

Rational Interpolation to e^x

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1. INTRODUCTION

We derive estimates for the error in interpolating e^x by rational functions of degree n on intervals of length less than two. Let π_n denote the class of all polynomials of degree at most n with real coefficients. Our main result is the following:

THEOREM 1. *Let $\gamma_1, \gamma_2, \dots, \gamma_{2n+1}$ be points (not necessarily distinct) in $[0, \alpha]$, where $\alpha < 2$. Choose $P_n, Q_n \in \pi_n$ so that*

$$P_n(\gamma_i) - Q_n(\gamma_i) e^{-\gamma_i} = 0 \quad \text{for } i = 1, 2, \dots, 2n + 1.$$

Then, for $x \in [0, \alpha]$,

$$\begin{aligned} & |P_n(x)/Q_n(x) - e^{-x}| \\ & \leq \left(\frac{2e\sqrt{n}e^{2\sqrt{\alpha n}}}{2-\alpha} \right) \frac{n!(n+1)!}{(2n)!(2n+1)!} \left| \prod_{i=1}^{2n+1} (x - \gamma_i) \right|. \end{aligned}$$

Furthermore, Q_n has positive coefficients.

Let

$$\lambda_{m,n}[a, b] = \min_{p \in \pi_m, q \in \pi_n} \|e^x - p(x)/q(x)\|_{[a,b]},$$

where $\|\cdot\|$ denotes the supremum norm on $[a, b]$.

The following conjecture was made by G. Meinardus in 1964.

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CONJECTURE [3, p. 168].

$$\lambda_{m,n}[-1, 1] = \frac{m! n!}{2^{m+n}(m+n)!(m+n+1)!} (1 + o(1)).$$

D. J. Newman, through some clever manipulation of the Padé approximant, has recently proved

THEOREM A [5, p. 24].

$$\lambda_{m,n}[-1, 1] \leq \frac{8m! n!}{2^{m+n}(m+n)!(m+n+1)!}.$$

G. Németh ([4], see also Braess [1]) has shown

THEOREM B.

$$\lambda_{n,n}[-1, 1] = \frac{n! n!}{4^n(2n)!(2n+1)!} (1 + O(1)).$$

If we choose the γ_i in Theorem 1 to be the zeros of the $(2n+1)$ st Chebyshev polynomial (shifted to $[0, \alpha]$) then we see that, up to the "slowly growing" $e^{2\sqrt{\alpha n}}$ term, we get essentially the right order of approximation. In light of Theorems A and B it seems plausible that the initial bracketed term of the error estimate is superfluous.

2. PRELIMINARIES

Suppose that $P_n, Q_n \in \pi_n$ and suppose that $P_n(x) - Q_n(x)e^{-x}$ has $2n+1$ zeros on the interval $[0, \alpha]$. If $Q_n(x) = q_0 + q_1x + \dots + q_nx^n$ then

$$\begin{aligned} & (P_n(x) - Q_n(x)e^{-x})^{(n+1)} \\ &= (Q_n(x)e^{-x})^{(n+1)} \\ &= \sum_{k=0}^n \binom{n+1}{k} Q_n^{(k)} e^{-x} (-1)^{(n+1-k)} \\ &= (-1)^{n+1} e^{-x} \sum_{k=0}^n \frac{x^k}{k!} \sum_{j=0}^{n-k} \binom{n+1}{j} (-1)^j (k+j)! q_{k+j}. \quad (1) \end{aligned}$$

Since $(Q_n(x)e^{-x})^{(n+1)}$ has n zeros on $[0, \alpha]$, we deduce that there exist $\beta_1, \dots, \beta_n \in [0, \alpha]$ so that

$$\sum_{k=0}^n \frac{x^k}{k!} \sum_{j=0}^{n-k} \binom{n+1}{j} (-1)^j (k+j)! q_{k+j} = q_n \prod_{i=1}^n (x - \beta_i).$$

