MÜNTZ SYSTEMS AND ORTHOGONAL MÜNTZ–LEGENDRE POLYNOMIALS

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§1. INTRODUCTION

Let $0 \le \lambda_0 < \lambda_1 < \cdots \to \infty$. The classical Müntz–Szász Theorem states that the Müntz polynomials of the form $\sum_{k=0}^{n} a_k x^{\lambda_k}$ with real coefficients are dense in $L^2[0,1]$ if and only if

$$\sum_{k=1}^{\infty} \lambda_k^{-1} = +\infty. \tag{1.1}$$

If the constant function 1 is also in the system, that is, $\lambda_0 = 0$, then the denseness of the Müntz polynomials in $C[0,1]$ in the uniform norm is also characterized by (1.1). It is our intention to examine various facets of the Müntz space

$$M = \text{span}\{x^{\lambda_0}, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

and for its subspaces

$$M_n = \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\},$$

where the span is taken over all real numbers (§4 and §5, where real properties are studied) or complex numbers (§2 and §3). It has been observed [25, Mathematical Review 88e:33008] and [13], but does not appear to be particularly well–known, that the orthogonal polynomials associated with a Müntz system (with respect to Lebesgue measure) on $[0,1]$ can be explicitly written down. These orthogonal

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polynomials are called Müntz–Legendre polynomials. This is the key tool for the analysis we undertake. We prove for example the L^2 Markov inequality

$$\frac{\|xp'(x)\|_2}{\|p\|_2} \leq \frac{1}{\sqrt{2}} \sum_{k=0}^{n} (1 + 2\lambda_k)$$

for all Müntz polynomials p from M_n. Compare this with the L^∞ result in [17]

$$\frac{\|xp'(x)\|_\infty}{\|p\|_\infty} \leq 11 \sum_{k=0}^{n} \lambda_k.$$

Both of these are sharp up to the constants. In order to prove this result and various of its relatives we first derive some explicit formulae and recursions for the sequence of Müntz–Legendre polynomials. Since this orthogonalization is not well–known, and for the sake of completeness we briefly reprove some of the basic formulae, some of which may be found in [13, 25]. This is contained in Section 2. Section 3 offers some inequalities for Müntz polynomials, mainly, the above mentioned L^2 Markov inequality. In Section 4, we study the interlacing and lexicographical properties of the zeros of Müntz–Legendre polynomials . Also in this section, universal estimates of the smallest and largest zeros of Müntz–Legendre polynomials are obtained via the zeros of Laguerre polynomials. Finally in the last section, we study the properties of the Christoffel functions, whose pointwise or uniform convergence on closed subintervals of $[0, 1)$ turns out to give a characterization of the nondenseness of the Müntz space on $[0, 1]$.

Proofs of the Müntz–Szász Theorem can be found in [6], [8], and [10], and various new developments are in [1], [2], [3], [4], [5], [7], [8], [11], [17], [18], [21], [22], [23], [26], [28]. A very special class of Müntz systems, the incomplete polynomials of the form $x^m p(x)$ with ordinary polynomials p has been studied intensively (cf. [12, 20]).

§2. Basic Properties of Müntz–Legendre Polynomials

Throughout this paper, we adopt the following definition for x^λ:

$$x^\lambda = e^{\lambda \log x}, \quad x \in (0, \infty),\ \lambda \in \mathbb{C} \tag{2.1}$$

and the value at $x = 0$ is defined to be the limit of x^λ as $x \to 0$ from $(0, \infty)$ whenever the limit exists. Given a complex sequence $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$, a linear combination of the Müntz system $\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$ is called a *Müntz polynomial,* or a Λ–*polynomial.* Denote the set of all such polynomials by $M_n(\Lambda)$, that is,

$$M_n(\Lambda) = \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}, \tag{2.2}$$

where the linear span is over the real numbers (§4 and §5) or over the complex numbers (this section and §3), according to context. The union of all M_n is denoted by $M(\Lambda)$, that is,

$$M(\Lambda) = \cup_{n=0}^{\infty} M_n(\Lambda). \tag{2.3}$$

For the L^2 theory of a Müntz system, we consider

$$\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}, \qquad \Re(\lambda_k) > -1/2, \quad \text{and} \quad \lambda_k \neq \lambda_j \, (k \neq j), \tag{2.4}$$

where $\Re(\lambda)$ is the real part of λ. This ensures that every Λ–polynomial is in $L^2[0,1]$. We can then define the orthogonal Λ–polynomials with respect to the Lebesgue measure, the *Müntz–Legendre polynomials.* Although we almost always assume (2.4), the following definition does not require the distictness of λ_k.

Definition 2.1. Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ be a complex sequence. We define the n–th *Müntz–Legendre polynomial* on $(0,1]$ to be [cf. 25]

$$L(\lambda_0, \lambda_1, \dots, \lambda_n; x) = \frac{1}{2\pi i} \int_\Gamma \prod_{k=0}^{n-1} \frac{t + \bar{\lambda}_k + 1}{t - \lambda_k} \frac{x^t dt}{t - \lambda_n}, \quad n = 0, 1, 2, \dots, \tag{2.5}$$

where the simple contour Γ surrounds all the zeros of the denominator in the integrand, and $\bar{\lambda}$ denotes the conjugate of λ.

The orthogonality of the above functions with respect to the Lebesgue measure will be proved in Corollary 2.3. Here we first record an immediate consequence of the definition and the residue theorem.

Corollary 2.2. *Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ satisfy (2.4). Then for every $n = 0, 1, 2, \dots$,*

$$L(\lambda_0, \dots, \lambda_n; x) = \sum_{k=0}^{n} c_{k,n} x^{\lambda_k}, \qquad c_{k,n} = \frac{\prod_{j=0}^{n-1} (\lambda_k + \bar{\lambda}_j + 1)}{\prod_{j=0, j\neq k}^{n} (\lambda_k - \lambda_j)} \tag{2.6}$$

with $L(\lambda_0, \dots, \lambda_n; x)$ defined by (2.5).

So, $L(\lambda_0, \dots, \lambda_n)$ is indeed a Λ–polynomial provided that $\lambda_0, \lambda_1, \dots, \lambda_n$ are distinct. Its value at $x = 0$ is defined if for all k either $\Re(\lambda_k) > 0$ or $\lambda_k = 0$. For example, if $\lambda_0 = 0$ and $\Re(\lambda_k) > 0$ $(1 \le k \le n)$, then $L(\lambda_0, \dots, \lambda_n; 0) = c_{0,n}$, and it is 0 if $\Re(\lambda_0) > 0$ also holds.

Remark. From either Definition 2.1 or Corollary 2.2, it is obvious that in $L(\lambda_0, \dots, \lambda_n)$, the order of $\lambda_0, \dots, \lambda_{n-1}$ does not make any difference, as long as λ_n is kept last. For example, $L(\lambda_0, \lambda_1, \lambda_2) = L(\lambda_1, \lambda_0, \lambda_2)$, but both are usually different from $L(\lambda_0, \lambda_2, \lambda_1)$. For a fixed (ordered) sequence Λ, we will use $L_n(\Lambda)$, or simply L_n to denote the n–th Müntz–Legendre polynomial $L(\lambda_0, \dots, \lambda_n)$, whenever there is no ambiguity.

In (2.6), repeated indices (for example, $\lambda_0 = \lambda_1$) cause a problem. But in the original definition, $\lambda_k = \lambda_j$ is allowed. We can view this also as a limiting case ($\lambda_k \to \lambda_j$). We state a very special case when all indices are the same, which turns out to be closely related to the Laguerre polynomials. Notice also that the result is actually no longer a Λ–polynomial, with $\log x$ coming into the picture.

Corollary 2.3. *Let $L(\lambda_0, \dots, \lambda_n; x)$ be defined by (2.5). If $\lambda_0 = \cdots = \lambda_n = \lambda$, then*

$$L(\lambda_0, \dots, \lambda_n; x) = x^\lambda \mathcal{L}_n \left(-(1 + \lambda + \bar{\lambda}) \log x\right), \tag{2.7}$$

where $\mathcal{L}_n$ is the n–th Laguerre polynomial orthogonal with respect to the weight e^{-x} on $(0, \infty)$ and with $\mathcal{L}_n(0) = 1$.

