

**Markov-Bernstein Type Inequalities for Classes  
of Polynomials with Restricted Zeros**

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**Abstract.** We prove that there exists an absolute constant  $c > 0$  such that

$$|p'(y)| \leq c \min \left\{ n(k+1), \left( \frac{n(k+1)}{1-y^2} \right)^{1/2} \right\} \max_{-1 \leq x \leq 1} |p(x)|, \quad -1 \leq y \leq 1$$

for every real algebraic polynomial of degree at most  $n$  having at most  $k$  zeros in the open unit disk  $\{z \in \mathbb{C} : |z| < 1\}$ . This inequality, which has been conjectured for at least a decade, improves and generalizes several earlier results. Up to the multiplicative absolute constant  $c$ , it is a sharp generalization of both Markov's and Bernstein's inequalities.

## 1. INTRODUCTION, NOTATIONS

Bernstein's inequality [LO1, pp. 39-41] asserts that

$$(1.1) \quad \max_{-\pi \leq t \leq \pi} |p'(t)| \leq n \max_{-\pi \leq t \leq \pi} |p(t)|$$

for every  $p \in \mathcal{T}_n$ , where  $\mathcal{T}_n$  denotes the set of all trigonometric polynomials of degree at most  $n$  with real coefficients. The corresponding algebraic result [LO1, pp. 39-41], known as Markov's inequality, states that

$$(1.2) \quad \max_{-1 \leq x \leq 1} |p'(x)| \leq n^2 \max_{-1 \leq x \leq 1} |p(x)|$$

for all  $p \in \mathcal{P}_n$ , where  $\mathcal{P}_n$  denotes the set of all algebraic polynomials of degree at most  $n$  with real coefficients. The Chebyshev polynomials  $Q_n \in \mathcal{T}_n$  and  $T_n \in \mathcal{P}_n$  defined by

$$(1.3) \quad Q_n(t) := \cos(nt + \alpha), \quad \alpha \in \mathbb{R}$$

and

$$(1.4) \quad T_n(x) := \cos(n \arccos x), \quad -1 \leq x \leq 1$$

show that inequalities (1.1) and (1.2) are sharp. The substitution  $x = \cos t$  in (1.1), together with (1.2), yields

$$(1.5) \quad |p'(y)| \leq \min \left\{ n^2, \frac{n}{\sqrt{1-y^2}} \right\} \max_{-1 \leq x \leq 1} |p(x)|, \quad -1 \leq y \leq 1$$

for every  $p \in \mathcal{P}_n$ . Markov-Bernstein type inequalities in weighted spaces and in  $L_p$  norms play a key role in proving inverse theorems of approximation and of course have their own intrinsic interest.

Denote by  $\mathcal{P}(n, k)$  the set of all  $p \in \mathcal{P}_n$  having at most  $k$  zeros (by counting multiplicities) in the open unit disk  $\{z \in \mathbb{C} : |z| < 1\}$ . Markov-Bernstein type inequalities for constrained polynomials have been the subject of many research papers and the classes  $\mathcal{P}(n, k)$ ,  $0 \leq k \leq n$ , have been of special interest. One might correctly suspect that the restrictions on the zeros of a polynomial imply an improvement in inequality (1.5). In 1940 Erdős [ERDÖS] proved that there is an absolute constant  $c > 0$  such that

$$(1.6) \quad |p'(y)| \leq \min \left\{ \frac{en}{2}, \frac{c\sqrt{n}}{(1-y^2)^2} \right\} \max_{-1 \leq x \leq 1} |p(x)|, \quad -1 \leq y \leq 1$$

for every  $p \in \mathcal{P}(n, 0)$  having only real zeros. By taking the polynomials  $p_n \in \mathcal{P}(n, 0)$  defined by  $p_n(x) := (1+x)^{n-1}(1-x)$ , it is easy to see that the constant  $e/2$  in (1.6) is asymptotically sharp. In 1963 G. G. Lorentz [LO2] showed that there is an absolute constant  $c > 0$  such that

$$(1.7) \quad |p'(y)| \leq c \min \left\{ n, \frac{\sqrt{n}}{\sqrt{1-y^2}} \right\} \max_{-1 \leq x \leq 1} |p(x)|, \quad -1 \leq y \leq 1$$

for every  $p \in \mathcal{P}_n$  of the form

$$(1.8) \quad p(x) = \sum_{j=0}^n a_j (1+x)^j (1-x)^{n-j} \text{ with all } a_j \geq 0 \text{ or all } a_j \leq 0.$$

By an observation of G. G. Lorentz [SCH], every  $p \in \mathcal{P}(n, 0)$  is of the form (1.8), therefore (1.7) holds for every  $p \in \mathcal{P}(n, 0)$ . Inequality (1.7) is sharp up to the multiplicative absolute constant  $c > 0$ ; namely it was shown in [ER1] that there is an absolute constant  $c > 0$  such that

$$(1.9) \quad \sup_{p \in \mathcal{P}(n, 0)} \frac{|p'(y)|}{\max_{-1 \leq x \leq 1} |p(x)|} \geq c \min \left\{ n, \frac{\sqrt{n}}{\sqrt{1-y^2}} \right\}$$

for every  $n \in \mathbb{N}$  and  $y \in [-1, 1]$ . In 1972 Scheick [SCH] found the best possible constant in Lorentz's Markov-type inequality. Extending Erdős's Markov-type inequality, he proved that

