int_{[-\pi,\pi]\setminus A} (f_{n,k}(t))^2\,dt. \tag{34}$$

Since $f_{n,k}(t) = \alpha F_{n,k}(\cos t)$, there is an absolute constant $c_{17} > 0$ so that with the notation

$$\delta_{n,k} := 1 - (c_{17}\, n(k+3))^{-1} \tag{35}$$

we have

$$\int_{-1}^{1} (F_{n,k}(x))^2 (1-x^2)^{-1/2}\,dx \leqslant 2 \int_{-\delta_{n,k}}^{\delta_{n,k}} (F_{n,k}(x))^2 (1-x^2)^{-1/2}\,dx. \tag{36}$$

Now

$$\begin{aligned}(f_{n,k}(0))^2 &= (F_{n,k}(1))^2 \left(\int_{-\pi}^{\pi} (F_{n,k}(\cos t))^2\,dt\right)^{-1} \\ &\geqslant c_{16}\, n(k+3) \left(2 \int_{-1}^{1} (F_{n,k}(x))^2 (1-x^2)^{-1/2}\,dx\right)^{-1} \\ &\geqslant c_{16}\, n(k+3) \left(4 \int_{-\delta_{n,k}}^{\delta_{n,k}} (F_{n,k}(x))^2 (1-x^2)^{-1/2}\,dx\right)^{-1} \\ &\geqslant c_{16}\, n(k+3)\,(c_{17}\, n(k+2))^{-1/2}\, 4^{-1} \left(\int_{-\delta_{n,k}}^{\delta_{n,k}} (F_{n,k}(x))^2\,dx\right)^{-1} \\ &\geqslant c_{15}(n(k+3))^{1/2} \left(\int_{-1}^{1} (F_{n,k}(x))^2\,dx\right)^{-1} = c_{15}(n(k+2))^{1/2} \end{aligned} \tag{37}$$

with $c_{15} := 4^{-1} c_{16}\, c_{17}^{-1/2}$, and the lemma is proved.

Proof of Theorem 1. Let f be of the form (1), let $l := [2p^{-1}] + 1$ for $0 < p \leqslant 1$, and let

$$g := f(f_{n,k})^l. \tag{38}$$

Applying Lemma 1.7 to g, and then to $f_{n,k}$, we obtain

$$\begin{aligned}
&\max_{\tau\in\mathbb{R}}|f(\tau)(f_{n,k}(\tau))^l|^p \\
&\leqslant c_{14}(l+1)^2(n(k+3))^{1/2}\int_{-\pi}^{\pi}|f(\tau)|^p|f_{n,k}(\tau)|^2|f_{n,k}(\tau)|^{lp-2}\,d\tau \\
&\leqslant c_{14}(l+1)^2(n(k+3))^{1/2}\max_{\tau\in\mathbb{R}}|f_{n,k}(\tau)|^{lp-2}\int_{-\pi}^{\pi}|f(\tau)|^p(f_{n,k}(\tau))^2\,d\tau \\
&\leqslant c_{14}(l+1)^2(n(k+3))^{1/2}\left(c_{14}(n(k+3))^{1/2}\int_{-\pi}^{\pi}(f_{n,k}(\tau))^2\,d\tau\right)^{(lp-2)/2} \\
&\quad\times\int_{-\pi}^{\pi}|f(\tau)|^p(f_{n,k}(\tau))^2\,d\tau \\
&\leqslant (l+1)^2c_{14}^{lp/2}(n(k+3))^{lp/4}\int_{-\pi}^{\pi}|f(\tau)|^p(f_{n,k}(\tau))^2\,d\tau, \qquad (39)
\end{aligned}$$

where $lp\geqslant 2$ and Lemma 1.8 (i) were also used. Applying Corollary 1.6 to g defined by (38), we get

$$|g'(t)|^p\leqslant c_{14}^p(l+1)^{2p}(n(k+3))^{p/2}\max_{\tau\in\mathbb{R}}|g(\tau)|^p \qquad (40)$$

for every $t\in\mathbb{R}$. Putting $t=0$ and using $f'_{n,k}(0)=0$ and (39), we conclude that

$$\begin{aligned}
&|f'(0)|^p|f_{n,k}(0)|^{lp} \\
&\leqslant c_{13}^p(l+1)^{2p}(n(k+3))^{p/2}\max_{\tau\in\mathbb{R}}|f(\tau)(f_{n,k}(\tau))^l|^p \\
&\leqslant c_{13}^p(l+1)^{2p}(n(k+3))^{p/2}(l+1)^2c_{14}^{lp/2}(n(k+3))^{lp/4}\int_{-\pi}^{\pi}|f(\tau)|^p(f_{n,k}(\tau))^2\,d\tau. \qquad (41)
\end{aligned}$$