Thus, if $q_n \prod_{i=1}^n (x - \beta_i) = b_0 + b_1 x + \dots + b_n x^n$, we have

$$\begin{bmatrix} \binom{n+1}{0}, -\binom{n+1}{1}, +\binom{n+1}{2}, \dots, (-1)^n \binom{n+1}{n} \\ 0, \binom{n+1}{0}, -\binom{n+1}{1}, \dots, (-1)^{n-1} \binom{n+1}{n-1} \\ 0, 0, \binom{n+1}{0}, \dots, (-1)^{n-2} \binom{n+1}{n-2} \\ \vdots \\ 0, 0, 0, \dots, \binom{n+1}{0} \end{bmatrix} \begin{bmatrix} q_0 0! \\ q_1 1! \\ q_2 2! \\ \vdots \\ q_n n! \end{bmatrix} = \begin{bmatrix} b_0 0! \\ b_1 1! \\ b_2 2! \\ \vdots \\ b_n n! \end{bmatrix} \tag{2}$$

We can invert (2) to obtain

$$\begin{bmatrix} \binom{n}{n}, \binom{n+1}{n}, \binom{n+2}{n}, \dots, \binom{2n}{n} \\ 0, \binom{n}{n}, \binom{n+1}{n}, \dots, \binom{2n-1}{n} \\ 0, 0, \binom{n}{n}, \dots, \binom{2n-2}{n} \\ \vdots \\ 0, 0, 0, \dots, \binom{n}{n} \end{bmatrix} \begin{bmatrix} b_0 0! \\ b_1 1! \\ b_2 2! \\ \vdots \\ b_n n! \end{bmatrix} = \begin{bmatrix} q_0 0! \\ q_1 1! \\ q_2 2! \\ \vdots \\ q_n n! \end{bmatrix} \tag{3}$$

We observe that (3) can be easily derived from (2) combined with the facts that the (m, n) Padé approximant to e^{-x} is given by

$$\sum_{v=0}^m \frac{\binom{m}{v}}{\binom{m+n}{v}} \frac{(-x)^v}{v!} \bigg/ \sum_{v=0}^n \frac{\binom{n}{v}}{\binom{n+m}{v}} \frac{x^v}{v!}.$$

and that for the Padé approximant $b_0 = b_1 = \dots = b_{n-1} = 0$.

We are now in a position to prove the following:

LEMMA 1. *Suppose that $P_n(x) \in \pi_n$ and suppose that $Q_n = q_0 +$*

$q_1 x + \dots + q_n x^n$, where $q_0 > 0$. Suppose also that $P_n(x) - Q_n(x) e^{-x}$ has $2n + 1$ zeros at $\gamma_1, \dots, \gamma_{2n+1} \in [0, \alpha]$. Then, if $\alpha < 2$, Q_n has positive coefficients and

$$q_n \leq \left(\frac{2}{2-\alpha} \right) \frac{n!}{(2n)!} q_0.$$

Proof. The first part follows from an examination of (3) using the facts that for $i \leq n$,

$$(i-1)! |b_{i-1}| \leq \alpha(i!) |b_i| \quad \text{and} \quad \binom{n+i-1}{n} \leq \frac{1}{2} \binom{n+i}{n}.$$

The second part is proved by noting that

$$\begin{aligned} q_0 &\geq n! |b_n| \binom{2n}{n} - (n-1)! |b_{n-1}| \binom{2n-1}{n} \\ &\geq \left(1 - \frac{\alpha}{2} \right) \frac{(2n)!}{n!} q_n. \quad \blacksquare \end{aligned}$$

The next lemma is a slight adaptation of a result of S. N. Bernstein [2, p. 38].

LEMMA 2. Suppose that f and g are $m + 1$ times continuously differentiable on $[a, b]$ and suppose that $f(x) = g(x) = 0$ has $m + 1$ solutions on $[a, b]$. If

$$|f^{(m+1)}(x)| \leq |g^{(m+1)}(x)| \quad \text{for } x \in [a, b]$$

then

$$|f(x)| \leq |g(x)| \quad \text{for } x \in [a, b].$$

LEMMA 3 [3, pp. 16 and 165]. (a) If $\gamma_1, \dots, \gamma_{m+n+1} \in [a, b]$ then there exist $P_m \in \pi_m$, $Q_n \in \pi_n$, so that

$$P_m(\gamma_i) - Q_n(\gamma_i) e^{-\gamma_i} = 0 \quad \text{for } i = 1, 2, \dots, m+n+1.$$

(b) If $P_m^* \in \pi_m$, $Q_n^* \in \pi_n$ and

$$\|e^{-x} - P_m^*/Q_n^*\|_{[a,b]} = \min_{P_m \in \pi_m, Q_n \in \pi_n} \|e^{-x} - P_m/Q_n\|_{[a,b]}$$

then P_m^*/Q_n^* interpolates e^{-x} at exactly $m + n + 1$ points in $[a, b]$.