$$(1.10) \quad \max_{-1 \leq x \leq 1} |p'(x)| \leq \frac{en}{2} \max_{-1 \leq x \leq 1} |p(x)|$$

for every  $p \in \mathcal{P}_n$  of the form (1.8), hence for every  $p \in \mathcal{P}(n, 0)$ . In 1980 Szabados and Varma showed that there is a constant  $c(k) > 0$  depending only on  $k$  such that

$$(1.11) \quad \max_{-1 \leq x \leq 1} |p'(x)| \leq c(k)n \max_{-1 \leq x \leq 1} |p(x)|$$

for every  $p \in \mathcal{P}(n, k)$ ,  $0 \leq k \leq n$ , having only real zeros. Subsequently Máté proved that

$$(1.12) \quad \max_{-1 \leq x \leq 1} |p'(x)| \leq 6n \exp(\pi\sqrt{k}) \max_{-1 \leq x \leq 1} |p(x)|$$

for every  $p \in \mathcal{P}(n, k)$ ,  $1 \leq k \leq n$ , having  $n - k$  zeros in  $\mathbb{R} \setminus (-1, 1)$ . Szabados' conjecture, proved by P. Borwein, [BO] establishes the Markov-type inequality

$$(1.13) \quad \max_{-1 \leq x \leq 1} |p'(x)| \leq 9n(k+1) \max_{-1 \leq x \leq 1} |p(x)|$$

for every  $p \in \mathcal{P}(n, k)$ ,  $0 \leq k \leq n$ , having  $n - k$  zeros in  $\mathbb{R} \setminus (-1, 1)$ . Inequality (1.13) was extended in [ER2] to all  $p \in \mathcal{P}(n, k)$ ,  $0 \leq k \leq n$ . Another proof of (1.13) for all  $p \in \mathcal{P}(n, k)$ ,  $0 \leq k \leq n$ , is obtained in [ER3] with the constant 11 instead of 9. The fact that (1.13) is sharp up to the multiplicative constant was shown by Szabados [SZA, Example 1]. While (1.13) is essentially sharp, it is only a good estimate for  $|p'(y)|$  with  $|y|$  close to 1.

It was proved in [ER-SZA] that there is an absolute constant  $c > 0$  such that

$$(1.14) \quad |p'(y)| \leq \frac{c\sqrt{n}(k+1)^2}{\sqrt{1-y^2}} \max_{-1 \leq x \leq 1} |p(x)|, \quad -1 < y < 1$$

for every  $p \in \mathcal{P}(n, k)$ ,  $0 \leq k \leq n$ . Subsequently it was shown in [ER3] that there is an absolute constant  $c > 0$  such that

$$(1.15) \quad |p'(y)| \leq \frac{c\sqrt{n}(k+1)}{1-y^2} \max_{-1 \leq x \leq 1} |p(x)|, \quad -1 < y < 1$$

for every  $p \in \mathcal{P}(n, k)$ ,  $0 \leq k \leq n$ . When  $y = 0$ , inequality (1.15) is sharp up to the multiplicative constant  $c > 0$ ; namely it was verified in [ER2] that there is an absolute constant  $c > 0$  such that

$$(1.16) \quad \sup_{p \in \mathcal{P}(n, k)} \frac{|p'(0)|}{\max_{-1 \leq x \leq 1} |p(x)|} \geq c\sqrt{n(k+1)}$$

for every  $0 \leq k \leq n$ . The unpleasant thing about the Bernstein-type inequalities (1.14) and (1.15) is the fact that none of them matches inequality (1.5) in the unrestricted case  $k = n$  (note that  $\mathcal{P}(n, n) = \mathcal{P}_n$ ). To formulate a Markov-Bersnstein type inequality for  $\mathcal{P}(n, k)$ ,  $0 \leq k \leq n$ , which contains all of the earlier results as special cases, the following result was conjectured in [ER-SZA].

**Theorem.** *There is an absolute constant  $c > 0$  such that*

$$|p'(y)| \leq c \min \left\{ n(k+1), \left( \frac{n(k+1)}{1-y^2} \right)^{1/2} \right\} \max_{-1 \leq x \leq 1} |p(x)|, \quad -1 \leq y \leq 1$$

for every  $p \in \mathcal{P}(n, k)$ ,  $0 \leq k \leq n$ .

The purpose of this paper is to prove the above theorem, which seems to be the “right” Markov-Bernstein type inequality for the classes  $\mathcal{P}(n, k)$ ,  $0 \leq k \leq n$ . The proof relies on a series of lemmas, some of which are interesting for their own merit.

## 2. PROOF OF THE THEOREM

First we prove the following Bernstein-type inequality for trigonometric polynomials  $p \in \mathcal{T}_n$  having  $2n - 2k$  zeros at 0.

**Theorem 1.** *Let  $0 \leq k \leq n$ ,  $n \geq 1$  be integers. There is an absolute constant  $c_1 > 0$  such that*

$$(2.1) \quad \max_{t \in \mathbb{R}} |p'(t)| \leq c_1 \sqrt{n(k+1)} \max_{t \in \mathbb{R}} |p(t)|$$

for every  $p \in \mathcal{T}_n$  of the form

$$(2.2) \quad p(t) = (\sin(t/2))^{2n-2k} q(t), \quad q \in \mathcal{T}_k.$$

Theorem 1 implies immediately the following result for higher derivatives by induction on  $s$ .