Combining this with $(f_{n,k}(0))^2\geqslant c_{15}(n(k+3))^{1/2}$ (see Lemma 1.8 (iii)) and $l+1\leqslant 2(1+p^{-1})$, we deduce that

$$\begin{aligned}
|f'(0)|^p &\leqslant c_{13}^p(l+1)^{2p}(n(k+3))^{p/2}(l+1)^2c_{14}^{lp/2}c_{15}^{-lp/2}\int_{-\pi}^{\pi}|f(\tau)|^p(f_{n,k}(\tau))^2\,d\tau \\
&\leqslant c_{18}p^{-2}(n(k+3))^{p/2}\int_{-\pi}^{\pi}|f(\tau)|^p(f_{n,k}(\tau))^2\,d\tau \qquad (42)
\end{aligned}$$

with an absolute constant $c_{18}>0$. Applying (42) to $f_1(\tau):=f(\tau+t)$, we obtain

$$(c_{18}\sqrt{3})^{-1}p^2\left(\frac{|f'(t)|}{\sqrt{(n(k+1))}}\right)^p\leqslant\int_{-\pi}^{\pi}|f(\tau)|^p(f_{n,k}(\tau-t))^2\,d\tau. \qquad (43)$$

Since

$$\int_{-\pi}^{\pi}(f_{n,k}(\tau-t))^2\,d\tau=1 \quad \text{for } t\in\mathbb{R} \qquad (44)$$

and $\chi:[0,\infty)\to\mathbb{R}$ is a convex and nondecreasing function, (44) and an application of the Jensen inequality yield

$$\begin{aligned}
\chi\left((c_{18}\sqrt{3})^{-1}p^2\left(\frac{|f'(t)|}{\sqrt{(n(k+1))}}\right)^p\right) &\leqslant \chi\left(\int_{-\pi}^{\pi}|f(\tau)|^p(f_{n,k}(\tau-t))^2\,d\tau\right) \\
&\leqslant \int_{-\pi}^{\pi}\chi(|f(\tau)|^p)(f_{n,k}(\tau-t))^2\,d\tau. \qquad (45)
\end{aligned}$$

Integrating both sides of (45) with respect to t, and using

$$\int_{-\pi}^{\pi} (f_{n,k}(\tau - t))^2 \, dt = 1, \tag{46}$$

we get the desired inequality by the Fubini Theorem.

Proof of Corollary 2. This result follows from Theorem 1 with $\tilde{p} := p$ and $\chi(x) := x$ if $0 < p < 1$, and with $\tilde{p} := 1$ and $\chi(x) := x^p$ if $p > 1$.

To prove Theorem 3 we need some lemmas.

LEMMA 3.1 ([2], [6, Corollary 1.3], [9, Theorem 1], or [4, Theorem 3.4]). *There is an absolute constant $c_{19} > 0$ such that*

$$\max_{-1 \leqslant x \leqslant 1} |f'(x)| \leqslant c_{19} n(k+1) \max_{-1 \leqslant x \leqslant 1} |f(x)|$$

for all $f \in \mathscr{P}_n$ having at most k (with $0 \leqslant k \leqslant n$) zeros (counting multiplicities) in the open unit disk.

LEMMA 3.2 [3, Theorem 3.3]. *There is an absolute constant $c_{20} > 0$ such that*

$$\max_{-1 \leqslant x \leqslant 1} |f(x)|^p \leqslant c_{20}(1+p^2) n(k+1) \int_{-1}^{1} |f(x)|^p \, dx$$

for all $f \in \mathscr{P}_n$ having at most k (with $0 \leqslant k \leqslant n$) zeros (counting multiplicities) in the open unit disk and for all $p > 0$.

