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On the Generating Function of the Integer Part: $[n\alpha+\gamma]$

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We show that

$$G(z,w):=\sum_{n=1}^{\infty} z^n w^{[n\alpha+\gamma]}=\frac{z}{1-z}+\frac{1-w}{w}\times\sum_{n=0}^{\infty}\frac{(-1)^{n+1}z^{t_n^{**}}w^{s_n^{**}}}{(1-z^{q_n}w^{p_n})(1-z^{q_{n+1}}w^{p_{n+1}})}.$$

Here p_n and q_n are the numerators and denominators of the convergents of the continued fraction expansion of α and t_n^{**} and s_n^{**} are particular algorithmically generated sequences of best approximates for the non-homogeneous diophantine approximation problem of minimizing $|n\alpha+\gamma-m|$. This generalizes results of Böhmer and Mahler, who considered the special case where $\gamma=0$. This representation allows us to easily derive various transcendence results. For example, $\sum_{n=1}^{\infty}[ne+\frac{1}{2}]/2^n$ is a Liouville number. Indeed the first series is Liouville for rational z, $w\in[-1,1]$ with $|zw|\neq 1$ provided α has unbounded continued fraction expansion. A second application, which generalizes a theorem originally due to Lord Raleigh, is to give a new proof of a theorem of Fraenkel, namely $[n\alpha+\gamma]_{n=1}^{\infty}$ and $[n\alpha'+\gamma']_{n=1}^{\infty}$ partition the non-negative integers if and only if $1/\alpha+1/\alpha'=1$ and $\gamma/\alpha+\gamma'/\alpha'=0$ (provided some sign and integer independence conditions are placed on α, β, γ, γ'). The analysis which leads to the results is quite delicate and rests heavily on a functional equation for G. For this a natural generalization of the simple continued fraction to Kronecker's forms $|n\alpha+\gamma-m|$ is required.

0. INTRODUCTION

Consider the two examples

$$\sum_{n=1}^{\infty}\left[n\left(\frac{1+\sqrt{5}}{2}\right)+\frac{1}{2}\right]z^n=\sum_{n=0}^{\infty}\frac{(-1)^n z^{F_{n+3}-(1/2)F_{3[n/3]+3}}}{(1-z^{F_{n+1}})(1-z^{F_{n+2}})} \tag{0.1}$$

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and

$$\frac{1-w}{w}\sum_{n=1}^{\infty} z^n w^{[n((1+\sqrt{5})/2)]}$$

$$= \cfrac{1}{\cfrac{(zw)^{-1}-1}{w^{-1}-1} + \cfrac{1}{z^{-F_0}w^{-F_1} + \cfrac{1}{z^{-F_1}w^{-F_2} + \cfrac{1}{z^{-F_1}w^{-F_2} + \cfrac{1}{\vdots}}}}}. \tag{0.2}$$

Here $F_0 := 0$, $F_1 := 1$, and $F_{n+1} := F_n + F_{n-1}$ are the Fibonnaci numbers and $|z|$, $|w| < 1$. An obvious consequence of (0.2) is that numbers like

$$\sum_{n=1}^{\infty} \frac{1}{3^n} \cdot \frac{1}{2^{[n((1+\sqrt{5})/2)]}}$$

are irrational numbers, because the continued fraction expansion is clearly unbounded. These two wildly improbable looking identities are specializations of the following theorems. We suppose throughout that $0 < \alpha$ is irrational, that $|\gamma| < 1$, and that $n\alpha + \gamma$ is never integral (unless otherwise specifically noted). This is primarily to avoid special case considerations in the continued fraction expansions. We use the notation $[x]$ for the integer part function and x^+ to denote $x^+ := \max(0, x)$.

THEOREM 0.1. *Let* $|z| \leqslant 1$, $|w| \leqslant 1$, *and* $|zw| \neq 1$. Suppose that $\alpha + \gamma < 1$, $1 > \alpha > 0$, $\gamma \geqslant 0$. Then

$$G_{\alpha,\gamma}(z, x) := \sum_{n=1}^{\infty} z^n w^{[n\alpha+\gamma]}$$

$$= \frac{z}{1-z} + \frac{1-w}{w} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^{t_n^{**}} w^{s_n^{**}}}{(1-z^{q_n}w^{p_n})(1-z^{q_{n+1}}w^{p_{n+1}})} \tag{0.3}$$

and

$$F_{\alpha,\gamma}(z, w) := \sum_{n=1}^{\infty} z^n \sum_{m=1}^{[n\alpha+\gamma]} w^m$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{t_n^{**}} w^{s_n^{**}}}{(1-z^{q_n}w^{p_n})(1-z^{q_{n+1}}w^{p_{n+1}})}. \tag{0.4}$$

Here $\{p_n\}$, $\{q_n\}$, $\{t_n^{**}\}$, and $\{s_n^{**}\}$ are given by the non-homogeneous continued fraction algorithm, Algorithm 0.3 (p_n/q_n are just the partial quotients of the continued fraction for α).

We have picked a convenient domain of convergence but not the most general possible. This theorem is proved in Section 3 without the assumption $\alpha+\gamma<1$. Dropping this assumption may make necessary the inclusion of some additional initial terms. Moreover, whenever gamma is positive the positive part in (0.3) and (0.5) is superfluous.

Note that

$$F_{\alpha,\gamma}(z,1)=\sum_{n=1}^{\infty}[n\alpha+\gamma]^{+}z^{n} \tag{0.5}$$

and (0.1) follows from (0.5) and explicit computation of the quantities in Algorithm 0.3.

Note also the duality relation

$$F_{\alpha,\gamma}(z,w)=\frac{zw}{(1-z)(1-w)}-\frac{w}{1-w}G_{\alpha,\gamma}(z,w). \tag{0.6}$$

From Theorem 0.1, with $\gamma=0$, we can deduce the following continued fraction expansion.

THEOREM 0.2.

$$\left(\frac{1-w}{w}\right)\sum_{n=1}^{\infty}z^{n}w^{[n\alpha]}=\cfrac{1}{T_0+\cfrac{1}{T_1+\cfrac{1}{T_2+\cdots}}}, \tag{0.7}$$

where

$$T_0:=\frac{1/zw^{a_0}-1}{1/w-1}$$

and

$$T_n:=\frac{1/(z^{q_{n-2}}w^{p_{n-2}})[1/(z^{q_{n-1}}w^{p_{n-1}})^{a_n}-1]}{[1/z^{q_{n-1}}w^{p_{n-1}}-1]}$$

with $\{a_n\}$ *the convergents and* $\{p_n/q_n\}$ *the partial quotients of the continued fraction expansion of* α.

The continued fraction (0.2) is now an obvious corollary of this theorem with $\alpha:=(1+\sqrt{5})/2$. Theorem 0.2 is proved in Section 4.