3. PROOF OF THEOREM 1

Lemma 3 guarantees the existence of P_n and Q_n with the desired interpolation property. We may assume that

$$Q_n(x) = q_0 + \cdots + q_{n-1}x^{n-1} + x^n.$$

Then, as in (1), there exist $\beta_1, \dots, \beta_n \in [0, \alpha]$ so that

$$\begin{aligned} (Q_n(x) e^{-x})^{(n+1)} &= (-1)^{n+1} e^{-x} \prod_{i=1}^n (x - \beta_i) \\ &= (-1)^{n+1} e^{-x} R_n(x). \end{aligned}$$

Hence,

$$(Q_n(x) e^{-x})^{(2n+1)} = (-1)^{n+1} \sum_{k=0}^n \binom{n}{k} (-1)^k e^{-x} R_n^{(n-k)}(x).$$

Since $R_n^{(n-k)}(x) = n!/k! \prod_{i=1}^k (x - \rho_{i,k})$, where $\rho_{i,k} \in [0, \alpha]$, we have

$$\begin{aligned} |(Q_n(x) e^{-x})^{(2n+1)}| &\leq \sum_{k=0}^n \binom{n}{k} \frac{n!}{k!} \alpha^k \\ &\leq n! \sum_{k=0}^n \frac{n! \alpha^k}{k! k!(n-k)!}. \end{aligned}$$

By Stirling's formula, $n^n e^{-n} < n! < e \sqrt{n} n^n e^{-n}$,

$$\begin{aligned} \frac{n! \alpha^k}{k! k!(n-k)!} &\leq \frac{e \sqrt{n} \alpha^k e^k n^n}{k^k k^k (n-k)^{n-k}} \\ &= e \sqrt{n} \alpha^k e^k \frac{n^k}{k^{2k}} \left(1 + \frac{k}{n-k}\right)^{n-k} \\ &\leq e \sqrt{n} \left(\frac{ae^2 n}{k^2}\right)^k. \end{aligned}$$

A little elementary calculus reveals that $(ae^2 n/k^2)^k$ has a maximum at $k = \sqrt{an}$ and hence,

$$|(Q_n(x) e^{-x})^{(2n+1)}| \leq (n+1)! e \sqrt{n} e^{2\sqrt{an}}.$$

We apply Lemma 2 using $m = 2n + 1$,

$$f(x) = P_n(x) - Q_n(x) e^{-x},$$

and

$$g(x) = e \sqrt{n} e^{2\sqrt{an}} \frac{(n+1)!}{(2n+1)!} \prod_{i=1}^{2n+1} (x - \gamma_i)$$

and deduce that for $x \in [0, \alpha]$,

$$|P_n(x) - Q_n(x) e^{-x}| \leq e \sqrt{n} e^{2\sqrt{an}} \frac{(n+1)!}{(2n+1)!} \left| \prod_{i=1}^{2n+1} (x - \gamma_i) \right|.$$

We complete the result by appealing to Lemma 1 to show that for $x \geq 0$,

$$Q_n(x) \geq q_0 \geq \frac{(2-\alpha)(2n)!}{2n!}. \quad \blacksquare$$

The (1, 1) Padé approximant to e^{-x} has denominator $Q(x) = 1 + \frac{1}{2}x$. It follows that the (1, 1) rational function that interpolates e^{-x} with multiplicity three at any point β will have denominator $Q_\beta(x) = 1 + \frac{1}{2}(x - \beta)$. In particular if $\beta \geq 2$ then Q_β does not have positive coefficients. This shows that $\alpha < 2$ is essential, at least for the $n = 1$ case, in Theorem 1.

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