Proof. Since $\lambda_k = \lambda$ for $k = 0, 1, \dots, n$, (2.5) yields,

$$L(\lambda_0, \lambda_1, \dots, \lambda_n; x) = \frac{1}{2\pi i} \int_\Gamma \frac{x^t (t + \bar{\lambda} + 1)^n}{(t - \lambda)^{n+1}} \, dt,$$

where the contour Γ can be taken to be any circle centered at λ. By the residue theorem,

$$\begin{aligned} L(\lambda_0, \dots, \lambda_n; x) &= \frac{d^n}{n! \, dt^n} \left[x^t (t + \bar{\lambda} + 1)^n\right]_{t=\lambda} \\ &= \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} x^\lambda (\log x)^k n(n-1) \cdots (k+1)(\lambda + \bar{\lambda} + 1)^k \\ &= x^\lambda \sum_{k=0}^n \frac{1}{k!} \binom{n}{k} (1 + \lambda + \bar{\lambda})^k \log^k x. \end{aligned}$$

These are just the Laguerre polynomials $\{\mathcal{L}_n\}$ in $(-\log x)$ which are orthogonal with respect to the weight function e^{-t} on $(0, \infty)$ with the normalization $\mathcal{L}_n(0) = 1$ (cf. [24, p. 100]), and we obtain (2.7). □

The name Müntz–Legendre polynomial is justified by the following theorem, where the orthogonality of $\{L_n\}$ with respect to the Lebesgue measure is proved.

Theorem 2.4. *Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ satisfy $\Re(\lambda_k) > -1/2$ for $k = 0, 1, 2, \dots$. Assume that L_n, $n = 0, 1, 2, \dots$, are defined by (2.5). Then*

$$\int_0^1 L_n(x) \overline{L_m(x)} = \delta_{n,m} / (1 + \lambda_n + \bar{\lambda}_n) \tag{2.8}$$

holds for every $m, n = 0, 1, 2, \dots$.

Remark. In the orthogonality (2.8), repeated indices are allowed.

Proof. We provide a proof here for the sake of completeness. It suffices to consider $0 \le m \le n$. Also, we just need to prove (2.8) for distinct indices, since from the definition in (2.5), $L(\lambda_0, \dots, \lambda_n; x)$ is uniformly continuous in $\lambda_0, \dots, \lambda_n$ for x in closed subintervals of $(0,1]$, and the non–distinct case is a limiting argument. Since $\Re(\lambda_k) > -1/2$, we can pick a contour Γ in the integral (2.5) such that Γ lies completely to the right of the vertical line $\Re(t) = -1/2$, and Γ surrounds all zeros of the denominator. When $t \in \Gamma$, we have $\Re(t + \bar\lambda_m) > -1$, and $\int_0^1 x^{t+\bar\lambda_m} dx = 1/(1 + t + \bar\lambda_m)$, for every $m \ge 0$. Hence,

$$\int_0^1 L_n(x)\overline{x^{\lambda_m}}\,dx = \int_\Gamma \prod_{k=0}^{n-1} \frac{t + \bar\lambda_k + 1}{t - \lambda_k} \frac{dt}{(t - \lambda_n)(t + \bar\lambda_m + 1)}.$$

Notice that for $m < n$, the new term $t + \bar\lambda_m + 1$ in the denominator can be cancelled, and for $m = n$ the new pole $-(\bar\lambda_n + 1)$ is in the outside of Γ, because $\Re(-\bar\lambda_n - 1) < -1/2$. Changing the contour from Γ to $|t| = R$ with $R > \max\{|\lambda_0|+1, \dots, |\lambda_n|+1\}$, we have for $0 \le m \le n$ that

$$\int_0^1 L_n(x)\overline{x^{\lambda_m}}\,dx = \frac{1}{2\pi i} \int_{|t|=R} \prod_{k=0}^{n-1} \frac{t + \bar\lambda_k + 1}{t - \lambda_k} \frac{dt}{(t - \lambda_n)(t + \bar\lambda_n + 1)} - \frac{\delta_{m,n}}{-\bar\lambda_n - 1 - \lambda_n} \prod_{k=0}^{n-1} \frac{-\bar\lambda_n + \lambda_k}{-\bar\lambda_n - 1 - \lambda_k}.$$

Letting $R \to \infty$, we see that the integral on the right–hand side is actually 0, which gives

$$\int_0^1 L_n(x)\overline{x^{\lambda_m}}\,dx = \frac{\delta_{n,m}}{\bar\lambda_n + \lambda_n + 1} \prod_{k=0}^{n-1} \frac{\bar\lambda_n - \lambda_k}{\bar\lambda_n + \lambda_k + 1}.$$

Now with (2.6), we have for $0 \le m \le n$ that

$$\int_0^1 L_n(x)\overline{L_m(x)}dx = \int_0^1 L_n(x) \sum_{k=0}^{m} \overline{c_{k,m} x^{\lambda_k}}\,dx$$

$$= \overline{c_{m,m}} \int_0^1 L_n(x) x^{\bar\lambda_m}\,dx = \delta_{m,n}/(\lambda_n + \bar\lambda_n + 1),$$

where the last step comes from the formula for $c_{k,n}$ in (2.6) □

An alternative and probably easier proof of orthogonality follows from (2.10) below, integration by parts and induction. Later we will see that $L_n(1) = 1$. This can be viewed as the normalization for Müntz–Legendre polynomial L_n. Clearly, if we let

$$L_n^* = (1 + \lambda_n + \bar\lambda_n)^{1/2} L_n, \tag{2.9}$$

then we get an orthonormal system, that is,

$$\int_0^1 L_n^*(x)\overline{L_m^*(x)}dx = \delta_{m,n}, \quad m,n = 0,1,2,\dots .$$

These L_n^*, $n = 0,1,2,\dots$, will be called *orthonormal Müntz–Legendre polynomials.*

There is also a Rodrigues formula for the Müntz–Legendre polynomials [13]. Let

$$p_n(x) = \sum_{k=0}^{n} x^{\lambda_k} / \prod_{j=0, j\neq k}^{n} (\lambda_k - \lambda_j),$$

then

$$L_n(x) = D_{\lambda_0} \cdots D_{\lambda_{n-1}} p_n(x),$$

where the differential operators D_λ are defined by $D_\lambda f = x^{-\bar{\lambda}} \frac{d}{dx} x^{1+\bar{\lambda}} f$. Notice also that p_n and its first $n-1$ derivatives vanish at $x = 1$ (cf. [13]). This formula follows easily from Corollary 2.2.

Now we state the differential recurrence formulae for $\{L_n\}$.

Theorem 2.5. *Assume that Λ is a complex sequence satisfying $\Re(\lambda_k) > -1/2$ for all k. Then*

$$xL_n'(x) - xL_{n-1}'(x) = \lambda_n L_n(x) + (1+\bar{\lambda}_{n-1})L_{n-1}(x), \quad n = 1,2,3,\dots, \tag{2.10}$$

where L_n, $n = 0,1,2,\dots$, are the associated Müntz–Legendre polynomials defined by (2.5).