The analysis of Theorem 0.1 requires, indeed leads to, the following non-homogeneous continued fraction algorithm for approximations of form $|n\alpha+\gamma-m|$.

ALGORITHM 0.3. Suppose $\alpha, \gamma \in [0, 1)$ and α irrational.
Let

$$\alpha_0 := \alpha, \qquad \gamma_0 := \gamma$$

and

$$\alpha_{n+1} := \frac{1}{\alpha_n} - \left[\frac{1}{\alpha_n}\right], \qquad \gamma_{n+1} := \frac{\gamma_n}{\alpha_n} - \left[\frac{\gamma_n}{\alpha_n}\right].$$

Let

$$a_{n+1} := \left[\frac{1}{\alpha_n}\right], \qquad c_{n+1} := \left[\frac{\gamma_n}{\alpha_n}\right].$$

Let

$$\begin{aligned}
p_{n+1} &:= a_{n+1} p_n + p_{n-1}, \qquad p_0 := 0, \qquad p_{-1} := 1\\
q_{n+1} &:= a_{n+1} q_n + q_{n-1}, \qquad q_0 := 1, \qquad q_{-1} := 0\\
s_n &:= s_{n-1} + (a_{n+1} + (-1)^{n+1} c_{n+1}) p_n, \qquad s_{-1} := 1\\
t_n &:= t_{n-1} + (a_{n+1} + (-1)^{n+1} c_{n+1}) q_n, \qquad t_{-1} := 1.
\end{aligned}$$

Finally, define t_n^{**} and s_n^{**} by

$$t_{2n}^{**} := \begin{cases} t_{2n} & \text{if } c_{2n+2} = 0 \text{ and } t_{2n} \leqslant q_{2n} + q_{2n+1} \\ t_{2n} - q_{2n} & \text{else} \end{cases}$$

$$t_{2n+1}^{**} := \begin{cases} t_{2n+1} & \text{if } c_{2n+2} = 0 \text{ and } t_{2n+1} \leqslant q_{2n+1} + q_{2n+2} \\ t_{2n+1} - q_{2n+2} & \text{else} \end{cases}$$

$$s_{2n}^{**} := \begin{cases} s_{2n} & \text{if } c_{2n+2} = 0 \text{ and } t_{2n} \leqslant q_{2n} + q_{2n+1} \\ s_{2n} - p_{2n} & \text{else} \end{cases}$$

$$s_{2n+1}^{**} := \begin{cases} s_{2n+1} & \text{if } c_{2n+2} = 0 \text{ and } t_{2n+1} \leqslant q_{2n+1} + q_{2n+2} \\ s_{2n+1} - p_{2n+2} & \text{else.} \end{cases}$$

Note that $\{a_n\}$, $\{\alpha_n\}$, $\{p_n\}$, $\{q_n\}$ are the quantities that arise in the usual simple continued fraction for $\alpha \in (0, 1)$.

The properties of this algorithm are explored somewhat in Appendixes 1–3. In particular

$$t_n^{**}\alpha + \gamma - s_n^{**}$$

are best one sided approximants in the appropriate range. Also, $\{s_n^{**}\}$ and $\{t_n^{**}\}$ are algorithmic solutions to the Minkowski problem

$$|t_n^{**}\alpha+\gamma-s_n^{**}|<\frac{1}{t_n^{**}}. \tag{0.8}$$

These sequences have various other continued fraction like properties. For example, α is a quadratic irrational and γ is of the form $u\alpha+v$, with $u, v\in\mathbf{Q}$ if and only if the convergents $\{a_n\}$ and $\{c_n\}$ are periodic. This however is tangential and will not be proved here.

The hard work in this paper is the derivation of Theorem 3.2 and its equivalence to expansion (1.6). Theorem 0.1 is a special case of this. This rests heavily on the functional equation (2.1) and the rather delicate analysis of Algorithm 0.3. This constitutes Sections 1 to 3 and Appendixes 1 to 3. Two pretty applications which follow from this have proofs presented in Sections 5 and 6. Section 7 contains some examples.

THEOREM 0.4. *Suppose $z:=1/p$ and $w:=1/q$, where p and q are positive integers and $pq>1$. Then*

(a) *if α is irrational then $G_{\alpha,\gamma}(1/p, 1/q)$ is irrational and if the partial quotients satisfy* $\limsup q_{n+1}/q_n>3$, *then $G_{\alpha,\gamma}(1/p, 1/q)$ is transcendental (and not of normal rate of rational approximation)*;

(b) *if α has an unbounded continued fraction then $G_{\alpha,\gamma}(1/p, 1/q)$ is Liouville. In particular $F_{\alpha,\gamma}(1/p, 1/q)$ as a function of α in $[0, 1]$ is Liouville on a set of measure* 1.

The transcendence of the $F_{\alpha,0}(u, v)$ when u and v are algebraic is treated by Loxton and van der Poorten [11, 12], who follow up work of Mahler [13].

The second application is to give a proof of Fraenkel's generalization of a theorem of Lord Raleigh. Raleigh's theorem is the $\gamma:=0$ case of the theorem below, which is sometimes also referred to as Beatty's theorem (see Schoenberg [16, p. 32]). Fraenkel gives a proof of Theorem 0.5 and more in [7]. Our proof, once we have developed the other machinery of this paper, is considerably shorter.

THEOREM 0.5. *Assume $\alpha>1$, α irrational, $\gamma\in[0, 1)$, and $n\alpha+\gamma$ never integral. Then*

$$\{[n\alpha+\gamma]\}_{n=1}^{\infty} \quad \textit{and} \quad \{[n\alpha'+\gamma']\}_{n=1}^{\infty}$$

partition the positive integers if and only if

$$\frac{1}{\alpha}+\frac{1}{\alpha'}=1 \qquad \textit{and} \qquad \frac{\gamma}{\alpha}+\frac{\gamma'}{\alpha'}=0.$$

Theorem 0.1 in the case where $\gamma=0$, and hence $t_n^{**}=q_n+q_{n+1}$ and $s_n^{**}=p_n+p_{n+1}$, is due to Mahler [13] and is discussed by Loxton and van der Poorten [11]. A version of it is implicit in Hardy and Littlewood [10]. A different development is given in [3].

A single variable version of Theorem 0.2 ($z:=1$) is due to Böhmer [1]. It was rediscovered by Davison [6] in the more special case $\alpha=(1+\sqrt{5})/2$, which is presented in [8].

Algorithm 0.3 resembles the multi-dimensional Jacobi–Perron algorithm [4, 14]. Though it is not truly multi-dimensional some of the same complications of analysis arise.

The series $F_{\alpha,0}(z,1)$ has been studied from various points of view. For example, it represents a rational function if and only if α is rational [14, 15, 19].

The standard number theory we need is available in [2, 5, 9, 17, 18].