**Corollary 2.** *Let  $0 \leq k \leq n$ ,  $n \geq 1$  and  $s \geq 1$  be integers. Then*

$$(2.3) \quad \max_{t \in \mathbb{R}} |p^{(s)}(t)| \leq \prod_{j=1}^s (c_1 \sqrt{n(k+j)}) \max_{t \in \mathbb{R}} |p(t)|$$

for every  $p \in \mathcal{T}_n$  of the form (2.2).

The proof of Theorem 1 rests on the following.

**Theorem 3.** *Let  $0 \leq k \leq n$  and  $n \geq 1$  be integers. There is an absolute constant  $c_2 > 0$  such that*

$$(2.4) \quad |p(z)| \leq c_2 \max_{t \in \mathbb{R}} |p(t)|$$

for every  $p \in \mathcal{T}_n$  of the form (2.2), and for every  $z \in \mathbb{C}$  such that  $|\operatorname{Im}(z)| \leq (n(k+1))^{-1/2}$ .

To prove Theorem 3 we need four lemmas.

**Lemma 4.** *Let  $a < b$  and  $r > 0$  be arbitrary real numbers. Assume that  $P \in \mathcal{P}_{2n}$  and*

$$(2.5) \quad |P(b)| = \max_{a \leq x \leq b} |P(x)|.$$

*Then  $P$  has at most  $c_3 nr^{1/2}(b-a)^{-1/2}$  zeros (by counting multiplicities) in the interval  $[b-r, b]$  with some  $0 < c_3 \leq 4$ .*

Lemma 4 is proved in [ER4, Corollary 1] with a slightly larger constant. The fact that  $c_3 \leq 4$  was pointed out by Lorentz and von Golitschek. Our next lemma is a special case of [BO-ER, Theorem 3.2].

**Lemma 5.** *Let  $0 \leq k \leq n$  and  $n \geq 1$  be integers. For every  $c_4 > 0$  there is a constant  $c_5 = c_5(c_4)$  depending only on  $c_4$  such that*

$$(2.6) \quad \max_{a \leq x \leq b + c_4(b-a)(n(k+1))^{-1}} |P(x)| \leq c_5 \max_{a \leq x \leq b} |P(x)|$$

*holds for every  $P \in \mathcal{P}_{2n}$  of the form*

$$(2.7) \quad P(x) = (x-a)^{2n-2k}Q(x), \quad Q \in \mathcal{P}_{2k}.$$

From Lemmas 4 and 5 we deduce

**Lemma 6.** *Let  $0 \leq k \leq n$ ,  $n \geq 1$  be integers. There is an absolute constant  $c_6 > 0$  such that*

$$(2.8) \quad \max_{-1 \leq x \leq 1 + (n(k+1))^{-1}} |P(x)| \leq c_6 \max_{-1 \leq x \leq 1} |P(x)|$$

*for every polynomial of the form*

$$(2.9) \quad P(x) = (x-a)^{2n-2k}Q(x), \quad Q \in \mathcal{P}_{2k}, \quad a \in [-1, 1].$$

*Proof.* If  $n/2 \leq k \leq n$ , then inequality (2.8) holds for every  $P \in \mathcal{P}_{2n}$  by Bernstein's inequality [LO1, pp. 42-43]. Therefore, in the sequel we assume that  $0 \leq k \leq n/2$ . Let  $b \geq 1$  be the smallest real number for which

$$(2.10) \quad |P(b)| = \max_{-1 \leq x \leq 1} |P(x)| = \max_{-1 \leq x \leq b} |P(x)|.$$

Since  $0 \leq k \leq n/2$ ,  $P$  has  $2n - 2k \geq n$  zeros at  $a$ . On the other hand, Lemma 4 implies that  $P$  has at most  $c_3 n(b-a)^{1/2}(b+1)^{-1/2}$  zeros (by counting multiplicities) in  $[a, b]$ , hence

$$c_3 n(b-a)^{1/2}(b+1)^{-1/2} \geq n,$$

thus

$$(2.11) \quad (b-a)^{-1} \leq \frac{c_3^2}{2} =: c_7.$$

Now Lemma 5 and inequality (2.11) yield that

$$(2.12) \quad \begin{aligned} \max_{-1 \leq x \leq 1 + (n(k+1))^{-1}} |P(x)| &\leq \max_{a \leq x \leq b + (n(k+1))^{-1}} |P(x)| \\ &\leq \max_{a \leq x \leq b + c_7(b-a)(n(k+1))^{-1}} |P(x)| \leq c_6 \max_{a \leq x \leq b} |P(x)| = c_6 \max_{-1 \leq x \leq 1} |P(x)|, \end{aligned}$$

where  $c_6 = c_5(c_7) > 0$  is an absolute constant, and the lemma is proved.  $\square$

From Lemma 6 we easily obtain.