Proof. From (2.5), we get

$$\frac{d}{dx}\left(x^{-\lambda_n} L_n(x)\right) = \frac{1}{2\pi i}\int_\Gamma \frac{\prod_{k=0}^{n-2}(t+\bar{\lambda}_k+1)}{\prod_{k=0}^{n-1}(t-\lambda_n)}(t+\bar{\lambda}_{n-1}+1)x^{t-\lambda_n-1}dt.$$

Multiplying both sides by $x^{\lambda_n+\bar{\lambda}_{n-1}+1}$, we obtain

$$x^{\lambda_n+\bar{\lambda}_{n-1}+1}\left(x^{-\lambda_n}L_n(x)\right)' = \frac{1}{2\pi i}\int_\Gamma \frac{\prod_{k=0}^{n-2}(t+\bar{\lambda}_k+1)}{\prod_{k=0}^{n-1}(t-\lambda_k)}(t+\bar{\lambda}_{n-1}+1)x^{t+\bar{\lambda}_{n-1}}dt,$$

and again by the definition of L_{n-1} (cf. (2.5)),

$$x^{\lambda_n+\bar{\lambda}_{n-1}+1}\left(x^{-\lambda_n}L_n(x)\right)' = \left(x^{\bar{\lambda}_{n-1}+1}L_{n-1}(x)\right)'.$$

Simplifying by the product rule and dividing both sides by $x^{\bar{\lambda}_{n-1}}$, we get (2.10). □

Corollary 2.6. *Let a complex sequence Λ satisfy (2.4), and let the associated Müntz–Legendre polynomials L_n and the orthonormal Müntz–Legendre polynomials L_n^* be defined by (2.5) and (2.9), respectively. Then*

$$xL_n'(x) = \lambda_n L_n(x) + \sum_{k=0}^{n-1} (\lambda_k + \bar{\lambda}_k + 1) L_k(x), \tag{2.11}$$

$$xL_n^{*'}(x) = \lambda_n L_n^*(x) + \sqrt{\lambda_n + \bar{\lambda}_n + 1} \sum_{k=0}^{n-1} \sqrt{\lambda_k + \bar{\lambda}_k + 1} L_k^*(x), \tag{2.12}$$

and

$$xL_n''(x) = (\lambda_n - 1) L_n'(x) + \sum_{k=0}^{n-1} (\lambda_k + \bar{\lambda}_k + 1) L_k'(x) \tag{2.13}$$

for every $x \in (0, \infty)$ and every $n = 0, 1, 2, \ldots$.

Proof. The first equality (2.11) follows from Theorem 2.4 by writing $xL_n'(x) - xL_0'(x)$ as a telescoping sum. From (2.11) and the relation $L_k^* = (\lambda_k + \bar{\lambda}_k + 1)^{1/2} L_k$ (cf. (2.9)), we get (2.12). Differentiating (2.11), we obtain (2.13). □

The values and derivative values of the Müntz–Legendre polynomials at 1 can all be calculated. They are useful in locating the zeros of Müntz–Legendre polynomials (cf. §4).

Corollary 2.7. *Let L_n be the n–th Müntz–Legendre polynomial defined by (2.5) (or by (2.6) from Λ satisfying (2.4)), then*

$$L_n(1) = 1, \quad L_n'(1) = \lambda_n + \sum_{k=0}^{n-1} (\lambda_k + \bar{\lambda}_k + 1), \quad n = 0, 1, 2, \ldots, \tag{2.14}$$

and

$$L_n''(1) = (\lambda_n - 1) L_n'(1) + \sum_{k=0}^{n-1} (\lambda_k + \bar{\lambda}_k + 1) L_k'(1) \quad n = 0, 1, 2, \ldots . \tag{2.15}$$

Proof. It suffices to show that $L_n(1) = 1$, for the rest follows from Corollary 2.6. Notice that from (2.5),

$$L_n(1) = \frac{1}{2\pi i} \int_\Gamma \prod_{k=0}^{n-1} \frac{t + \bar{\lambda}_k + 1}{t - \lambda_n} \frac{dt}{t - \lambda_n}.$$

Since Γ surrounds all zeros of the denominator, and the degree of the denominator is 1 higher than that of the numerator, let Γ be the circle $|t| = R$ and let $R \to \infty$. ¿From this we get $L_n(1) = 1$. □

The recurrence formula can also be expressed in an integral form.

Corollary 2.8. *Let a complex sequence* Λ *satisfy (2.4), and let* L_n, $n = 0, 1, 2, \dots$, *be the Müntz–Legendre polynomials defined by (2.5). Then,*

$$L_n(x) = L_{n-1}(x) - (\lambda_n + \bar{\lambda}_{n-1} + 1)x^{\lambda_n} \int_x^1 x^{-\lambda_n - 1} L_{n-1}(t)\, dt, \quad x \in (0, 1]. \tag{2.16}$$

Proof. Rewriting the recurrence formula (2.10) as

$$xL_n(x) - \lambda_n L_n(x) = xL'_{n-1}(x) + (1 + \bar{\lambda}_{n-1})L_{n-1}(x),$$

and multiplying both sides by $x^{-\lambda_n - 1}$, we obtain

$$\left(x^{-\lambda_n} L_n(x)\right)' = x^{-\lambda_n} L'_{n-1}(x) + (1 + \bar{\lambda}_{n-1})x^{-\lambda_n - 1} L_{n-1}(x).$$

On taking the definite integral of the above on $[x, 1]$, and using the fact that $L_k(1) = 1$ for all $k \geq 0$, we conclude

$$\begin{aligned} 1 - x^{-\lambda_n} L_n(x) = 1 - x^{-\lambda_n} L_{n-1}(x) &- \int_x^1 (t^{-\lambda_n})' L_{n-1}(t)\, dt \\ &+ (\bar{\lambda}_{n-1} + 1) \int_x^1 t^{-\lambda_n - 1} L_{n-1}(t) dt, \end{aligned}$$

which implies (2.16). □

Another observation is that if $0 \leq \lambda_n \to \infty$ very fast, then $x = 1$ is the unique maximal point of the Müntz–Legendre polynomial on $[0, 1]$. A reasonable conjecture seems to be that the maximum of $|L_n|$ always attains at one of the endpoints of $[0, 1]$ when $\lambda_n \geq 0$.

Corollary 2.9. *If* $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ *is a nonnegative sequence such that*

$$\lambda_n \geq \sum_{k=0}^{n-1} (1 + 2\lambda_k), \qquad n = 1, 2, 3, \dots, \tag{2.17}$$

then

$$|L_n(x)| < L_n(1) = 1, \qquad x \in [0, 1), \ n = 2, 3, 4, \dots . \tag{2.18}$$

Remark. If $\lambda_k = \rho^k$, then (2.17) holds if and only if $\rho \geq 2 + \sqrt{3}$.

Proof. We assume $\lambda_0 = 0$. (The proof for $\lambda_0 > 0$ is essentially the same.) In this case, $L_0(x) \equiv 1$, and (2.18) fails for $n = 0$. From (2.17), $\lambda_1 \geq 1$, and $\lambda_k \geq 2 + \lambda_{k-1}$ for $k \geq 2$. By (2.6),

$$|L_n(0)| = |c_{0,n}| = \frac{\prod_{j=0}^{n-1} |1 + \lambda_j|}{\prod_{j=1}^{n} |\lambda_j|} = \prod_{j=1}^{n} \frac{1 + \lambda_{j-1}}{\lambda_j}.$$

Hence, $|L_1(0)| \leq 1$, and $L_n(0) < 1$ for every $n \geq 2$. Now we use induction to show that $|L_n(x)| < 1$ on $(0,1)$ for every $n \geq 1$. Indeed, for $n = 1$, because $|L_1(0)| \leq 1 = L_1(1)$, and $L_1(x) = c_{0,1} + c_{1,1}x^{\lambda_1}$ is monotone on [0,1], we have $|L_1(x)| < 1$ on (0,1). Assume that $n \geq 2$, and $|L_k(x)| < 1$ for $1 \leq k \leq n-1$. Let x be a local maximal point of $|L_n|$ in $(0,1)$, then $L_n'(x) = 0$. Hence Corollary 2.6 yields

$$\lambda_n L_n(x) + \sum_{k=0}^{n-1}(1+2\lambda_k)L_k(x) = 0.$$

Therefore

$$|L_n(x)| = \frac{1}{\lambda_n}\left|\sum_{k=0}^{n-1}(1+2\lambda_k)L_k(x)\right| < \sum_{k=0}^{n-1}(1+2\lambda_k)/\lambda_n \leq 1. \quad \square$$

We finish this section by introducing the reproducing kernels. They are similar to the Dirichlet kernels in the trigonometric theory, or to the reproducing kernels for ordinary polynomials (cf. [24, p. 40]).

Corollary 2.10. *Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ be as in (2.4), and let L_n and L_n^* be defined by (2.5) and (2.9). Then for every Λ–polynomial $p(x) = \sum_{k=0}^{n} a_k x^{\lambda_k}$ in $M_n(\Lambda)$, we have*

$$p(x) = \int_0^1 K_n(x,t)p(t)dt, \tag{2.19}$$

where

$$K_n(x,t) = \sum_{k=0}^{n} L_k^*(x)\overline{L_k^*(t)} \tag{2.20}$$

is the n–th reproducing kernel.