Finally, we need a few further pieces of notation. Let

$$G_{\alpha,\gamma}^{N}(z,w):=\frac{\sum_{n=1}^{q_N} z^n w^{[n\alpha+\gamma]}}{1-z^{q_N}w^{p_N}} \tag{0.9}$$

and

$$F_{\alpha,\gamma}^{N}(z,w):=\frac{zw}{(1-z)(1-w)}-\frac{w}{1-w}G_{\alpha,\gamma}^{N}(z,w). \tag{0.10}$$

These will provide rational approximations to $G_{\alpha,\gamma}$ and $F_{\alpha,\gamma}$. Also, let

$$B_N:=1-z^{q_N}w^{p_N} \tag{0.11}$$

and

$$A_N:=\sum_{n=1}^{q_N} z^n w^{[n\alpha+\gamma]}. \tag{0.12}$$

1. The Rational Approximation

We commence by constructing simple Padé like approximants for F and G. For $\gamma\geqslant 0$, and α irrational in $[0,1]$ we consider

$$\begin{aligned} B_N G-A_N &= \sum_{n=q_N+1}^{\infty} z^n w^{[n\alpha+\gamma]}-\sum_{n=1}^{\infty} z^{q_N+n} w^{[n\alpha+\gamma+p_N]} \\ &= z^{q_N}w^{p_N}\sum_{n=1}^{\infty} z^n\left\{w^{[n\alpha+\gamma+\varepsilon_N]}-w^{[n\alpha+\gamma]}\right\} \end{aligned}$$

with $\varepsilon_N:=\alpha q_N-p_N$.

Now Lemma A2.1 shows that there is at most one integer $1 \leqslant n_N \leqslant q_{N+1}$ with $[n_N\alpha+\gamma] \neq [n_N\alpha+\gamma+\varepsilon_N]$, where n_N is defined to be the smallest such integer (and may exceed q_{N+1}). Hence

$$B_N G - A_N = (-1)^{N+1} \frac{1-w}{w} z^{q_N+n_N} w^{p_N+m_N} + O(z^{q_N+q_{N+1}+1} w^{p_N+p_{N+1}-1}) \tag{1.1}$$

where $m_N := [n_N\alpha+\gamma] + (1+(-1)^N)/2$. We may now write

$$B_N A_{N+1} - B_{N+1} A_N = (-1)^{N+1} \frac{1-w}{w} \{ z^{q_N+n_N} w^{p_N+m_N} + z^{q_{N+1}+n_{N+1}} w^{p_{N+1}+m_{N+1}} + O(z^{q_N+q_{N+1}+1}) \}. \tag{1.2}$$

Now the left-hand side is of degree q_N+q_{N+1} in z and Proposition A2.2 shows that at most one of the two terms on the right-hand side has degree that low. The left-hand side cannot vanish identically and so

$$\frac{A_{N+1}}{B_{N+1}} - \frac{A_N}{B_N} = (-1)^{N+1} \frac{1-w}{w} \frac{z^{t_N^*} w^{s_N^*}}{(1-z^{q_N} w^{p_N})(1-z^{q_{N+1}} w^{p_{N+1}})}, \tag{1.3}$$

where

$$t_N^* := \begin{cases} n_N + q_N & \text{if } n_N \leqslant q_{N+1} \\ n_{N+1} + q_{N+1} = n_N & \text{if } n_N > q_{N+1}. \end{cases}$$
$$s_N^* := \begin{cases} m_N + p_N & \text{if } n_N \leqslant q_{N+1} \\ m_{N+1} + p_{N+1} & \text{if } n_N > q_{N+1}. \end{cases} \tag{1.4}$$

(See also Appendix 3.) Since A_N/B_N converges to $G_{\alpha,\gamma}$ we have for each N

$$G_{\alpha,\gamma}(z,w) = G_{\alpha,\gamma}^N(z,w) + \frac{1-w}{w} \sum_{n=N}^{\infty} (-1)^{n+1} \times \frac{z^{t_n^*} w^{s_n^*}}{(1-z^{q_n} w^{p_n})(1-z^{q_{n+1}} w^{p_{n+1}})}. \tag{1.5}$$

We provide an explicit construction of t_n^* and s_n^* in the next section. Meanwhile, we observe that if $\gamma = 0$ then $t_n^* = q_N + q_{N+1}$, $s_n^* = p_N + p_{N+1}$, as follows from Lemma 4.1 in [3] or as in Lemma A2.1.

From (1.5) and (0.6) we have

$$F_{\alpha,\gamma}(z,w) = F_{\alpha,\gamma}^N(z,w) + \sum_{n=N}^{\infty} (-1)^n \frac{z^{t_n^*} w^{s_n^*}}{(1-z^{q_n} w^{p_n})(1-z^{q_{n+1}} w^{p_{n+1}})}, \tag{1.6}$$

which for $\gamma := 0$, $N := 0$ yields Mahler's series since $F^0 = 0$ in this case. In the next section we derive a similar series via a functional equation approach and in Section 3 we see when the two are the same.

2. A Functional Equation Approach

Assuming as before that $n\alpha + \gamma$ is never integral ($n \geqslant 1$) we derive

$$F_{\alpha,\gamma}(z, w) + F_{\alpha^{-1}, -\gamma\alpha^{-1}}(w, z) = \frac{zw}{(1-z)(1-w)}, \tag{2.1}$$

since either $1 \leqslant m \leqslant [n\alpha + \gamma]$ or $1 \leqslant n \leqslant [m\alpha^{-1} - \gamma\alpha^{-1}]$ and not both. This is the basic functional equation. There is a more complicated version of (2.1) if $n\alpha + \gamma = m$. We now proceed to use the *non-homogeneous continued fraction*

$$\alpha_0 := \alpha,\ \gamma_0 := \gamma \geqslant 0 \left| \begin{array}{ll} \alpha_{n+1} + a_{n+1} = \alpha_n^{-1}, & a_{n+1} := [\alpha_n^{-1}] \\ \gamma_{n+1} + c_{n+1} = \gamma_n \alpha_n^{-1}, & c_{n+1} := [\gamma_n \alpha_n^{-1}] \end{array} \right. \tag{NHA}$$

detailed in Algorithm 0.3 and Appendix 1. Some tedious but straightforward manipulation allows us to replace α^{-1} by α_1, and $-\gamma\alpha^{-1}$ by $-\gamma_1$ in (2.1). Indeed, in general with $z_0 := z$, $w_0 := w$

$$F_{\alpha_0,\gamma_0}(z_0, w_0) + z^{-c_1} F_{\alpha_1, -\gamma_1}(z_1, w_1) = z^{-c_1} \frac{z_1 w_1}{(1-z_1)(1-w_1)} D_1, \tag{2.2}$$

where $z_1 := z_0^{a_1} w_0$, $w_1 := z_0$, and

$$D_1 := z_1^{c_2} + (1 - z_1^{c_2})/w_1.$$

Similarly

$$F_{\alpha_1, -\gamma_1}(z_1, w_1) + z_1^{c_2} F_{\alpha_2,\gamma_2}(z_2, w_2) = z_1^{c_2} \frac{z_2 w_2}{(1-z_2)(1-w_2)}, \tag{2.3}$$

where $z_2 := z_1^{\alpha_2} w_1$, $w_2 := z_1$.