**Lemma 7.** *Let  $0 \leq k \leq n$ ,  $n \geq 1$  be integers. There is an absolute constant  $c_8 > 0$  such that*

$$(2.13) \quad |p(i\delta)| \leq c_8 \max_{t \in \mathbb{R}} |p(t)|$$

( $i$  denotes the imaginary unit) for every  $p \in \mathcal{T}_n$  of the form

$$(2.14) \quad p(t) = (\sin((t - \alpha)/2))^{2n-2k} q(t), \quad q \in \mathcal{T}_k, \quad \alpha \in \mathbb{R}$$

and for every  $\delta \in \mathbb{R}$  such that

$$(2.15) \quad |\delta| \leq (n(k+1))^{-1/2}.$$

*Proof.* Let

$$(2.16) \quad \tilde{p}(t) := p(t)p(-t) = \frac{1}{4^{n-k}} (\cos t - \cos \alpha)^{2n-2k} q(t)q(-t).$$

Since  $q(t)q(-t) \in \mathcal{T}_{2k}$  is an even trigonometric polynomial, by the substitutions  $x = \cos t$  and  $a = \cos \alpha$ , we obtain that

$$(2.17) \quad \tilde{p}(t) = (x - a)^{2n-2k} Q(x), \quad Q \in \mathcal{P}_{2k}, \quad a \in [-1, 1].$$

For the sake of brevity let

$$(2.18) \quad P(x) := (x - a)^{2n-2k}Q(x).$$

Now (2.15) – (2.18) and Lemma 6 yield

$$(2.19) \quad \begin{aligned} |p(i\delta)|^2 &= |p(i\delta)p(-i\delta)| = |\tilde{p}(i\delta)| \\ &= |(\cosh \delta - a)^{2n-2k}Q(\cosh \delta)| \leq \max_{-1 \leq x \leq 1+\delta^2} |P(x)| \\ &\leq c_6 \max_{-1 \leq x \leq 1} |P(x)| = c_6 \max_{t \in \mathbb{R}} |\tilde{p}(t)| \\ &= c_6 \max_{t \in \mathbb{R}} |p(t)p(-t)| \leq c_6 \max_{t \in \mathbb{R}} |p(t)|^2, \end{aligned}$$

and the lemma is proved.  $\square$

Theorem 3 follows from Lemma 7 immediately, while Theorem 1 can be obtained from Theorem 3 by the Cauchy integral formula. Corollary 2, which follows from Theorem 1 by induction on  $s$ , plays a key role in the proof of our next lemma.

**Lemma 8.** *Let  $0 \leq k \leq n$ ,  $n \geq 1$  be integers, let  $p \in \mathcal{T}_n$  be of the form*

$$(2.20) \quad p(t) = (\sin(t/2))^{2n-2k}q(t), \quad q \in \mathcal{T}_k,$$

let  $t_0 \in \mathbb{R}$  be such that

$$(2.21) \quad p(t_0) = \max_{t \in \mathbb{R}} |p(t)| = 1.$$

Then  $p$  has at most  $2ec_1\sqrt{n(k+1)h}$  zeros (by counting multiplicities) in the interval

$[t_0 - h, t_0 + h]$  for any  $0 < h < (2ec_1)^{-1}(k+1)^{1/2}n^{-1/2}$ , where  $c_1 > 0$  is the same as in Theorem 1 (and Corollary 2).

*Proof.* For the sake of brevity let

$$(2.22) \quad s := [2ec_1\sqrt{n(k+1)h}] + 1,$$

where  $c_1 > 0$  is the same as in Theorem 1. Assume that we can find  $t_1 < t_2 < \dots < t_\nu$  in  $[t_0 - h, t_0 + h]$  such that  $p$  has  $\mu_i$  repeated zeros at  $t_i$ ,  $1 \leq i \leq \nu$ , and  $\sum_{i=1}^{\nu} \mu_i = s$ . Now let

$$(2.23) \quad \Omega(x) := \prod_{i=1}^{\nu} (x - t_i)^{\mu_i}.$$



Then, by a well-known relation for the remainder of the Hermite interpolation polynomial, there exists a  $\xi \in [t_1, t_\nu] \subset [t_0 - h, t_0 + h]$  such that

$$(2.24) \quad p(t_0) - H(t_0) = \frac{1}{s!} p^{(s)}(\xi) \Omega(t_0),$$

where  $H \in \mathcal{P}_{s-1}$  and  $H^{(j)}(t_i) = p^{(j)}(t_i) = 0$  for every  $1 \leq i \leq \nu$  and  $0 \leq j \leq \mu_i - 1$ , so  $H \equiv 0$ . Hence, by  $s! > (s/e)^s$  and from (2.21) – (2.24), we can deduce that

$$(2.25) \quad \begin{aligned} |p^{(s)}(\xi)| &\geq s! h^{-s} > \left(\frac{s}{e}\right)^s \left(\frac{2ec_1 \sqrt{n(k+1)}}{s}\right)^s \\ &\geq (2c_1)^s (n(k+1))^{s/2} \geq \prod_{j=1}^s (c_1 \sqrt{n(k+j)}) \\ &= \prod_{j=1}^s (c_1 \sqrt{n(k+j)}) \max_{t \in \mathbb{R}} |p(t)| \end{aligned}$$

for any  $0 < h < (2ec_1)^{-1} (k+1)^{1/2} n^{-1/2}$  (these bounds for  $h$  imply  $1 \leq s \leq k+1$  by (2.22), which is used in the last inequality). Since (2.25) contradicts Corollary 2, our assumption is false, and the lemma is proved.  $\square$

By the substitution  $x + 1 = 2 \sin^2(t/2)$  (i.e.  $x = -\cos t$ ), from Lemma 8 we obtain

**Corollary 9.** *Let  $0 \leq k \leq n$  and  $n \geq 1$  be integers, let  $p \in \mathcal{P}_n$  be of the form*

$$(2.26) \quad p(x) = (x+1)^{n-k} q(x), \quad q \in \mathcal{P}_k,$$

and let

$$(2.27) \quad |p(1)| = \max_{-1 \leq x \leq 1} |p(x)|.$$

Then  $p$  has at most  $4ec_1 \sqrt{n(k+1)h}$  zeros in  $[1-h, 1]$  for any

$$0 < h < (4ec_1)^{-2} (k+1)n^{-1}$$

.