Proof. This is a well–known consequence of orthogonality. Since L_n^*, $n = 0,1,2,\dots,$ form an orthogonal system,

$$\int_0^1 K_n(x,t)L_k^*(t)dt = L_k^*(t), \qquad 0 \leq k \leq n.$$

Note that $\{L_0^*, \dots, L_n^*\}$ is a basis of $M_n(\Lambda)$, and the above is equivalent to (2.19). $\square$

Later in §3 and §5, we will see the importance of K_n in solving an extremal problem for Λ–polynomials, and in the characterization of denseness of Müntz systems.

§3. Inequalities for Müntz Systems

Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ satisfy (2.4) and let the Müntz spaces $M(\Lambda)$ and $M_n(\Lambda)$ be defined by (2.2) and (2.3). With the help of Müntz–Legendre polynomials, we establish some inequalities for Λ–polynomials.

We now record an estimate of a Λ–polynomial p and its derivative at a point $y \in (0,1]$ in terms of its L^2 norm ($\|p\|_2 = (\int_0^1 |p(t)|^2\,dt)^{1/2}$). First we state a more general theorem in terms of linear functionals.

Theorem 3.1. *Suppose that Λ satisfies (2.4) and that L_n^*, $n = 0,1,2,\dots,$ are the orthonormal Müntz–Legendre polynomials. Then*

$$|\phi(p)| \le \left[\sum_{k=0}^{n} |\phi(L_k^*)|^2\right]^{1/2} \|p\|_2 \tag{3.1}$$

for every linear functional ϕ defined on the Müntz space $M_n(\Lambda)$, and for every $p \in M_n(\Lambda)$. If $\Phi \not\equiv 0$, then the equality holds if and only if $p(x) = const \sum_{k=0}^{n} \overline{\phi(L_k^*)} L_k^*(x)$.

Proof. This is also a well–known consequence of the orthogonality of L_n^*, $n = 0,1,2,\dots$ (cf. [24, p. 39], where $\phi(p) = p(x)$ is considered.) To show (3.1) we write

$$p(x) = \sum_{k=0}^{n} c_k L_k^*(x)$$

with $\sum_{k=0}^{n} |c_k|^2 = \|p\|_2^2$. Hence, by the linearity of ϕ, we have

$$\phi(p) = \sum_{k=0}^{n} c_k \phi(L_k^*).$$

The theorem now follows from the Cauchy–Schwartz inequality. □

If the linear functional is $\phi(p) = p^{(\nu)}(y)$ for some fixed $y \in (0,1]$ and fixed integer ν, then the above becomes

Corollary 3.2. *Suppose that Λ satisfies (2.4) and that L_n^*, $n = 0,1,2,\dots,$ are the orthonormal Müntz–Legendre polynomials. Then*

$$|p^{(\nu)}(y)| \le \left[\sum_{k=0}^{n} |L_k^{*(\nu)}(y)|^2\right]^{1/2} \|p\|_2 \tag{3.2}$$

for every Λ–polynomial $p \in M_n(\Lambda)$, $\nu = 0,1,2,\dots,$ and $y \in (0,1]$. Equality holds if and only if $p(x) = const \sum_{k=0}^{n} \overline{L_k^{*(\nu)}(y)} L_k^*(x)$.

Remark. An equivalent expression of (3.2) is

$$\left[\sum_{k=0}^{n} |L_k^{*(\nu)}(y)|^2\right]^{1/2} = \max\{\, |p^{(\nu)}(y)| : p \in M_n(\Lambda), \|p\|_2 = 1\}. \tag{3.3}$$

By letting $n \to \infty$, this leads to

$$\left[\sum_{k=0}^{\infty} |L_k^{*(\nu)}(y)|^2\right]^{1/2} = \sup\{\, |p^{(\nu)}(y)| : p \in M(\Lambda), \|p\|_2 = 1\} \tag{3.4}$$

which may be finite or infinite. We will return to this in §5.

More explicit estimates than those in Corollary 3.2 can be obtained by combining Corollary 3.2 and Corollary 2.7. For simplicity, we only consider the cases $\nu = 0$ and 1, that is, we only state the estimates for $|p(y)|$ and $|p'(y)|$ in terms of $\|p\|_2$ and the index set Λ. These are of the flavour of Nikolskii and Berstein type inequalities.

Corollary 3.3. *Under the conditions of Corollary 3.2, we have*

$$|y^{1/2}p(y)| \le \left[\sum_{k=0}^{n} 1 + 2\Re(\lambda_k)\right]^{1/2} \|p\|_2, \tag{3.5}$$

and

$$|y^{3/2}p'(y)| \le \left[\sum_{k=0}^{n}(1+2\Re(\lambda_k))\left|\lambda_k + \sum_{j=0}^{k-1}(1+2\Re(\lambda_j))\right|^2\right]^{1/2} \|p\|_2 \tag{3.6}$$

hold for every $p \in M_n(\Lambda)$ and $y \in (0,1]$.

Proof. When $y = 1$, the above is a simple combination of Corollaries 2.7 and 3.3. For $0 < y < 1$, the scaling $x \to yx$ reduces the problem to the case $y = 1$. □

We now focus on one of the principal results, the L^2 Markov inequalities for Müntz polynomials, whose L^∞ version is in [17].

Theorem 3.4. *Assume that $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ is given as in (2.4). Then,*

$$\sup_{p \in M_n(\Lambda)} \frac{\|xp'(x)\|_2}{\|p\|_2} \le \left[\sum_{j=0}^{n} |\lambda_j|^2 + \sum_{j=0}^{n}(1+2\Re(\lambda_j)) \sum_{k=j+1}^{n}(1+2\Re(\lambda_k))\right]^{1/2} \tag{3.7}$$

If, in addition, Λ consists of nonnegative real numbers, then

$$\frac{1}{\sqrt{120}} \sum_{j=0}^{n} \lambda_j \le \sup_{p \in M_n(\Lambda)} \frac{\|xp'(x)\|_2}{\|p\|_2} \le \frac{1}{\sqrt{2}} \sum_{j=0}^{n}(1+2\lambda_j) \tag{3.8}$$

where n is an arbitrary nonnegative integer.

Remark. It is easy to see that the imaginary part of λ_j's does not affect the Markov factor as much as their real parts. For example, if $\lambda_j = ij$, then the Markov bound on the right–hand side of (3.7) is $[\sum_{j=0}^{n}(j^2+n-j)]^{1/2} = O(n^{3/2})$, while $\lambda_j = j$ results in $O(n^2)$.

Proof. Let $p \in M_n(\Lambda)$ be arbitrary, and $\|p\|_2 = 1$. Then $p(x) = \sum_{k=0}^{n} a_k L_k^*(x)$, and $\|p\|_2^2 = \sum_{k=0}^{n} |a_k|^2 = 1$. Thus,

$$xp'(x) = \sum_{k=0}^{n} a_k x L_k^{*'}(x).$$

If we use the recurrence formula (2.12) for the terms $xL_k^{*'}(x)$ in the above and rearrange the sum, we get

$$xp'(x) = \sum_{j=0}^{n} \left[a_j\lambda_j + \sqrt{1+\lambda_j+\bar{\lambda}_j} \sum_{k=j+1}^{n} a_k \sqrt{1+\lambda_k+\bar{\lambda}_k} \right] L_j^*(x).$$

Hence,

$$\int_0^1 |xp'(x)|^2\, dx = \sum_{k=0}^{n} \Big| a_j\lambda_j + \sqrt{1+\lambda_j+\bar{\lambda}_j} \sum_{k=j+1}^{n} a_k \sqrt{1+\lambda_k+\bar{\lambda}_k} \Big|^2.$$

Applying the Cauchy–Schwartz inequality for each term in the sum, and recalling that $\sum_{k=0}^{n} |a_k|^2 = 1$, we obtain

$$\int_0^1 |xp'(x)|^2 dx \le \sum_{j=0}^{n} \left[|\lambda_j|^2 + (1+\lambda_j+\bar{\lambda}_j) \sum_{k=j+1}^{n} (1+\lambda_k+\bar{\lambda}_k) \right] \le \frac{1}{2} \left[\sum_{j=0}^{n} (1+2|\lambda_j|) \right]^2 .$$