This is simpler since $[n\alpha_2 + \gamma_2]$ is always positive while $[n\alpha_1 - \gamma_1]$ is not (for $n \leqslant c_2$). Combining (2.2) and (2.3) with 0 replaced by any even integer gives

$$\begin{aligned} &F_{\alpha_{2n},\gamma_{2n}}(z_{1n}, w_{2n}) - z_{2n+1}^{c_{2n+2}} z_{2n}^{-c_{2n+1}} F_{\alpha_{2n+2},\gamma_{2n+2}}(z_{2n+2}, w_{2n+2}) \\ &\quad = z_{2n+1}^{c_{2n+2}} z_{2n}^{-c_{2n+1}} \left\{ \frac{z_{2n+1} w_{2n+1}}{(1-z_{2n+1})(1-w_{2n+1})} - \frac{z_{2n+2} w_{2n+2}}{(1-z_{2n+2})(1-w_{2n+2})} \right\} \\ &\qquad + \frac{1 - z_{2n+1}^{c_{2n+2}}}{1 - z_{2n+1}} \frac{z_{2n+2}}{1 - z_{2n+2}}, \end{aligned} \tag{2.4}$$

where $z_{k+1} := z_k^{a_k+1} w_k$, $w_{k+1} := z_k$, and $\alpha_k, \gamma_k, a_k, c_k$ are generated by (NHA). We observe that the right-hand side of (2.4) can be rewritten as

$$\Delta_{2n} := \frac{z_{2n+1} z_{2n}^{-c_{2n+1}}}{(1-z_{2n+1})(1-z_{2n})} - \frac{z_{2n+1}^{c_{2n+2}+1} z_{2n}^{-c_{2n+1}}}{(1-z_{2n+2})(1-z_{2n+1})} \tag{2.4'}$$

while if $c_{2n+2}=0$ this is also equal to

$$\Delta_{2n} := \frac{z_{2n+1} z_{2n} z_{2n}^{-c_{2n+1}}}{(1-z_{2n+1})(1-z_{2n})} - \frac{z_{2n+2} z_{2n+1} z_{2n}^{-c_{2n+1}}}{(1-z_{2n+2})(1-z_{2n+1})}. \tag{2.4''}$$

Let

$$x_n := \prod_{k=0}^{n-1} z_k^{(-1)^{k+1} c_{k+1}} = x_{n-1} z_{n-1}^{(-1)^n c_n}$$

and write

$$F_{\alpha,\gamma}(z, w) - x_{2N} F_{\alpha_{2N},\gamma_{2N}}(z_{2N}, w_{2N}) = \sum_{n=0}^{N-1} x_{2n} \Delta_{2n}. \tag{2.5}$$

Next we make x_k, z_k explicit. Inductively, we deduce that $z_k = z^{q_k} w^{p_k}$ and then that $x_k = z^{n_{k-1}} w^{\rho_{k-1}}$, where

$$\begin{aligned} n_k - n_{k-1} &= (-1)^{k+1} c_{k+1} q_k, \qquad n_{-1} := 0 \\ \rho_k - \rho_{k-1} &= (-1)^{k+1} c_{k+1} p_k, \qquad \rho_{-1} := 0. \end{aligned} \tag{2.6}$$

Let $t_k := q_k + q_{k+1} + n_k$ and $s_k := p_k + p_{k+1} + \rho_k$ so that

$$\begin{aligned} t_k &= t_{k-1} + (a_{k+1} + (-1)^{k+1} c_{k+1})\, q_k, \qquad t_{-1} := 1 \\ s_k &= s_{k-1} + (a_{k+1} + (-1)^{k+1} c_{k+1})\, p_k, \qquad s_{-1} := 1. \end{aligned} \tag{2.7}$$

Then some more manipulation with (2.4) yields

$$x_{2n}\Delta_{2n} = \frac{z^{t_{2n}-q_{2n}} w^{s_{2n}-p_{2n}}}{(1-z_{2n+1})(1-z_{2n})} - \frac{z^{t_{2n+1}-q_{2n+2}} w^{s_{2n+1}-p_{2n+2}}}{(1-z_{2n+2})(1-z_{2n+1})}.$$

Moreover, since $|F_{\alpha_{2N},\gamma_{2N}}(z_{2N}, w_{2N})| \leq |z_{2N}|\,|w_{2N}|$ while $x_{2N} z_{2N} w_{2N} = z^{t_{2N-1}} w^{s_{2N-1}}$, and $t_N \to \infty$ (Appendix 3) we see that (2.5) yields

$$F_{\alpha,\gamma}(z, w) = \sum_{n=0}^{\infty} \left\{ \frac{z^{t_{2n}-q_{2n}} w^{s_{2n}-p_{2n}}}{(1-z_{2n+1})(1-z_{2n})} - \frac{z^{t_{2n+1}-q_{2n+2}} w^{s_{2n+1}-p_{2n+2}}}{(1-z_{2n+2})(1-z_{2n+1})} \right\}. \tag{2.8}$$

In the case $c_{2n+2}=0$ we observe from (2.7) that the bracketed term equals

$$\frac{z^{t_{2n}}w^{s_{2n}}}{(1-z_{2n+1})(1-z_{2n})}-\frac{z^{t_{2n+1}}w^{s_{2n+1}}}{(1-z_{2n+2})(1-z_{2n+1})}.$$

We now define t_n^{**} and s_n^{**}, as in Algorithm 0.3, by

$$\begin{aligned}
t_{2n}^{**} &:= \begin{cases} t_{2n} & \text{if } c_{2n+2}=0 \text{ and } t_{2n}\leqslant q_{2n}+q_{2n+1} \\ t_{2n}-q_{2n} & \text{else} \end{cases} \\
t_{2n+1}^{**} &:= \begin{cases} t_{2n+1} & \text{if } c_{2n+2}=0 \text{ and } t_{2n+1}\leqslant q_{2n+1}+q_{2n+2} \\ t_{2n+1}-q_{2n+2} & \text{else} \end{cases} \\
s_{2n}^{**} &:= \begin{cases} s_{2n} & \text{if } c_{2n+2}=0 \text{ and } t_{2n}\leqslant q_{2n}+q_{2n+1} \\ s_{2n}-p_{2n} & \text{else} \end{cases} \\
s_{2n+1}^{**} &:= \begin{cases} s_{2n+1} & \text{if } c_{2n+2}=0 \text{ and } t_{2n+1}\leqslant q_{2n+1}+q_{2n+2} \\ s_{2n+1}-p_{2n+2} & \text{else.} \end{cases}
\end{aligned} \tag{2.9}$$