LEMMA 3.3. *There is an absolute constant $c_{21} > 0$ such that*

$$\int_{-1}^{1} |f(x)|^p (1-x^2)^{-\alpha} \, dx \leqslant c_{21}(1-\alpha)^{-1}(1+p^2)(n(k+1))^{\alpha} \int_{-1}^{1} |f(x)|^p \, dx$$

for all $f \in \mathscr{P}_n$ having at most k (with $0 \leqslant k \leqslant n$) zeros (counting multiplicities) in the open unit disk, and for all $p > 0$ and $0 < \alpha < 1$.

Proof. Let $\delta_{n,k} := 1 - (n(k+1))^{-1}$. Using Lemma 3.2, we obtain

$$\begin{aligned}
&\int_{-1}^{1} |f(x)|^p (1-x^2)^{-\alpha} \, dx \\
&\quad \leqslant (1-\delta_{n,k})^{-\alpha} \int_{-\delta_{n,k}}^{\delta_{n,k}} |f(x)|^p \, dx + \int_{[-1,1] \setminus [-\delta_{n,k}, \delta_{n,k}]} (1-x^2)^{-\alpha} dx \max_{-1 \leqslant y \leqslant 1} |f(y)|^p \\
&\quad \leqslant (n(k+1))^{\alpha} \int_{-1}^{1} |f(x)|^p \, dx + 2(1-\alpha)^{-1} (1-\delta_{n,k})^{1-\alpha} c_{20}(1+p^2) n(k+1) \int_{-1}^{1} |f(x)|^p \, dx \\
&\quad \leqslant c_{21}(1-\alpha)^{-1}(1+p^2)(n(k+1))^{\alpha} \int_{-1}^{1} |f(x)|^p \, dx,
\end{aligned}$$

where $c_{21} := 1 + 2c_{20}$, and the lemma is proved.

Proof of Theorem 3. We distinguish two cases.

Case 1: $p \geqslant 1$. Corollary 2 and the substitution $x = \cos t$ yield

$$\int_{-1}^{1} |f'(x)|^p (1-x^2)^{(p-1)/2}\,dx \leqslant c_2(p)\,(n(k+1))^{p/2} \int_{-1}^{1} |f(x)|^p (1-x^2)^{-1/2}\,dx \tag{47}$$

for every $f \in \mathscr{P}_n$ of the form (2). Now let $\delta_{n,k} := 1-(n(k+1))^{-1}$. Then, by (47) and Lemma 3.3, we get

$$\begin{aligned}
&\int_{-\delta_{n,k}}^{\delta_{n,k}} |f'(x)|^p\,dx \\
&\leqslant (n(k+1))^{(p-1)/2} \int_{-\delta_{n,k}}^{\delta_{n,k}} |f'(x)|^p (1-x^2)^{(p-1)/2}\,dx \\
&\leqslant (n(k+1))^{(p-1)/2} c_2(p)\,(n(k+1))^{p/2} \int_{-1}^{1} |f(x)|^p (1-x^2)^{-1/2}\,dx \\
&\leqslant (n(k+1))^{(p-1)/2} c_2(p)\,(n(k+1))^{p/2} 2c_{21}(1+p^2)\,(n(k+1))^{1/2} \int_{-1}^{1} |f(x)|^p\,dx \\
&\leqslant c_{22}(p)\,(n(k+1))^p \int_{-1}^{1} |f(x)|^p\,dx,
\end{aligned} \tag{48}$$

where $c_{22}(p) := 2c_{21}(1+p^2)\,c_2(p) = 2c_{21}(1+p^2)\,c_3^{p+1}(1+p^{-2})$ for every $f \in \mathscr{P}_n$ of the form (2). Using Lemmas 3.1 and 3.2 we can easily deduce that

$$\begin{aligned}
\int_{[-1,1]\setminus[-\delta_{n,k},\delta_{n,k}]} |f'(x)|^p\,dx &\leqslant 2(1-\delta_{n,k}) \max_{-1 \leqslant x \leqslant 1} |f'(x)|^p \\
&\leqslant 2(n(k+1))^{-1} (c_{19}\,n(k+1))^p \max_{-1 \leqslant x \leqslant 1} |f(x)|^p \\
&\leqslant 2(n(k+1))^{-1} (c_{19}\,n(k+1))^p c_{20}(1+p^2)\,n(k+1) \int_{-1}^{1} |f(x)|^p\,dx \\
&\leqslant c_{23}(p)\,(n(k+1))^p \int_{-1}^{1} |f(x)|^p\,dx,
\end{aligned} \tag{49}$$

where $c_{23}(p) := 2c_{19}^p\,c_{20}(1+p^2)$, for every $f \in \mathscr{P}_n$ of the form (2). Now (48) and (49) yield the theorem.