The above proves (3.7) and the right half of (3.8). To prove the sharpness for the case $\lambda_k \ge 0$ $(k \ge 0)$, we need to find a Λ–polynomial $p \ne 0$ in $M_n(\Lambda)$, such that

$$\|xp'\|_2^2 \ge \frac{1}{120} \left(\sum_{k=0}^{n} \lambda_k \right)^2 \|p\|_2^2. \tag{3.9}$$

Corollary 3.2 suggests that a possible candidate is $\sum_{k=0}^{n} \overline{L_k^{*'}(1)} L_k^*(x)$, and indeed, this works. However, a slight alternation makes the estimation easier. We consider

$$p(x) = \sum_{k=0}^{n} \sqrt{\lambda_k} \left(\sum_{j=0}^{k} \lambda_j \right) L_k^*(x).$$

Since the system $\{L_k^*\}_{k=0}^{\infty}$ is orthonormal, we have

$$\int_0^1 |p(x)|^2 dx = \sum_{k=0}^{n} \lambda_k \left(\sum_{j=0}^{k} \lambda_j \right)^2 \le \left(\sum_{j=0}^{n} \lambda_j \right)^3 . \tag{3.10}$$

Now

$$xp'(x) = \sum_{k=0}^{n} \sqrt{\lambda_k} \left(\sum_{j=0}^{k} \lambda_j \right) xL_k^{*'}(x) = \sum_{m=0}^{n} b_m L_m^*(x),$$

where by the recurrence formula (2.12)

$$b_m = \lambda_m \sqrt{\lambda_m} \sum_{j=0}^{m} \lambda_j + \sqrt{1+2\lambda_m} \sum_{k=m+1}^{n} \sqrt{\lambda_k(1+2\lambda_k)} \sum_{j=0}^{k} \lambda_j \geq \sqrt{\lambda_m} \sum_{k=m}^{n} \lambda_k \sum_{j=0}^{k} \lambda_j \geq 0.$$

Hence

$$\int_0^1 |xp'(x)|^2 dx = \sum_{m=0}^{n} |b_m|^2 \geq \sum_{m=0}^{n} \lambda_m \left(\sum_{k=m}^{n} \lambda_k \sum_{j=0}^{k} \lambda_j \right)^2$$

$$= \sum_{\substack{0 \leq m \leq n \\ m \leq k, k' \leq n}} \sum_{\substack{0 \leq j \leq k \\ 0 \leq j' \leq k'}} \lambda_m \lambda_k \lambda_j \lambda_{k'} \lambda_{j'} \geq \sum_{0 \leq m \leq j \leq j' \leq k \leq k' \leq n} \lambda_m \lambda_k \lambda_j \lambda_{k'} \lambda_{j'} \geq \frac{1}{5!} \left(\sum_{k=0}^{n} \lambda_k \right)^5 .$$

This, together with (3.10), proves (3.9), and hence the left–hand side of the inequality (3.8). □

We believe that the general L^p analogue of Theorem 3.4 is true. When the index sequence $\{\lambda_0, \lambda_1, \lambda_2, \dots\}$ is lacunary, the proof can be obtained from (5.10) in Lemma 5.6.

§4. On Zeros of Müntz–Legendre Polynomials

In this section we always assume that $\lambda_0, \lambda_1, ..., \lambda_n$ are real numbers (not necessarily distinct) greater than $-1/2$. We make several observations on the zeros of Müntz–Legendre polynomials, some of them are interesting for their own right. The main result is a lexicographic property of the zeros given by Theorem 4.7.

Proposition 4.1. *For a function $f \in C(0,1)$, let $S^-(f)$ and $Z(f)$ denote the number of sign changes and the number of zeros, respectively, of f in $(0,1)$ (in the count we count the zeros where there is no sign change, twice). Let Φ and Ψ be in $C(0,1)$. If*

$$n \leq S^-(\alpha\Phi + \beta\Psi) \leq Z(\alpha\Phi + \beta\Psi) \leq n+1$$

for every real α and β, then the zeros of Φ and Ψ strictly interlace.

A proof may be found in [19, Theorem 1.1 and Corollary 2].

Proposition 4.2. *Assume that*

$$\{\lambda_0, \lambda_1, ..., \lambda_n\} = \{\tilde{\lambda}_0, \tilde{\lambda}_1, ..., \tilde{\lambda}_m\},$$

where the numbers $\tilde{\lambda}_0, \tilde{\lambda}_1, ..., \tilde{\lambda}_m$ *are distinct, and let* m_j, $j = 0, 1, ..., m$, *be the number of indices* $i = 0, 1, ..., n$ *for which* $\lambda_i = \tilde{\lambda}_j$. *Then* $L_n(\lambda_0, \lambda_1, ..., \lambda_n)$ *is in the Chebyshev space*

$$H_n = span\{x^{\lambda_j}(\log x)^i : j = 0, 1, ..., m,\ i = 0, 1, ..., m_j - 1\}.$$

This follows from the definition (cf. (2.5)) and the Residue Theorem.

Proposition 4.3. $\{L_k(\lambda_0, \lambda_1, ..., \lambda_k)\}_{k=0}^{n}$ *is a basis of the Chebyshev space* H_n *defined in Proposition 4.2.*

This follows from orthogonality (cf. Theorem 2.4).

Proposion 4.4. $L_n = L_n(\lambda_0, \lambda_1, ..., \lambda_n)$ *has exactly* n *distict zeros in* $(0, 1)$, *and* L_n *changes sign at each of these zeros.*

Proof. Assume to the contrary that the number of sign changes of L_n in $(0, 1)$ is less than n. By Proposion 4.3, there is a function $p \in \text{span}\{L_k\}_{k=0}^{n-1}$, which changes sign exactly at those points in $(0, 1)$ where L_n changes sign. Then $\int_0^1 L_n p \neq 0$ which contradicts Theorem 2.4.

Proposion 4.5. *Let* $1 \le k \le n$ *be fixed and* $\lambda_k < \lambda_k^*$. *Then the zeros of*

$$\Phi = L_n(\lambda_0, ..., \lambda_{k-1}, \lambda_k, \lambda_{k+1}, ..., \lambda_n)$$

and

$$\Psi = L_n(\lambda_0, ..., \lambda_{k-1}, \lambda_k^*, \lambda_{k+1}, ..., \lambda_n)$$

in $(0, 1)$ *strictly interlace.* □

Proof. Note that Theorem 2.4 implies

$$\int_0^1 (\alpha\Phi + \beta\Psi)p = 0$$

for every $p \in H_{n-1}$, where H_{n-1} is defined in Proposition 4.2. As in the proof of Propostion 4.4, $\alpha\Phi + \beta\Psi$ has at least n sign changes in $(0, 1)$, whenever α and β are real with $\alpha^2 + \beta^2 > 0$. Proposition 4.2 implies that $\alpha\Phi + \beta\Psi$ cannot have more than $n + 1$ zeros in $(0, 1)$ whenever α and β are real with $\alpha^2 + \beta^2 > 0$. Now the proof can be finished by Proposition 4.1. □

Propostion 4.6. *Let $\lambda_0, ..., \lambda_{k-1}, \lambda_{k+1}, ..., \lambda_n$ be fixed distict numbers with some fixed integers $0 \le k \le n$. Suppose $\{\lambda_{k,i}\}_{i=1}^{\infty} \subset (-1/2, \infty)$ is a sequence with* $\lim_{i\to\infty} \lambda_{k,i} = \infty$. *Then the largest zero of*

$$L_{n,k,i} = L_n(\lambda_0, ..., \lambda_{k-1}, \lambda_{k,i}, \lambda_{k+1}, ..., \lambda_n)$$

in (0,1) tends to 1.

Proof. Assume, without loss of generality, that $\lambda_{k,i}$ is greater than each of the numbers λ_j, $j = 0, , 1, ..., n$, $j \neq k$. We distinguish two cases.