Then Lemma A3.1 and Proposition A3.3 show that $q_n \leqslant t_n^{**} \leqslant q_n+q_{n+1}$ and (2.8) becomes

$$F_{\alpha,\gamma}(z,w)=\sum_{n=0}^{\infty}\frac{(-1)^n z^{t_n^{**}}w^{s_n^{**}}}{(1-z^{q_n}w^{p_n})(1-z^{q_{n+1}}w^{p_{n+1}})}. \tag{2.10}$$

We now have two representations for $F_{\alpha,\gamma}$, namely (1.6) with $N=0$ and (2.10). We define

$$F_{**}^{N} := \sum_{n=0}^{N-1}(-1)^n\frac{z^{t_n^{**}}w^{s_n^{**}}}{(1-z^{q_n}w^{p_n})(1-z^{q_{n+1}}w^{p_{n+1}})}. \tag{2.11}$$

For sufficiently large n in Appendix 3 we show that $t_n^{**}=t_n^*$, $s_n^{**}=s_n^*$ and so derive our general identities. Correspondingly we let

$$F_{*}^{N} := \sum_{n=0}^{N-1}(-1)^n\frac{z^{t_n^{*}}w^{s_n^{*}}}{(1-z^{q_n}w^{p_n})(1-z^{q_{n+1}}w^{p_{n+1}})}. \tag{2.12}$$

3. The Main Identity

We wish to see when (1.6) and (2.10) generate the same series expansions.

Lemma 3.1. *With B_n as in* (0.11), *with t_n^*, s_n^* as in* (1.4), *and with t_n^{**}, s_n^{**} as in* (2.9) *we have*

(i) $F_*^N - F_*^M = \sum_{n=M}^{N-1} (-1)^n (z^{t_n^*} w^{s_n^*}/B_n B_{n+1})$

(ii) $F_{**}^N - F_{**}^M = \sum_{n=M}^{N-1} (-1)^n (z^{t_n^{**}} x^{s_n^{**}}/B_n B_{n+1})$

(iii) *if* $a_{2k+1} > c_{2k+1}$ *then* $t_n^* = t_n^{**}$ *and* $s_n^* = s_n^{**}$ *for all* $n \geqslant 2k+1$ *and so*

$$F_*^N - F_*^M = F_{**}^N - F_{**}^M \qquad \text{for all} \quad N \geqslant M \geqslant 2k+1 \tag{3.1}$$

and

$$F_*^M = F_{**}^M. \tag{3.2}$$

Proof. Part (i) follows from definition (2.12) and (ii) from (2.11). Part (iii) follows from Proposition A3.4. Since both F_*^N and F_{**}^N converge to $F_{\alpha,\gamma}$, (3.1) implies (3.2). ∎

THEOREM 3.2. *Assume* $0 < \alpha < 1, \alpha$ *irrational,* $0 \leqslant \gamma < 1, n\alpha + \gamma$ *never integral for* $n > 0$. *Then*

(a) *For* $N \geqslant 2k+1$ *if* $a_{2k+1} > c_{2k+1}$

$$\sum_{n=0}^{N-1} (-1)^n \frac{z^{t_n^{**}} w^{s_n^{**}}}{(1 - z^{q_n} w^{p_n})(1 - z^{q_{n+1}} w^{p_{n+1}})} = \frac{zw}{(1-z)(1-w)} - \frac{w}{1-w} \frac{\sum_{n=1}^{q_N} z^n w^{[n\alpha+\gamma]}}{1 - z^{q_N} w^{p_N}} \tag{3.3}$$

with t_n^{**}, s_n^{**} *given by* (2.7), (2.9).

In particular if $a_1 > c_1$ *as happens if* $\alpha + \gamma < 1$ *or* $\gamma < \alpha$ *and so if* $\gamma \in [0, \frac{1}{2}]$, *then* (3.3) *holds for all* $N = 1, 2, 3, \ldots$.

(b) *Moreover,*

$$\frac{zw}{(1-z)(1-w)} - \frac{w}{1-w} \frac{\sum_{n=1}^{q_N} z^n w^{[n\alpha+\gamma]}}{1 - z^{q_N} w^{p_N}} = \frac{I_N}{(1-z)(1 - z^{q_N} w^{p_N})},$$

where I_N *is an integral polynomial in* z, w *with degree*$_z(I_N) < q_{N+1}$.

Proof. Part (a) follows from Lemma 3.1 and in (b) we write

$$I_N(z, w) = wz\left\{\frac{1 - w^{[\alpha+\gamma]}}{1-w} - z^{q_N} w^{p_N} \left(\frac{1 - w^{[\gamma+\varepsilon_N]}}{1-w}\right) + \sum_{n=1}^{q_N - 1} ([(n+1)\alpha + \gamma] - [n\alpha + \gamma]) z^{n-1} w^{[n\alpha+\gamma]}\right\}. \quad ∎$$

Note that $[\alpha+\gamma] = 0$ or 1 while $[\gamma + \varepsilon_N]$ is 0 or -1, actually zero for N even or N large and $\gamma > 0$.

For $w := 1$ we have, for $\gamma \geqslant 0$,

$$\sum_{n=1}^{\infty} [n\alpha + \gamma]\, z^n = \sum_{n=0}^{\infty} (-1)^n \frac{z^{t_n^{**}}}{(1 - z^{q_n})(1 - z^{q_{n+1}})} \tag{3.4}$$

and

$$\sum_{n=0}^{N-1} \frac{(-1)^n z^{t_n^{**}}}{(1 - z^{q_n})(1 - z^{q_{n+1}})} = \frac{I_N(z, 1)}{(1 - z)(1 - z^{q_N})}$$

is an approximation of order $t_N^{**} \geqslant q_N$.

4. Proof of the Continued Fraction Expansion 0.2

Let

$$Q_n := z^{-q_n} w^{-p_n} - 1$$

$$P_n := z^{-q_n} w^{-p_n} \sum_{n=1}^{q_n} z^n w^{[n\alpha]}.$$

Let T_n be as in Theorem 0.2. Observe that by (1.3) (with $\gamma = 0$)

$$\frac{P_{n+1}}{Q_{n+1}} - \frac{P_n}{Q_n} = \frac{(-1)^{n+1}\,((1 - w)/w)}{Q_{n+1} Q_{n+1}}$$

and hence

$$\frac{P_{n+2}}{Q_{n+2}} - \frac{P_n}{Q_n} = \frac{(-1)^{n+1}\,((1 - w)/w)\, T_{n+2}}{Q_{n+2} Q_n}.$$