Case 2: $0 < p < 1$. Let $u := [p^{-1}]$. Since $0 < p < 1$, we have $|a+b|^p \leqslant |a|^p + |b|^p$ for any two real numbers a and b. Combining this with the product rule and Corollary 2, we obtain

$$\begin{aligned}
&\int_{-\pi}^{\pi} |f'(t) \sin^u t|^p\,dt \\
&\leqslant \int_{-\pi}^{\pi} \left|\frac{d}{dt}(f(t) \sin^u t)\right|^p dt + \int_{-\pi}^{\pi} |f(t)\,u \sin^{u-1} t \cos t|^p\,dt \\
&\leqslant c_2(p)\,((n+u)\,(k+u+1))^{p/2} \int_{-\pi}^{\pi} |f(t) \sin^u t|^p\,dt + u \int_{-\pi}^{\pi} |f(t) \sin^{u-1} t|^p\,dt
\end{aligned} \tag{50}$$

for every $f \in \mathscr{T}_n$ of the form (1). Substituting $x = \cos t$, we get

$$\int_{-1}^{1} |f'(x)|(1-x^2)^{(u+1)p/2-1/2}\,dx$$
$$\leqslant c_2(p)\,((n+u)(k+u+1))^{p/2} \int_{-1}^{1} |f(x)|^p(1-x^2)^{up/2-1/2}\,dx$$
$$+u\int_{-1}^{1} |f(x)|^p(1-x^2)^{(u-1)p/2-1/2}\,dx \tag{51}$$

for every $f \in \mathscr{P}_n$ of the form (2). Observe that $0 < p < 1$ and $u = [p^{-1}]$ imply that

$$\tfrac{1}{2}(u+1)p - \tfrac{1}{2} \geqslant 0, \tag{52}$$
$$-\tfrac{1}{2} < \tfrac{1}{2}(u-1)p - \tfrac{1}{2} < \tfrac{1}{2}up - \tfrac{1}{2} \leqslant 0. \tag{53}$$

Let $\delta_{n,k} := 1-(n(k+1))^{-1}$. Using (51), (52), (53), and Lemma 3.3, we can deduce that

$$\int_{-\delta_{n,k}}^{\delta_{n,k}} |f'(x)|^p\,dx$$
$$\leqslant (n(k+1))^{(u+1)p/2-1/2} \int_{-\delta_{n,k}}^{\delta_{n,k}} |f'(x)|^p(1-x^2)^{(u+1)/2-1/2}\,dx$$
$$\leqslant (n(k+1))^{(u+1)p/2-1/2} c_2(p)\,((n+u)(k+u+1))^{p/2} \int_{-1}^{1} |f(x)|^p(1-x^2)^{up/2-1/2}\,dx$$
$$+(n(k+1))^{(u+1)p/2-1/2} u \int_{-1}^{1} |f(x)|^p\,(1-x^2)^{(u-1)p/2-1/2}\,dx$$
$$\leqslant c_2(p)\,(1+p^{-1})^p 4c_{21}(n(k+1))^{((u+1)p/2-1/2)+p/2+(1/2-up/2)} \int_{-1}^{1} |f(x)|^p\,dx$$
$$+4c_{21}p^{-1}(n(k+1))^{((u+1)p/2-1/2)+(1/2-(u-1)p/2)} \int_{-1}^{1} |f(x)|^p\,dx$$
$$\leqslant c_{24}(p)\,(n(k+1))^p \int_{-1}^{1} |f(x)|^p\,dx \tag{54}$$

for every $f \in \mathscr{P}_n$ of the form (2), where $c_{24}(p) := c_2(p)\,(1+p^{-1})^p 4c_{21} + 4c_{21}p^{-1} \leqslant c_{25}p^{-2}$ with a suitable absolute constant $c_{25} > 0$. Further, using Lemmas 3.1 and 3.2 we get

$$\int_{[-1,1]\setminus[-\delta_{n,k},\delta_{n,k}]} |f'(x)|^p\,dx \leqslant 2(1-\delta_{n,k}) \max_{-1 \leqslant x \leqslant 1} |f'(x)|^p$$
$$\leqslant 2(n(k+1))^{-1}(c_{19}n(k+1))^p \max_{-1 \leqslant x \leqslant 1} |f(x)|^p$$
$$\leqslant 2(n(k+1))^{-1}(c_{19}n(k+1))^p 2c_{20}n(k+1) \int_{-1}^{1} |f(x)|^p\,dx$$
$$\leqslant c_{26}(n(k+1))^p \int_{-1}^{1} |f(x)|^p\,dx, \tag{55}$$

where $c_{26} := 4c_{20}(1+c_{19})$ for every $f \in \mathscr{P}_n$ of the form (2). Now (54) and (55) give the theorem.