Case 1: $k = n$. Let

$$g_i(x) = \lambda_{n,i} \left(L_{n,n,i}(x) - c_{n,n}^{(i)} x^{\lambda_{n,i}} \right),$$

where

$$c_{n,n}^{(i)} = \frac{\prod_{j=0}^{n-1} (\lambda_{n,i} + \lambda_j + 1)}{\prod_{j=0}^{n-1} (\lambda_{n,i} - \lambda_j)}$$

is the coefficient of $x^{\lambda_{n,i}}$ in $L_{n,n,i}$. Now Corollary 2.2 implies that the functions g_i converge uniformly on $[\delta, 1]$, $\delta \in (0,1)$, to a function

$$0 \not\equiv g \in H_{n-1} = \text{span}\{x^{\lambda_0}, x^{\lambda_1}, ..., x^{\lambda_{n-1}}\}.$$

By using $L_{n,n,i}(1) = 1$ (cf. Corollary 2.7), and the explicit formula for $c_{n,n}^{(i)}$, it follows that $g(1) \le 0$ and

$$\lambda_{n,i} L_{n,n,i}(x) = \lambda_{n,i}(L_{n,n,i}(x) - c_{n,n}^{(i)} x^{\lambda_{n,i}}) + \lambda_{n,i} c_{n,n}^{(i)} x^{\lambda_{n,i}}$$

converge to $g(x)$, as $i \to \infty$, for every $x \in (0,1)$.

Now assume that the statement of the proposition is false. Then there is an $\epsilon \in (0,1)$ and a subsequence $\{\lambda_{n,i_j}\}_{j=1}^{\infty}$ of $\{\lambda_{n,i}\}_{i=1}^{\infty}$ so that the Müntz-Legendre polynomials L_{n,n,i_j} have no zeros in $[1-\epsilon, 1]$. From this we can deduce that g is nondecreasing on $[1-\epsilon, 1]$, which together with $0 \not\equiv g \in H_{n-1}$ and $g(1) \le 0$, implies that $g(1-\epsilon) < 0$. Therefore, $L_{n,n,i}(1-\epsilon) < 0$ if i is large enough. Since $L_{n,n,i}(1) = 1$ (cf. Corollary 2.7), each $L_{n,n,i}$ has a zero in $(1-\epsilon, 1)$ if $i \ge i_0$, which contradicts our assumption.

Case 2: $0 \le k \le n-1$. Let

$$g_i(x) = \lambda_{k,i} \left(L_{n,k,i}(x) - c_{k,n}^{(i)} x^{\lambda_{k,i}} \right),$$

where

$$c_{k,n}^{(i)} = \frac{\prod_{j=0}^{n-1} (\lambda_{k,i} + \lambda_j + 1)}{\prod_{j=0}^{n} (\lambda_{k,i} - \lambda_j)}$$

is the coefficient of $x^{\lambda_{k,i}}$ in $L_{n,k,i}$. ¿From Corollary 2.2, we can deduce that the functions g_i converge uniformly on $[\delta, 1]$, $\delta \in (0,1)$, to a function

$$0 \not\equiv g \in H_{n-1} = \operatorname{span}\{x^{\lambda_0}, ..., x^{\lambda_{k-1}}, x^{\lambda_{k+1}}, ..., x^{\lambda_n}\}.$$

By using $L_{n,k,i}(1) = 1$ (cf. Corollary 2.7) and the explicit formula for $c_{k,n}^{(i)}$, it follows that $g(1) \leq 0$ and

$$L_{n,k,i}(x) = \left(L_{n,k,i}(x) - c_{k,n}^{(i)} x^{\lambda_{k,i}}\right) + c_{k,n}^{(i)} x^{\lambda_{k,i}}$$

converge to $g(x)$, as $i \to \infty$, for every $x \in (0,1)$. Now the proof can be finished as in Case 1. □

Theorem 4.7. *Let $\lambda_j \leq \mu_j$, $j = 0, 1, ..., n$, with strict inequality for at least one index j. Let*

$$x_1 < x_2 < ... < x_n \quad \textit{and} \quad x_1^* < x_2^* < ... < x_n^*$$

be the zeros of

$$L_n(\lambda_0, \lambda_1, ..., \lambda_n) \quad \textit{and} \quad L_n(\mu_0, \mu_1, ..., \mu_n),$$

respectively, in $(0,1)$. Then

$$x_j < x_j^*, \qquad j = 1, 2, ..., n.$$

Proof. First assume that $\lambda_i \neq \lambda_j$ and $\mu_i \neq \mu_j$ when ever $i \neq j$. Without loss of generality, we may assume that there is an index k, $0 \leq k \leq n$, so that $\lambda_j = \mu_j$ if $j \neq k$, and $\lambda_k < \mu_k$. Now Propositions 4.5 and 4.6 yield the conclusion of the theorem. Finally, a limiting argument and Propostion 4.5 can be used to drop the assumption that $\lambda_i \neq \lambda_j$ and $\mu_i \neq \mu_j$ whenever $i \neq j$. □

Proposition 4.8. *Let $\lambda_k \neq \lambda_n$. Then the zeros of*

$$\Phi = L_n(\lambda_0, ..., \lambda_{k-1}, \lambda_k, \lambda_{k+1}, ..., \lambda_{n-1}, \lambda_n)$$

and

$$\Psi = L_n(\lambda_0, ..., \lambda_{k-1}, \lambda_n, \lambda_{k+1}, ..., \lambda_{n-1}, \lambda_k)$$

in $(0,1)$ strictly interlace.

Proof. Propostion 4.1 and arguments similar to those in the proof of Proposition 4.5 gives the conclusion. The observations

$$\Phi(1) = \Psi(1) = 1 \quad \text{and} \quad \Psi'(1) - \Phi'(1) = \lambda_n - \lambda_k \neq 0$$

(cf. Corollary 2.7) garantee that $\alpha\Phi + \beta\Psi \not\equiv 0$ whenever α and β are real with $\alpha^2 + \beta^2 > 0$. □

Propostion 4.9. *Let* Φ *and* Ψ *be as in Propostion 4.8. Let*

$$x_1 < x_2 < \dots < x_n \quad \text{and} \quad x_1^* < x_2^* < \dots < x_n^*$$

be the zeros of Φ *and* Ψ *in* $(0,1)$*. Then* $\lambda_k < \lambda_n$ *implies that*

$$x_j < x_j^*, \qquad j = 1, 2, \dots, n.$$

Proof. By Proposition 4.8, it is sufficient to prove that $x_n < x_n^*$. Let H_n be the Chebyshev space defined in Propostion 4.2. Corollary 2.7 implies

$$\Phi(1) = \Psi(1) = 1 \quad \text{and} \quad \Psi'(1) - \Phi'(1) = \lambda_n - \lambda_k > 0.$$

¿From this, and Propostion 4.8, we can deduce that x_n^* would imply that $0 \not\equiv \Psi - \Phi \in H_n$ has at least $n+1$ distinct zeros in $(0,1]$, which is a contradiction. □

Propostion 4.10. *Let* $\lambda_0 < \lambda_n$*. Let*

$$x_1 < x_2 < \dots < x_n \quad \text{and} \quad x_1^* < x_2^* < \dots < x_n^*$$

be the zeros of

$$L_n(\lambda_0, \lambda_1, \dots, \lambda_n) \quad \text{and} \quad L_n(\lambda_n, \lambda_{n-1}, \dots, \lambda_0),$$

respectively, in $(0,1)$*. Then* $\{x_j\}_{j=1}^n$ *and* $\{x_j^*\}_{j=1}^n$ *strictly interlace and*

$$x_j < x_j^*, \qquad j = 1, 2, \dots, n.$$

Proof. This follows from Propositions 4.8 and 4.9 and the Remark given after Corollary 2.2. □

Proposition 4.11. *The zeros of*

$$\Phi = L_{n-1}(\lambda_0, \lambda_1, \dots, \lambda_{n-1}) \quad \text{and} \quad \Psi = L_n(\lambda_0, \lambda_1, \dots, \lambda_n)$$

in $(0,1)$ *strictly interlace.*

Proof. Propostion 4.1 and arguments similar to those given in the proof of Proposition 4.5 yield the theorem.

Corollary 4.12. *Assume that* $x_1 < \cdots < x_n$ *are the zeros of* $L(\lambda_0, \dots, \lambda_n)$ *in* $(0,1)$. *Then,*

$$\exp\left(-\frac{4n+2}{1+2\lambda_*}\right) < x_1 < \cdots < x_n < \exp\left(\frac{-j_1^2}{(1+2\lambda^*)(4n+2)}\right),$$

where $\lambda_* = \min\{\lambda_0, \dots, \lambda_n\}$, $\lambda^* = \max\{\lambda_0, \dots, \lambda_n\}$ *and* $j_1 > 3\pi/4$ *is the smallest positive zero of the Bessel function* $J_0(z) = \sum_{k=0}^{\infty}(-z^2)^k/(k!2^k)^2$.