We deduce the recursion

$$\begin{bmatrix} P_{n+1} \\ Q_{n+1} \end{bmatrix} = T_{n+1} \begin{bmatrix} P_n \\ Q_n \end{bmatrix} + \begin{bmatrix} P_{n-1} \\ Q_{n-1} \end{bmatrix}$$

with $P_{-1} = 0$, $P_0 = 1$, $Q_{-1} = w^{-1} - 1$, $Q_0 = z^{-1} w^{a_0} - 1$. It follows that the T_n are the convergents of a continued fraction. However, since

$$\frac{P_n}{Q_n} \to \sum_{n=1}^{\infty} z^n w^{[n\alpha]}$$

we deduce (on dividing by $w^{-1} - 1$ so that $Q_{-1}/(w^{-1} - 1) = 1$) that

$$T_0 + \cfrac{1}{T_1 + \cfrac{1}{T_2 + \cdots}} = \frac{-w}{(w - 1) \sum_1^{\infty} z^n w^{[n\alpha]}},$$

from which the result flows. (Note that the above normalization makes the $\{P_n\}$ the denominators and the $\{Q_n\}$ the numerators.) ∎

5. Transcendence

The transcendence estimates are fairly straightforward. Let

$$G := G_{\alpha,\gamma}\left(\frac{1}{p}, \frac{1}{q}\right).$$

We now use the approximation (1.1) in the form

$$\left|G - \frac{1}{B_N}\left(A_N + (-1)^{N+1}\frac{(1-w)}{w} z^{q_N + n_N} w^{p_N + m_N}\right)\right|$$
$$= O(z^{q_N + q_{N+1}} w^{p_N + p_{N+1} - 1}),$$

where $w := 1/q$, $z := 1/p$. We deduce that there exist approprimants to G that satisfy

$$0 < \left|G - \frac{h_N}{k_N}\right| = O(1/k_N^{((1 + q_{N+1}/q_N)/(2+\varepsilon)}). \tag{5.1}$$

To see (5.1) we argue as follows. If

$$\left(\frac{1}{p}\right)^{q_N + n_N}\left(\frac{1}{q}\right)^{p_N + m_N} \leqslant \left(\frac{1}{p}\right)^{(q_N + q_{N+1})/2}\left(\frac{1}{q}\right)^{(p_N + p_{N+1})/2}$$

we use

$$\left|G - \frac{A_N}{B_N}\right| = O\left(\frac{1}{p^{(q_N + q_{N+1})/2}\, q^{(p_N + p_{N+1})/2}}\right)$$

as an approximation. If

$$\left(\frac{1}{p}\right)^{q_N + n_N}\left(\frac{1}{q}\right)^{p_N + m_N} > \left(\frac{1}{p}\right)^{(q_N + q_{N+1})/2}\left(\frac{1}{q}\right)^{(p_N + p_{N+1})/2}$$

we use

$$\left|G - \frac{1}{B_N}\left(A_N + (-1)^{N+1}(1-q)\frac{1}{p^{q_N + n_N} q^{p_N + m_N}}\right)\right|$$
$$= O\left(\frac{1}{p^{q_N + q_{N+1}} q^{p_N + p_{N+1}}}\right)$$

as an approximation.

We consider the first case. Now

$$B_N := 1 - \frac{1}{p^{q_N}} \cdot \frac{1}{q^{p_N}}$$

and

$$A_N := \sum_{n=1}^{q_N} z^n w^{[n\alpha+\gamma]}.$$

So

$$k_N := p^{q_N} q^{p_N} B_N$$

is an integer and

$$|k_N| \leqslant p^{q_N} q^{p_N}$$

while $n_N := p^{q_N} q^{p_N} A_N$ is a rational number with bounded denominator of size $\leqslant p^{\gamma+1}$. Now, for large N

$$0 < \left| G\left(\frac{1}{p}, \frac{1}{q}\right) - \frac{n_N}{k_N} \right| \leqslant (q-1) \frac{1}{p^{q_N+n_N} q^{p_N+m_N}}$$

$$+ O\left(\frac{1}{p^{q_N+q_{N+1}} q^{p_N+p_{N+1}-1}}\right)$$

$$= O\left(\frac{1}{p^{(q_N+q_{N+1})/(2+\varepsilon)} q^{(p_N+p_{N+1})/(2+\varepsilon)}}\right)$$

and (5.1) follows since $|k_N| \leqslant p^{q_N} q^{p_N}$ and $p_N \approx \alpha q_N$. The non-vanishing of the form follows from (1.1) and above directly. The second case is similar.

So provided $q_{N+1} > (3+\varepsilon') q_N$ holds infinitely often we deduce from Roth's theorem [17] that ζ is transcendental (and not of usual (Roth class) rate of approximation). The irrationality is immediate from (5.1) since $\liminf q_{N+1}/q_N > 1$. Clearly if α has unbounded continued fraction then from (5.1) we deduce that ζ is Liouville.

6. Proof of the Generalized Raleigh Theorem 0.5

Let $s := \alpha + 1$, $t := \alpha' + 1$. Then

$$\{[ns+\gamma]\}_{n=1}^{\infty} \quad \text{and} \quad \{[nt+\gamma']\}_{n=1}^{\infty} \tag{6.1}$$

partition the positive integers if and only if

$$G_{\alpha,\gamma}(z,z)+G_{\alpha',\gamma'}(z,z)=\sum_{n=1}^{\infty} z^{[n(\alpha+1)+\gamma]}+\sum_{n=1}^{\infty} z^{[n(\alpha'+1)+\gamma']} \\ =\sum_{n=1}^{\infty} z^n=\frac{z}{1-z}. \tag{6.2}$$

Note by (2.1)

$$F_{\alpha,\gamma}(z,w)+F_{\alpha^{-1},-\gamma\alpha^{-1}}(w,z)=\frac{zw}{(1-z)(1-w)}. \tag{6.3}$$

So with the duality relation (0.6)

$$F_{\alpha,\gamma}(z,w)=\frac{z}{1-z}G_{\alpha^{-1},-\gamma\alpha^{-1}}(w,z) \tag{6.4}$$

and

$$F_{\alpha^{-1},-\gamma\alpha^{-1}}(w,z)=\frac{w}{1-w}G_{\alpha,\gamma}(z,w). \tag{6.5}$$

Thus (6.2) re-writes as

$$G_{\alpha^{-1},-\gamma\alpha^{-1}}(w,z)+\frac{1-z}{z}\frac{w}{1-w}G_{\alpha,\gamma}(z,w)=\frac{w}{1-w}. \tag{6.6}$$

If we set $z:=w$ in (6.6) we deduce on comparison to (6.2) that

$$G_{\alpha^{-1},-\gamma\alpha^{-1}}(z,z)=G_{\alpha',\gamma'}(z,z). \tag{6.7}$$

Thus (6.1) holds if and only if for large n

$$[n\alpha^{-1}-\gamma\alpha^{-1}]=[n\alpha'+\gamma']. \tag{6.8}$$

It follows easily (by uniform distribution) that

$$\alpha^{-1}=\alpha' \qquad \text{and} \qquad -\gamma\alpha^{-1}=\gamma'$$

or, with $s=\alpha+1$ and $t=\alpha'+1$,

$$\frac{1}{s}+\frac{1}{t}=1 \qquad \text{and} \qquad \frac{\gamma}{s}+\frac{\gamma'}{t}=0$$

and the result follows on noting that s and t play the role of α and α' in the statement of Theorem 0.5.