We prove only Example 5, the proof of Example 4 is quite similar.

Proof of Example 5. From [5, Corollary 3.3] it follows that there are $G_{n,k} \in \mathscr{P}_n$ of the form

$$G_{n,k}(x) = (1+x)^s Q_{n,k}(x) \quad \text{with } Q_{n,k} \in \mathscr{P}_k \tag{56}$$

such that

$$\int_{-1}^{1} (G_{n,k}(x))^2 \, dx = 1, \tag{57}$$

$$|G'_{n,k}(1)|^2 \geqslant c_{27}(n(k+1))^3, \tag{58}$$

$$|G_{n,k}(1)|^2 \geqslant c_{28}\, n(k+1) \tag{59}$$

for all integers $0 \leqslant k \leqslant n$, where $c_{27} > 0$ and $c_{28} > 0$ are absolute constants. Let $u := [2p^{-1}] + 1, \tilde{n} := [n/u]$ and $\tilde{k} := [k/u]$. We distinguish two cases.

Case 1: $\tilde{n} \geqslant 1$. Let $H_{n,k,p} := (G_{\tilde{n},\tilde{k}})^u$. Obviously $H_{n,k,p} \in \mathscr{P}_n$ and it is of the form (4) for all integers $0 \leqslant k \leqslant n$ and for all $p > 0$. Using Lemmas 3.2 and 3.1, and $\tilde{n} \geqslant 1$, we obtain

$$\begin{aligned}
\int_{-1}^{1} |H'_{n,k,p}(x)|^p \, dx &\geqslant (c_{20}(1+p^2)\, n(k+1))^{-1} \max_{-1 \leqslant x \leqslant 1} |H'_{n,k,p}(x)|^p \\
&\geqslant (c_{20}(1+p^2))^{-1}(n(k+1))^{-1} u \max_{-1 \leqslant x \leqslant 1} |(G_{\tilde{n},\tilde{k}}(x))^{u-1} G'_{\tilde{n},\tilde{k}}(x)|^p \\
&\geqslant (c_{20}(1+p^2))^{-1}(n(k+1))^{-1} u c_{27}^p c_{28}^{(u-1)p} (\tilde{n}(\tilde{k}+1))^{(u+2)p/2} \\
&\geqslant c_{29}^{p+1} p(p+1)^{-1}(n(k+1))^{(u+2)p/2-1}
\end{aligned} \tag{60}$$

with a suitable absolute constant $c_{29} > 0$. Further, $up \geqslant 2$, Lemma 3.2 and (57) imply that

$$\begin{aligned}
\int_{-1}^{1} |H_{n,k,p}(x)|^p \, dx &= \int_{-1}^{1} |G_{\tilde{n},\tilde{k}}(x)|^{up} \, dx \\
&\leqslant \int_{-1}^{1} (G_{\tilde{n},\tilde{k}}(x))^2 \, dx \max_{-1 \leqslant x \leqslant 1} |G_{\tilde{n},\tilde{k}}(x)|^{up-2} \\
&\leqslant \left(c_{20}(1+p^2)\, \tilde{n}(\tilde{k}+1) \int_{-1}^{1} (G_{\tilde{n},\tilde{k}}(x))^2 \, dx \right)^{(up-2)/2} \\
&\leqslant c_{30}^{p+1}(n(k+1))^{up/2-1}
\end{aligned} \tag{61}$$

with a suitable absolute constant $c_{30} > 0$ which, together with (60), gives the desired result.

Case 2: $\tilde{n} = 0$. Then $n < u < 2p^{-1} + 1$. Let $H_{n,k,p}(x) := 1 + x$ if $n > 0$, and $H_{n,k,p} := 1$ if $n = 0$. A simple calculation yields the desired inequality.

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