Our next lemma constructs a polynomial with some nice properties, which will be exploited in the proof of Lemma 11.

**Lemma 10.** *Let  $1 \leq k \leq m \leq n$ ,  $m + k \leq n$  be integers, let  $0 < c_9 \leq 1$  be real and let*

$$(2.28) \quad y := 1 - \frac{c_9 k}{4n} \quad \text{and} \quad 0 < z \leq 1 - \frac{c_9 m}{n}.$$

*Let  $Q_{2k}$  be the Chebyshev polynomial  $T_{2k} = \cos(2k \arccos x)$  transformed linearly from  $[-1, 1]$  to  $[y, 1]$ . Denote the zeros of  $Q_{2k}$  by*

$$(1 >) x_1 > x_2 > \cdots > x_k > \tilde{x}_k > \tilde{x}_{k-1} > \cdots > \tilde{x}_1 (> y)$$

*and define*

$$(2.29) \quad P(x) := (x + 1)^{n-m-k} (x - z)^m \prod_{j=1}^k (x - x_j).$$

*Let  $x_0 := 1$  and  $x_{k+1} := y$ . Let  $\xi_j \in (x_j, x_{j-1}]$ ,  $j = 1, 2, \dots, k + 1$ , be the (only) point for which*

$$(2.30) \quad |P(\xi_j)| = \max_{x_j \leq x \leq x_{j-1}} |P(x)|.$$

*Then*

$$(2.31) \quad |P(\xi_j)| < |P(\xi_{j+1})|, \quad j = 1, 2, \dots, k.$$

*Proof.* We have

$$(2.32) \quad \begin{aligned} P(x) &= (x + 1)^{n-m-k} (x - z)^m \prod_{j=1}^k (x - x_j) \\ &= (x + 1)^{n-m-k} (x - z)^m \prod_{j=1}^k (x - \tilde{x}_j)^{-1} Q_{2k}(x) \\ &= (x + 1)^{n-m-k} (x - z)^m (x - y)^{-k} \prod_{j=1}^k \frac{x - y}{x - \tilde{x}_j} Q_{2k}(x). \end{aligned}$$

For the sake of brevity let

$$(2.33) \quad F(x) := \prod_{j=1}^k \frac{x - y}{x - \tilde{x}_j}$$

and

$$(2.34) \quad G(x) := (x+1)^{n-m-k}(x-z)^m(x-y)^{-k}.$$

Then  $P = FGQ_{2k}$ . Since  $Q_{2k}$  equioscillates  $k+1$  times on  $[(1+y)/2, 1]$ , it is sufficient to prove that both  $F$  and  $G$  are decreasing on  $[(1+y)/2, 1]$ . Since each of its factors is decreasing on  $[(1+y)/2, \infty)$ , so is  $F$ . To show that  $G$  is decreasing on  $[(1+y)/2, 1]$  we write

$$(2.35) \quad G = G_1 G_2,$$

where

$$(2.36) \quad G_1(x) := (x+1)^{n-m-k}(x-y)^{-k/2}$$

and

$$(2.37) \quad G_2(x) := (x-z)^m(x-y)^{-k/2}.$$

We have

$$(2.38) \quad G_1'(x) = \frac{(x+1)^{n-m-k-1}(x-y)^{k/2-1}((n-m-k)(x-y) - \frac{k}{2}(x+1))}{(x-y)^k} \\ < (x+1)^{n-m-k-1}(x-y)^{-k/2-1} \left( n \frac{c_9 k}{2n} - \frac{k}{2} \right) \leq 0$$

and

$$(2.39) \quad G_2'(x) = \frac{(x-z)^{m-1}(x-y)^{k/2-1}(m(x-y) - \frac{k}{2}(x-z))}{(x-y)^k} \\ < (x-z)^{m-1}(x-y)^{-k/2-1} \left( m \frac{c_9 k}{4n} - \frac{k}{2} \frac{c_9 m}{2n} \right) = 0$$

for every  $x \in [(1+y)/2, 1]$ , and the lemma is proved.  $\square$

**Lemma 11.** *Let  $k, m, n, c_9, y, z, x_j$  ( $j = 1, 2, \dots, k$ ) and  $P$  be the same as in Lemma 10. Let*

$$(2.40) \quad P^*(x) = (x+1)^{n-m-k}(x-z)^m Q^*(x), \quad Q^* \in \mathcal{P}_k$$

be the constrained Chebyshev polynomials of degree  $n$  on  $[y, 1]$ , which equioscillates  $k + 1$  times on  $[y, 1]$ <sup>1</sup>. Denote the zeros of  $P^*$  on  $(y, 1)$  by  $1 > y_1 > y_2 > \dots > y_k (> y)$ . Then  $y_j \leq x_j$  for every  $j = 1, 2, \dots, k$ .