7. EXAMPLES

As one might expect explicit examples are fairly rare, as they are for simple continued fractions.

EXAMPLE 1. Let $\alpha := (e-1)/2$ and $\gamma := (e-1)/4$; then

$$\{a_i\}_{i=1} = [1, 6, 10, 14, 18, 22, 26, ...]$$
$$\{c_i\}_{i=1} = [0, 3, 0, 7, 0, 15, 0, ...]$$
$$\{t_i^{**}\}_{i=0} = [1, 4, 74, 571, 18589, 217568, 10590502, ...]$$
$$\{s_i^{**}\}_{i=0} = [1, 4, 64, 491, 15971, 186922, 9098734, ...].$$

If we generate the simple continued fraction to $(e-1)/4$ we get

$$\{a_i'\}_{i=1} = [2, 3, 20, 7, 36, 11, 52, 15, ...]$$
$$\{q_i\}_{i=0} = [1, 2, 7, 142, 1001, 36178, 398959, 20782046, ...]$$
$$\{p_i\}_{i=0} = [0, 1, 3, 61, 430, 15541, 171381, 8927353, ...].$$

The interesting observation, and the key to verifying these numbers, is that

$$t_n^{**} = \frac{q_n + q_{n+1} - 1}{2}$$

and

$$s_n^{**} = p_n + p_{n+1}.$$

EXAMPLE 2. Let $\alpha := 1/\sqrt{2}$ and $\gamma := 1/(2\sqrt{2})$; then

$$\{a_i\}_{i=1} = [1, 2, 2, 2, 2, ...]$$
$$\{c_i\}_{i=1} = [0, 1, 0, 1, 0, ...]$$
$$\{t_i^{**}\}_{i=0} = [1, 2, 8, 15, 49, 90, 288, ...]$$
$$\{s_i^{**}\}_{i=0} = [1, 2, 6, 11, 35, 64, 204, ...].$$

If, once again, we derive the simple continued fraction to $\gamma := 1/(2\sqrt{2})$ we have

$$\{a_i'\}_{i=1} = [2, 1, 4, 1, 4, 1, 4, 1, ...]$$
$$\{q_i\}_{i=0} = [1, 2, 3, 14, 17, 82, 99, 478, ...]$$
$$\{p_i\}_{i=0} = [0, 1, 1, 5, 6, 29, 35, 169, ...]$$

and, again

$$t_n^{**} = \frac{q_n + q_{n+1} - 1}{2}$$

$$s_n^{**} = p_n + p_{n+1}.$$

In each case given the form of $\{a_i'\}$ easy but somewhat tedious inductions establish the claimed form for t_n^{**} and s_n^{**}. This is also true in (0.1).

APPENDIX 1: A Non-homogeneous Continued Fraction Algorithm

For $0 < \alpha < 1$, $\alpha \notin Q$, and $0 \leqslant \gamma < 1$ consider $\alpha_0 := \alpha$, $\gamma_0 := \gamma$, and

$$\alpha_{n+1} := \frac{1}{\alpha_n} - \left[\frac{1}{\alpha_n}\right]; \qquad \gamma_{n+1} := \frac{\gamma_n}{\alpha_n} - \left[\frac{\gamma_n}{\alpha_n}\right]. \tag{A1.1}$$

Writing $a_{n+1} := [1/\alpha_n]$ and $c_{n+1} := [\gamma_n/\alpha_n]$ we have

$$a_{n+1} + \alpha_{n+1} := \frac{1}{\alpha_n}; \qquad \gamma_{n+1} + c_{n+1} := \frac{\gamma_n}{\alpha_n}, \tag{A1.2}$$

and $[0, a_1, ..., a_n, ...]$ is the standard simple continued fraction for α. In particular, $a_n \geqslant 1$ while $0 \leqslant c_n \leqslant a_n$.

Proposition A1.1. *Let sequences p_n, q_n, λ_n be defined by*

(i) $p_{n+1} := a_{n+1} p_n + p_{n-1}$, $p_0 := 0$, $p_{-1} = 1$

(ii) $q_{n+1} := a_{n+1} q_n + q_{n-1}$, $q_0 := 1$, $q_{-1} = 0$

(iii) $\lambda_{n+1} := a_{n+1} \lambda_n + \lambda_{n-1} + c_{n+1}$, $\lambda_0 := 0$, $\lambda_{-1} = 0$.

Then for all $n \geqslant 0$

$$\alpha = \frac{\alpha_{n+1} p_n + p_{n+1}}{\alpha_{n+1} q_n + q_{n+1}} \tag{A1.3}$$

and

$$\gamma = \frac{\alpha_{n+1} \lambda_n + \lambda_{n+1} + \gamma_{n+1}}{\alpha_{n+1} q_n + q_{n+1}}. \tag{A1.4}$$

Proof. Both are easy inductive arguments. ∎

Let s_n and t_n be as in (2.7):

$$s_n := s_{n-1} + (a_{n+1} + (-1)^{n+1} c_{n+1}) p_n, \qquad s_{-1} := 1$$
$$t_n := t_{n-1} + (a_{n+1} + (-1)^{n+1} c_{n+1}) q_n, \qquad t_{-1} := 1. \tag{A1.5}$$

PROPOSITION A1.2. *For all $N \geq 0$*

$$\text{(i)} \quad t_N\alpha + \gamma - s_N = \frac{\gamma_{N+1} + (-1)^N (1 - \alpha_{N+1})}{q_{N+1} + \alpha_{N+1} q_N}$$

$$\text{(ii)} \quad q_N\alpha - p_N = \frac{(-1)^N}{q_{N+1} + \alpha_{N+1} q_N} \qquad (= \varepsilon_N)$$

$$\text{(iii)} \quad q_{N+1}\alpha - p_{N+1} = \frac{(-1)^{N+1} \alpha_{N+1}}{q_{N+1} + \alpha_{N+1} q_N} \qquad (= \varepsilon_{N+1}).$$

In particular

$$\text{(iv)} \quad t_N\alpha + \gamma - s_N = \varepsilon_N + \varepsilon_{N+1} + (-1)^N \varepsilon_N \gamma_{N+1}. \tag{A1.6}$$