Proof. Let $\mathcal{L}_n$ be the n–th Laguerre polynomial with respect to the weight e^{-x} on $[0,\infty)$, and let the zeros of $\mathcal{L}_n$ be $z_1 < \cdots < z_n$. Then we have (cf. [24, p. 127–131])

$$\frac{j_1^2}{4n+2} < z_1 < \cdots < z_n < 4n+2, \tag{4.1}$$

where the upper estimate is asymptotically sharp, and the lower estimate is sharp up to a constant(not exceeding $4^4/9\pi^2$). Since n is fixed, we let $\epsilon > 0$ be sufficiently small that $\lambda_* - n\epsilon > -1/2$. Then all the zeros of $L(\lambda_0, \dots, \lambda_n)$ lie to the right of those of $L(\lambda_*, \lambda_* - \epsilon, \dots, \lambda_* - n\epsilon)$ by Theorem 4.7. ¿From the contour integral formula (2.5), $L(\lambda_*, \lambda_* - \epsilon, \dots, \lambda_* - n\epsilon)$ tends to $L(\lambda_*, \lambda_*, \dots, \lambda_*)$ uniformly on closed subintervals of $(0,1]$ as $\epsilon \to 0$. Recalling that (cf. Corollary 2.3) $L(\lambda_*, \lambda_*, \dots, \lambda_*) = x^{\lambda_*}\mathcal{L}_n\left(-(1+2\lambda_*)\log x\right)$, we conclude that $x_1 \ge y_1$, where y_1 is the smallest zero of $L(\lambda_*, \lambda_*, \dots, \lambda_*)$. Since $z_n = -(1+2\lambda_*)\log y_1$, we can combine this with (4.1) to get

$$x_1 \ge y_1 = \exp\left(\frac{-z_n}{1+2\lambda_*}\right) > \exp\left(-\frac{4n+2}{1+2\lambda_*}\right),$$

which is the left–hand side inequlity of this Corollary. It can be seen similarly that all zeros of $L(\lambda_0, \dots, \lambda_n)$ lie to the left of zeros of $L(\lambda^*, \dots, \lambda^*) = x^{\lambda^*}\mathcal{L}_n\left(-(1+2\lambda^*)\log x\right)$, which implies that $x_n < \exp\left(-j_1^2/(1+2\lambda^*)(4n+2)\right)$. □

§5. Christoffel Functions

Christoffel functions have been intensively studied, and their utility in the theory of orthogonal polynomials and approximation theory can be illustrated by their relation with polynomial inequalities, interpolation theory, quadrature formulae, zeros of orthogonal polynomials, etc (cf. [14]). In this section, we will study the Müntz–Christoffel functions and some of their applications.

We assume that $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ satisfies

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \to +\infty \tag{5.1}$$

The Christoffel function for the Müntz system $M(\Lambda)$ with respect to the Lebesgue weight is defined by either side of the following equality

$$\frac{1}{\sum_{k=0}^{\infty} |L_k^*(x)|^2} = \inf_{p \in M(\Lambda), p(x)=1} \int_0^1 |p(t)|^2 \, dt, \tag{5.2}$$

which is a well known consequence of the orthogonality (cf. [24, p. 39]). If the infimum is taken just over $M_n(\Lambda)$, then we have

$$\frac{1}{\sum_{k=0}^{n} |L_k^*(x)|^2} = \min_{p \in M_n(\Lambda), p(x)=1} \int_0^1 |p(t)|^2 \, dt, \tag{5.3}$$

and either side can be called the n–th Christoffel function. Recalling the reproducing kernel (2.20), we see that $1/K(x,x)$, and $1/K_n(x,x)$ are what we have just defined (cf. (3.3) and (3.4)). For convenience, we will defy the section title a little by stating results in terms of the reciprocal of the Christoffel functions, namely, in terms of $K(x) = K(x,x)$ and $K_n(x) = K_n(x,x)$.

The classical Müntz theorem characterizes the denseness of $M(\Lambda)$ by the divergence of the series $\sum_{k=1}^{\infty} \lambda_k^{-1}$. Now we can connect the Christoffel functions with the denseness. All results here are stated for integer sequences $\{\lambda_k\}$, but they hold for positive sequences.

Theorem 5.1. *Let $\Lambda = \{0 = \lambda_0 < \lambda_1 < \dots\}$ be an integer sequence. Then the following are equivalent:*

(1) *$M(\Lambda)$ is not dense in $C[0,1]$ in the uniform norm;*
(2) *$\sum_{k=1}^{\infty} \lambda_k^{-1} < +\infty$;*
(3) *There is an $x \in [0,1)$, such that $\sum_{k=0}^{\infty} |L_k^*(x)|^2 < +\infty$;*
(4) *$\sum_{k=0}^{\infty} |L_k^*(x)|^2$ converges uniformly on $[0, 1-\epsilon]$ for every $0 < \epsilon < 1$.*

The right endpoint 1 is quite different, where we always have (cf. (2.9) and (2.14)) $K(1) = \sum_{k=0}^{\infty} |L_k^*(1)|^2 = \sum_{k=0}^{\infty}(1 + 2\lambda_k) = +\infty$. The following lemma is extracted from the proof of [7, Theorem 3], see also [Bor, Lemma 2]. It estimates the function values and derivative values of Λ–polynomials on $[0, 1-\epsilon]$ by their $L^2[0,1]$ norms. The proof of Theorem 5.1 will follow this.

Lemma 5.2. *Let $\Lambda = \{0 = \lambda_0 < \lambda_1 < \dots\}$ be an integer sequence with $\sum_{k=1}^{\infty} \lambda_k^{-1} < +\infty$. Then*

$$\max_{x \in [0, 1-\epsilon]} |p^{(\nu)}(x)| \le C \left(\int_0^1 |p(x)|^2 \right)^{1/2}, \tag{5.4}$$

for every $p \in M(\Lambda)$, for every $\nu = 0, 1, 2, \dots$, and $0 < \epsilon < 1$. And $C = C(\Lambda, \epsilon, \nu)$ is a constant, depending only on Λ and ϵ.

Proof. Since $\{\lambda_k\}$ is an integer sequence, and $\sum_{k=1}^{\infty} \lambda_k^{-1} < +\infty$, from the proof of [7, Theorem 3], for every $\epsilon > 0$, there is constant $C_0 = C_0(\Lambda, \epsilon) > 0$ depending only on Λ and ϵ such that

$$|a_k| \le C_0(1+\epsilon)^{\lambda_k} \left(\int_0^1 |p(x)|^2 \, dx \right)^{1/2}, \qquad k = 0, 1, 2, \dots$$

hold for every Λ–polynomial $p(x) = \sum_{k=0}^{n} a_k x^{\lambda_k}$, and for every $n = 0, 1, 2, \dots$. (We remark that the above also holds if $\{\lambda_k\}$ is not integer but has fixed gaps.) Note in particular that C_0 is independent of n and p. Hence

$$|p^{(\nu)}(x)| \le \sum_{k=0}^{n} |a_k| \lambda_k^{\nu} x^{\lambda_k - \nu} \le C_0 \sum_{k=0}^{n} (1+\epsilon)^{\lambda_k} \left(\int_0^1 |p(x)|^2 \, dx \right)^{1/2} \lambda_k^{\nu} \, x^{\lambda_k - \nu}.$$

If $x \in [0, 1-\epsilon]$, then $(1+\epsilon)x \le 1 - \epsilon^2$, and the above implies that

$$|p^{(\nu)}(x)| \le C_0 (1+\epsilon)^{\nu} \sum_{k=0}^{\infty} (1-\epsilon^2)^k k^{\nu} \left(\int_0^1 |p(t)|^2 \, dt \right)^{1/2}.$$

Therefore, (5.4) holds with $C(\Lambda, \epsilon, \nu) = C_0(1+\epsilon)^{\nu} \sum_{k=0}^{\infty} (1-\epsilon^2)^k k^{\nu}$. □

An easy consequence of the above is a bounded Nikolskii–type inequality:

Corollary 5.3. *Under the condition of Lemma 5.2,*

$$\max_{x \in [0, 1-\epsilon]} |p(x)| \le C \int_0^1 |p(x)| dx, \qquad p \in M(\Lambda),$$

where $C = C(\Lambda, \epsilon)$ is depends only on Λ and ϵ.