*Proof.* This follows from Lemma 10, by proceeding exactly in the same way as in the proof of [BO, Lemma 4]. Suppose that the statement of the lemma is false. Choose the smallest  $j$  for which  $y_j > x_j$ . Then pick  $\eta$  so that

$$(2.41) \quad \max_{x_j \leq x \leq 1} |\eta P^*(x)| = \max_{x_j \leq x \leq 1} |P(x)|.$$

(We will specify the sign of  $\eta$  later.) We can deduce from the equioscillation of  $P^*$  that  $\eta P^* - P$  has at least  $j - 1$  zeros on  $[\beta, 1]$ , where  $\beta$  is the smallest number greater than  $y_j$ , where  $|\eta P^*|$  achieves its maximum on  $[y, 1]$ . From Lemma 10 we deduce that  $\eta P^* - P$  has at least  $k - j$  zeros on  $[y, \alpha]$ , where  $\alpha := \xi_{j+1}$  (see (2.30)). We need only to observe that if we choose the sign of  $\eta$  so that

$$(2.42) \quad \text{sign}(\eta P^*(\beta)) = -\text{sign}(P(\alpha)),$$

then  $\eta P^* - P$  must have at least two zeros in  $(\alpha, \beta)$ . Thus, together with the  $n - m - k$  zeros at  $-1$  and  $m$  zeros at  $z$ ,  $\eta P^* - P$  has at least

$$(n - m - k) + m + (j - 1) + (k - j) + 2 = n + 1$$

zeros, a contradiction. Therefore  $y_j \leq x_j$  for every  $j = 1, 2, \dots, k$ , and the lemma is proved.  $\square$

From Lemma 11 we prove the following

**Lemma 12.** *Let  $k, m, n, c_9, y$  and  $z$  be the same as in Lemma 10 and 11. There is a constant  $c_{10} = c_{10}(c_9) > 0$  depending only on  $c_9$  such that*

$$(2.43) \quad |p(\alpha)| \leq c_{10} \max_{y \leq x \leq 1} |p(x)|$$

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<sup>1</sup>It is well-known from the theory of Chebyshev approximation that there is a ‘‘constrained Chebyshev polynomial’’  $P^* \in \mathcal{P}_n$  of the form (2.40) such that  $P^*(\gamma_j) = (-1)^{k+1-j} \max_{y \leq x \leq 1} |P^*(x)|$  for every  $j = 1, 2, \dots, k + 1$  with some  $y \leq \gamma_1 < \gamma_2 < \dots < \gamma_{k+1} \leq 1$ .

for every  $p$  of the form

$$(2.44) \quad p(x) = (x+1)^{n-m-k}(x-z)^m q(x), \quad q \in \mathcal{P}_k,$$

and for every

$$(2.45) \quad \alpha \in [1, 1 + (nk)^{-1}].$$

*Proof.* By a compactness argument and a variational method it is routine to show that it is sufficient to prove (2.43) for  $p = P^*$ , where  $P^*$  is the constrained Chebyshev polynomial of degree  $n$  on  $[y, 1]$  defined in Lemma 11. Recalling the definition of  $y$  and  $z$ , and using Lemma 10, (2.45) and the well-known formula

$$T_{2k}(x) = \frac{1}{2}((x + \sqrt{x^2 - 1})^{2k} + (x - \sqrt{x^2 - 1})^{2k}), \quad x \in \mathbb{R} \setminus (-1, 1)$$

for the Chebyshev polynomial of degree  $2k$  on  $[-1, 1]$ , we obtain

$$(2.46) \quad \begin{aligned} \frac{|P^*(\alpha)|}{\max_{y \leq x \leq 1} |P^*(x)|} &= \frac{|P^*(\alpha)|}{|P^*(1)|} \\ &= \left(\frac{\alpha+1}{2}\right)^{n-m-k} \left(\frac{\alpha-z}{1-z}\right)^m \prod_{j=1}^k \frac{\alpha-y_j}{1-y_j} \\ &\leq \left(\frac{\alpha+1}{2}\right)^{n-m-k} \left(\frac{\alpha-z}{1-z}\right)^m \prod_{j=1}^k \frac{\alpha-x_j}{1-x_j} \\ &\leq \left(\frac{\alpha+1}{2}\right)^{n-m-k} \left(\frac{\alpha-z}{1-z}\right)^m \frac{Q_{2k}(\alpha)}{Q_{2k}(1)} \\ &\leq \left(1 + \frac{1}{2nk}\right)^{n-m-k} \left(1 + \frac{1}{c_9 mk}\right)^m \frac{T_{2k}(1 + 8c_9^{-1}k^{-2})}{T_{2k}(1)} \leq c_{10}, \end{aligned}$$

where  $Q_{2k}$  is the Chebyshev polynomial  $T_{2k} = \cos(2k \arccos x)$  transformed linearly from  $[-1, 1]$  to  $[y, 1]$ , and  $c_{10}$  depends only on  $c_9$ . The lemma is now proved.  $\square$

Lemma 12 and Corollary 9 allow us to prove the following.

**Lemma 13.** *Let  $1 \leq k \leq n$ ,  $0 \leq m \leq n - k$  be integers, let  $-1 \leq a \leq 1$ . There is an absolute constant  $c_{11} > 0$  such that*

$$(2.47) \quad |p(\alpha)| \leq c_{11} \max_{-1 \leq x \leq 1} |p(x)|$$

for every  $p \in \mathcal{P}_n$  of the form

$$(2.48) \quad p(x) = (x+1)^{n-m-k}(x-a)^m q(x), \quad q \in \mathcal{P}_k$$

and for every

$$(2.49) \quad \alpha \in [1, 1 + (nk)^{-1}].$$

*Proof.* Without loss of generality we may assume that

$$(2.50) \quad |p(1)| = \max_{-1 \leq x \leq 1} |p(x)|.$$

Any other case can be reduced to this by a linear transformation. We distinguish two cases.

Case 1:  $1 \leq k \leq m$ . We show that there is an absolute constant  $c_9 \in (0, 1]$  such that

$$(2.51) \quad a < 1 - \frac{c_9 m}{n}.$$

Indeed, let  $0 < c_9 < (4ec_1)^{-2}$ . By Corollary 9  $p$  has at most

$$4ec_1 \sqrt{n(k+m+1)c_9 mn^{-1}} < 8ec_1 \sqrt{c_9} m$$

zeros (by counting multiplicities) in  $[1 - c_9 mn^{-1}, 1]$ . Therefore (2.51) holds with  $c_9 = (8ec_1)^{-2}$ , since  $p$  has a zero at  $a$  with multiplicity  $m$ . Since (2.51) holds, Lemma 12 can be applied to yield the desired result.

Case 2:  $0 \leq m < k$ . Now  $p \in \mathcal{P}_n$  has all but  $m+k < 2k$  zeros at  $-1$ , hence [BO-ER, Theorem 3.2] (cf. Lemma 5) gives the conclusion of the lemma.  $\square$

Transforming the result of Lemma 13 linearly from  $[-1, 1]$  to  $[-|b|, 1]$  ( $-1 \leq b \leq 1$ ), we immediately obtain

**Corollary 14.** *Let  $1 \leq k \leq n, 0 \leq m \leq n - k$  be integers and let  $-1 \leq b \leq 1$ . There is an absolute constant  $c_{12} > 0$  such that*

$$(2.52) \quad |p(\alpha)| \leq c_{12} \max_{-|b| \leq x \leq 1} |p(x)| \leq c_{12} \max_{-1 \leq x \leq 1} |p(x)|$$

for every  $p \in \mathcal{P}_n$  of the form

$$(2.53) \quad p(x) = (x+b)^{n-m-k}(x-b)^m q(x), \quad q \in \mathcal{P}_k,$$

and for every

$$(2.54) \quad \alpha \in [1, 1 + (2nk)^{-1}].$$

Corollary 14 plays a key role in the proof of our next lemma.

**Lemma 15.** *Let  $1 \leq k \leq n$ ,  $0 \leq m \leq 2n - 2k$  be integers. We have*

$$(2.55) \quad |p(i\delta)| \leq \sqrt{c_{12}} \max_{t \in \mathbb{R}} |p(t)|$$

( $i$  denotes the imaginary unit) for every  $p \in \mathcal{T}_n$  of the form

$$(2.56) \quad p(t) = (\sin((t + \gamma)/2))^{2n-m-2k} (\sin((t + \gamma - \pi)/2))^m q(t),$$

with  $q \in \mathcal{T}_k$  and  $\gamma \in \mathbb{R}$ , provided

$$(2.57) \quad \delta \in (-(8kn)^{-1/2}, (8kn)^{-1/2}).$$

*Proof.* The proof is quite similar to that of Lemma 7. Let

$$(2.58) \quad \begin{aligned} \tilde{p}(t) &:= p(t)p(-t) \\ &= 4^{k-n} (\cos t - \cos \gamma)^{2n-m-2k} (\cos t + \cos \gamma)^m q(t)q(-t). \end{aligned}$$

Since  $q(t)q(-t) \in \mathcal{T}_{2k}$  is an even trigonometric polynomial, by the substitutions  $x = \cos t$  and  $b = \cos \gamma$ , we obtain that

$$(2.59) \quad \tilde{p}(t) = (x - b)^{2n-m-2k} (x + b)^m Q(x), \quad Q \in \mathcal{P}_{2k},$$

where  $Q \in \mathcal{P}_{2k}$ . Let

$$(2.60) \quad P(x) := (x - b)^{2n-m-2k} (x + b)^m Q(x).$$

Now (2.57) – (2.60) and Corollary 14 yield

$$(2.61) \quad \begin{aligned} |p(i\delta)|^2 &= |p(i\delta)p(-i\delta)| = |\tilde{p}(i\delta)| \\ &= |(\cosh \delta + b)^{2n-m-2k} (\cosh \delta - b)^m Q(\cosh \delta)| \\ &\leq \max_{-1 \leq x \leq 1 + \delta^2} |P(x)| \leq c_{12} \max_{t \in \mathbb{R}} |\tilde{p}(t)| \\ &= c_{12} \max_{t \in \mathbb{R}} |p(t)p(-t)| \leq c_{12} \max_{t \in \mathbb{R}} |p(t)|^2, \end{aligned}$$

and the lemma is proved.  $\square$

We will use a slightly different version of Lemma 15, namely the following.