Proof. Consider the new sequence $\Lambda^* = \{1, \lambda_0 + 1, \lambda_1 + 1, \dots\}$ and the Müntz space $M(\Lambda^*) = span\{1, x^{\lambda_0+1}, x^{\lambda_1+1}, \dots\}$. Apply Lemma 5.2 with $\nu = 1$ for the Λ^*–polynomials $\int_0^x p(t)dt$ with $p \in M(\Lambda)$, and use the simple fact that $|\int_0^x p(t)dt| \le \int_0^1 |p(t)| \, dt$. □

Proof of Theorem 5.1. The equivalence of (i) and (ii) is the classical Müntz–Szász Theorem. We will follow (ii) $\implies$ (iv) $\implies$ (iii) $\implies$ (i).

(ii) $\implies$ (iv). Since $\sum_{k=0}^{\infty} \lambda_k^{-1} < +\infty$, we have by Corollary 3.2 that

$$\sum_{k=0}^{\infty} |L_k^{*(\nu)}(x)|^2 = \sup\{ |p^{(\nu)}(x)|^2 : p \in M(\Lambda), \int_0^1 |p(x)|^2 dx = 1 \} \tag{5.5}$$

for every $x \in [0,1]$. Hence by Lemma 5.2, for every $\epsilon > 0$ there is a constant $C = C(\Lambda, \epsilon)$ such that

$$\sum_{k=0}^{\infty} |L^*(x)|^2 \leq C, \qquad \sum_{k=0}^{\infty} |L^{*'}(x)|^2 \leq C. \tag{5.6}$$

Since

$$\frac{d}{dx} \sum_{k=0}^{n} (L^*(x))^2 = \sum_{k=0}^{n} L_k^*(x) L_k^{*'}(x), \tag{5.7}$$

on applying the Cauchy–Schwartz inequality and (5.6), we see that (5.7) is uniformly bounded by C for $x \in [0, 1-\epsilon]$, and $n \geq 0$. Therefore $\sum_{k=0}^{n} |L^*(x)|^2$ is equicontinuous for $n = 0, 1, \dots$, which implies the uniform convergence of K_n to K on $[0, 1-\epsilon]$ by the Arzela–Ascoli Theorem.

(iv) $\Longrightarrow$ (iii) is trivial; We now finish the proof by showing (iii) $\Longrightarrow$ (i). Assume that $K(x_0) < +\infty$ for some $x_0 \in [0,1]$. Then $M(\Lambda)$ fails to be dense in $C[0,1]$. Otherwise, let $f \in C[0,1]$ be such that $|f(x_0)|^2 \geq K(x_0) + 2$ and $\int_0^1 |f(x)|^2 = 1$. Then by the density assumption, there is a $p \in M(\Lambda)$, such that $|p(x_0)|^2 \geq K(x_0) + 1$ and $\int_0^1 |p(x)|^2 dx = 1$, which means that $\sup\{\, |p(x_0)| : p \in M(\Lambda), \int_0^1 |p(x)|^2 dx = 1\} \geq K(x_0) + 1$, which contradicts (5.3). □

Actually when $M(\Lambda)$ is not dense, the uniform convergence also holds for higher derivatives, and in this case, we do not require $\lambda_0 = 0$.

Theorem 5.4. *Let $\Lambda = \{0 \leq \lambda_0 < \lambda_1 < \dots\}$ be a sequence of integers with $\sum_{k=1}^{\infty} \lambda_k^{-1} < +\infty$. Then*

$$\sum_{k=0}^{\infty} \left| L_k^{*(\nu)}(x) \right|^2 \quad \textit{converges uniformly on } [0, 1-\epsilon] \tag{5.8}$$

for every $\nu = 0, 1, 2, \dots$ and every $0 < \epsilon < 1$.

Proof. The method is exactly the same as in the proof of (ii) $\Longrightarrow$ (iv) of Theorem 5.1. Lemma 5.2 implies the uniform boundedness of the series in (5.8) and that of $\sum_{k=0}^{\infty} |L_k^{*(\nu+1)}|^2$ on $[0, 1-\epsilon]$, and the uniform boundedness of $\frac{d}{dx} \sum_{k=0}^{n} |L_k^{*(\nu)}(x)|^2$ on $[0, 1-\epsilon]$ follows by Cauchy-Schwarz inequality. Now the Arzela–Ascoli Theorem completes the proof. □

We obtain immediately from Theorem 5.4 that under the conditions of Theorem 5.4, the orthonormal Müntz–Legendre polynomials tend to 0 uniformly on closed subintervals of $[0,1)$. Whereas for orthogonal polynomials p_n, $n = 0, 1, 2, \dots$, orthonormal with respect to a measure supported on $[0,1]$, only the relative growth $|p_n|^2 / \sum_{k=0}^{n} |p_k|^2$ tends to 0 uniformly on $[0,1]$ (cf. [15, 16, 27]).

Corollary 5.5. *If $0 = \lambda_0 < \lambda_1 < \cdots \to \infty$, and the associated Müntz system is not dense in $C[0,1]$. Then*

$$\lim_{k\to\infty} \max_{x\in[0,1-\epsilon]} |L_k^{*(\nu)}(x)| = 0$$

holds for every $0 < \epsilon < 1$ and every $\nu = 0, 1, 2, \dots$.

When the index sequence is lacunary, that is,

$$\inf\{\lambda_{k+1}/\lambda_k \colon k = 0, 1, 2, \dots\} > 1, \tag{5.9}$$

we can say more about the boundness of the function K. To do this, we first give a bounded Bernstein–type and a bounded Nikolskii–type inequality for a lacunary system (cf. [4, Theorem 3.1]).

Lemma 5.6. *Let $\Lambda = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ be lacunary as in (5.9). Then*

$$|p'(x)| \le \frac{C}{1-x} \max_{t\in[0,1]} |p(t)| \qquad x \in [0,1), p \in M(\Lambda), \tag{5.10}$$

and

$$|p(x)| \le \frac{C}{1-x} \int_0^1 |p(t)|dt \qquad x \in [0,1), p \in M(\Lambda) \tag{5.11}$$

hold with the constant $C = C(\Lambda)$ depending only on the system.

Proof. The inequality (5.10) comes from [4, Theorem 3.1]. For (5.11), consider the new lacunary sequence $\Lambda^* = \{0, 1+\lambda_0, 1+\lambda_1, \dots\}$, and apply (5.10) for Λ^*–polynomials which are indefinite integrals of $p \in M(\Lambda)$. □

Theorem 5.7. *Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ be lacunary. Then there is a constant $C = C(\Lambda)$, such that*

$$K(x) = \sum_{k=0}^{\infty} |L_k^*(x)|^2 \le \frac{C}{(1-x)^2}, \qquad x \in [0,1).$$

Proof. Since Λ is lacunary, applying Lemma 5.6, we get

$$|p(x)|^2 \le \frac{C}{(1-x)^2} \left(\int_0^1 |p(t)|dt \right)^2 \le \frac{C}{(1-x)^2} \int_0^1 |p(t)|^2 dt.$$

By (5.2)–(5.3) or (3.2)–(3.4), we have $K(x) \le C/(1-x)^2$. □

As a last observation in this paper, we point out that if $\lambda_n \to \infty$, then there is a sequence $x_n \to 1^-$, such that $K(x_n) \ge C_1/(1-x_n)$. Indeed, let $x_n = 1 - 1/\lambda_n$, and consider $p(x) = x^{\lambda_n}$. Then by (5.2)–(5.3) or Corollary 3.2,

$$\begin{aligned} K(x_n) &\ge p(x_n)^2/\|p\|_2^2 = x_n^{2\lambda_n}(2\lambda_n + 1) \\ &= (1 - 1/\lambda_n)^{2\lambda_n}(2\lambda_n + 1) \ge C_1\lambda_n \ge C_1/(1-x_n). \end{aligned}$$

where $C_1 = \inf\{(1-1/\lambda_n)^{2\lambda_n} : n = 1, 2, 3, \dots\} > 0$.

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