**Corollary 16.** *Let  $1 \leq k \leq n$ ,  $0 \leq m \leq n$ ,  $m + 2k \leq 2n$  be integers. We have*

$$(2.62) \quad |p(z)| \leq \sqrt{c_{12}} \max_{t \in \mathbb{R}} |p(t)|$$

for every  $p \in \mathcal{T}_n$  of the form

$$(2.63) \quad p(t) = (\sin(t/2))^{2n-m-2k} (\sin((t-\pi)/2))^m q(t),$$

with  $q \in \mathcal{T}_k$ , provided

$$(2.64) \quad |\operatorname{Im}(z)| \leq (8kn)^{-1/2}.$$

*Proof.* Let  $\tilde{p}(t) := p(t - \gamma)$ ,  $\gamma \in \mathbb{R}$ , and apply Lemma 15.  $\square$

**Corollary 17.** *Let  $1 \leq k \leq n$ ,  $0 \leq m \leq 2n - 2k$  be integers. There is an absolute constant  $c_{13} > 0$  such that*

$$(2.68) \quad \max_{t \in \mathbb{R}} |p'(t)| \leq c_{13} \sqrt{nk} \max_{t \in \mathbb{R}} |p(t)|$$

for every  $p \in \mathcal{T}_n$  of the form

$$(2.69) \quad p(t) = \left( \sin \frac{t + \pi/2}{2} \right)^{2n-m-2k} \left( \sin \frac{t - \pi/2}{2} \right)^m q(t), \quad q \in \mathcal{T}_k.$$

By the substitution  $x = \sin t$ , from Corollary 17 we deduce

**Corollary 18.** *Let  $1 \leq k \leq n$ ,  $0 \leq m \leq n - k$  be integers. We have*

$$(2.70) \quad |p'(y)| \leq c_{13} \frac{\sqrt{nk}}{\sqrt{1-y^2}} \max_{-1 \leq x \leq 1} |p(x)|, \quad -1 < y < 1$$

for every  $p \in \mathcal{P}_n$  of the form

$$(2.71) \quad p(x) = (x+1)^{n-m-k} (x-1)^m q(x), \quad q \in \mathcal{P}_k,$$

where  $c_{13} > 0$  is the same as in Corollary 17.

Now we are ready to prove the Theorem stated in Section 1.



*Proof of the Theorem.* The case  $k = 0$  follows from the case  $k = 1$ , hence, without loss of generality, we may assume that  $1 \leq k$ . A simple compactness argument shows that for every  $-1 < y < 1$  there is a  $\tilde{p} \in \mathcal{P}(n, k)$  such that

$$(2.72) \quad \frac{|\tilde{p}'(y)|}{\max_{-1 \leq x \leq 1} |\tilde{p}(x)|} = \sup_{p \in \mathcal{P}(n, k)} \frac{|p'(y)|}{\max_{-1 \leq x \leq 1} |p(x)|}.$$

Observe that  $\tilde{p}$  cannot attain its maximum modulus on  $[-1, 1]$  at  $y$ , otherwise  $\tilde{p}'(y) = 0$ , a contradiction. We will show that  $\tilde{p}$  has at most  $k + 1$  zeros (by counting multiplicities) different from  $\pm 1$ . First we prove that  $\tilde{p}$  has only real zeros. Indeed, if  $\tilde{p}(a) = 0$  for an  $a \in \mathbb{C} \setminus \mathbb{R}$ , then  $\tilde{p}(\bar{a}) = 0$ , and a simple calculation shows that, with a sufficiently small  $\epsilon > 0$ ,

$$(2.73) \quad p_\epsilon(x) := \tilde{p}(x) \left( 1 - \epsilon \frac{(x - y)^2}{(x - a)(x - \bar{a})} \right)$$

is in  $\mathcal{P}(n, k)$  and contradicts the extremality of  $\tilde{p} \in \mathcal{P}(n, k)$ .

Next we show that  $\tilde{p}$  can not have more than one zero (by counting multiplicities) in  $\mathbb{R} \setminus [-1, 1]$ . Indeed, if there are  $a \neq b \in \mathbb{R} \setminus [-1, 1]$  such that  $\tilde{p}(a) = \tilde{p}(b) = 0$ , or there are  $a = b \in \mathbb{R} \setminus [-1, 1]$  such that  $\tilde{p}(a) = \tilde{p}'(a) = 0$ , then the polynomial

$$(2.74) \quad p_\epsilon(x) := \tilde{p}(x) \left( 1 - \epsilon \operatorname{sign}(ab) \frac{(x - y)^2}{(x - a)(x - b)} \right)$$

with sufficiently small  $\epsilon > 0$  is in  $\mathcal{P}(n, k)$ , and it contradicts the extremal property of  $\tilde{p}$ . Since  $\tilde{p} \in \mathcal{P}(n, k)$ ,  $\tilde{p}$  has at most  $k + 1$  zeros (by counting multiplicities) different from  $\pm 1$ , hence Corollary 18 and the extremal property of  $\tilde{p} \in \mathcal{P}(n, k)$  yield

$$(2.75) \quad |p'(y)| \leq c_{13} \frac{\sqrt{n(k+1)}}{\sqrt{1-y^2}} \max_{-1 \leq x \leq 1} |p(x)|$$

for every  $p \in \mathcal{P}(n, k)$ . The uniform bound

$$(2.76) \quad \max_{-1 \leq x \leq 1} |p'(x)| \leq 9n(k+1) \max_{-1 \leq x \leq 1} |p(x)|$$

for every  $p \in \mathcal{P}(n, k)$  is proved in [ER2, Corollary 1.3] extending [BO]. Other proofs of (2.76) are given in [ER3, Theorem 1] and [BO-ER, Theorem 3.4] with slightly larger multiplicative constants than 9. By (2.75) and (2.76) the theorem is proved.  $\square$

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