Proof. Parts (ii) and (iii) are standard and follow from (A1.3) and $q_{N+1} p_N - p_{N+1} q_N = (-1)^{N+1}$. To prove (i) we write

$$t_N\alpha + \gamma - s_N = \frac{\gamma_{N+1} + \alpha_{N+1} w_N + v_N}{\alpha_{N+1} q_N + q_{N+1}},$$

where $w_N := t_N p_N - s_N q_N + \lambda_N$, $v_N := t_N p_{N+1} - s_N q_{N+1} + \lambda_{N+1}$ as follows from (A1.3) and (A1.4). Now we check that $w_{-1} = 1$, $w_0 = -1$, $v_{-1} = -1$, $v_0 = 1$ and that

$$\text{(a)} \quad w_{N+1} = v_N, \qquad \text{(b)} \quad v_N = a_{N+1}(w_N + (-1)^N) + w_{N-1}.$$

It follows inductively that $v_N = (-1)^N$, $w_N = (-1)^{N+1}$. ∎

PROPOSITION A1.3. *For some $\bar{m}$, $a_{2\bar{m}+1} > c_{2\bar{m}+1}$. Precisely, if $c_{2m+1} = a_{2m+1}$ for $m = 0, \ldots, \bar{m}$ then $q_{2\bar{m}+3} < 2/(1-\gamma)$.*

Proof. If $c_{2m+1} = a_{2m+1}$ for $m \leq \bar{m}$ then $c_{2m+2} = 0$ and so

$$\begin{aligned} 0 = t_{-1} - q_0 = t_0 - q_0 = \cdots &= t_{2m-1} - q_{2m} \\ &= t_{2m} - q_{2m} - q_{2m}(a_{2m+1} - c_{2m+1}) = t_{2m} - q_{2m} \\ &= t_{2m+1} - q_{2m+2} - c_{2m+2} q_{2m+1} = t_{2m+1} - q_{2m+2}. \end{aligned}$$

Similarly

$$0 = s_{-1} - p_0 = s_0 - p_0 = \cdots = s_{2m} - p_{2m} = s_{2m+1} - p_{2m+2}.$$

Thus $t_{2m+1} = q_{2m+2}$ and $s_{2m+1} = p_{2m+2} + 1$. Hence Proposition A1.2(iv) gives

$$1 - \gamma = 2\varepsilon_{2m+2} + (1 - \gamma_{2m+1})\, \varepsilon_{2m+1} \leqslant 2\varepsilon_{2m+2}$$

so that

$$1 - \gamma < 2/q_{2\bar{m}+3}. \quad \blacksquare$$

Since $q_{2\bar{m}+3} \geqslant F_{2\bar{m}+4}$ (Fibonacci), independent of α and $F_4 = 3$, $F_6 = 8$: if $\gamma \leqslant \frac{3}{4}$ either $a_1 > c_1$ or $a_3 > c_3$.

APPENDIX 2: One Sided Approximations

Lemma A2.1. *Let a non-negative integer N and $1 > \alpha > 0$, $1 > \gamma \geqslant 0$ be given with α irrational. If $n_1 > n_2 \geqslant 1$ are such that*

$$[n\alpha + \gamma + \varepsilon_N] \neq [n\alpha + \gamma]\ n = n_1, n_2 \tag{A2.1}$$

then

$$n_1 - n_2 = kq_{N+1} + t_{q_N}, \qquad \textit{for some } k \geqslant 1,\ t \geqslant 0\ k, t \in Z.$$

[*We define n_N to be the smallest positive solution to* (A2.1) (*such an integer exists by uniform distribution*).]

Proof. We establish only the even case. The odd case is parallel. We have integers m_1 and m_2 with

$$n_i\alpha + \gamma + \varepsilon_N \geqslant m_i > n_i\alpha + \gamma$$

and so

$$(n_1 - n_2)\, \alpha + \varepsilon_N \geqslant m_1 - m_2 > (n_1 - n_2)\, \alpha - \varepsilon_N.$$

Setting $n := n_1 - n_2$, $m := m_1 - m_2$ and using $\alpha > p_N/q_N$ (N even) we derive

$$(n + q_N)\, \varepsilon_N > mq_N - np_N > -\varepsilon_N q_N.$$

Let $k := mq_N - np_N$. Since $|\varepsilon_N| < 1/q_{N+1}$, $k \geqslant 0$. If $k = 0$ then $m_1 - m_2 = tp_N$, $n_1 - n_2 = tq_N$ for some $t \geqslant 1$ in $\mathbf{N}$. Hence

$$m_2 + tp_N > (n_2 + tq_N)\, \alpha + \gamma$$

and

$$m_2 > n_2\alpha + \gamma + t\varepsilon_N \geqslant n_2\alpha + \gamma + \varepsilon_N \geqslant m_2,$$

a contradiction. Thus $k \geqslant 1$. Since $p_{N+1}q_N - q_{N+1}p_N = 1$ we have

$$mq_N - np_N = (kp_{N+1})\,q_N - (kq_{N+1})\,p_N$$

and as q_N and p_N are relatively prime $(N > 0)$

$$m = kp_{N+1} + tp_N$$
$$n = kq_{N+1} + tp_N$$

for $k \geqslant 1$ and t in Z. But $(n + q_N)\,\varepsilon_N \geqslant k$ implies $n > kq_{N+1} - q_N$ so that t must be non-negative, and the conclusion obtains (even for $N = 0$). ▮

PROPOSITION A2.2. *If $n_{N+1} \leqslant q_N$ then $n_N = n_{N+1} + q_{N+1}$. In particular either $n_N + q_N$ or $n_{N+1} + q_{N+1}$ exceeds $q_N + q_{N+1}$.*

Proof. Again we establish this only for N even. Since, for some m,

$$n_{N+1}\alpha + \gamma + \varepsilon_{N+1} < m \leqslant n_{N+1}\alpha + \gamma,$$

and since $\varepsilon_N > -\varepsilon_{N+1}$, we derive

$$(n_{N+1} + q_{N+1})\,\alpha + \gamma < m + p_{N+1} < (n_{N+1} + q_{N+1})\,\alpha + \gamma + \varepsilon_N.$$

Whence $\bar{n} := n_{N+1} + q_{N+1}$ satisfies (A2.1) and hence either $n_{N+1} + q_{N+1} = n_N$ as claimed or (by Lemma A2.1)

$$n_{N+1} + q_{N+1} - n_N = kq_{N+1} + tq_N, \qquad k \geqslant 1,\ t \geqslant 0.$$

Now, since $n_{N+1} \leqslant q_N$ and $n_N \geqslant 1$, $t = 0$ and $k = 1$. Equivalently $n_{N+1} = n_N$. But now for some m', m''

$$\begin{aligned} n_N\alpha + \gamma + \varepsilon_{N+1} = n_{N+1}\alpha + \gamma + \varepsilon_{N+1} &< m' \leqslant n_{N+1}\alpha + \gamma \\ &= n_N\alpha + \gamma < m'' \leqslant n_N\alpha + \gamma + \varepsilon_N. \end{aligned}$$

This implies that $\varepsilon_N - \varepsilon_{N+1} > 1$, which is impossible. (Indeed $1 + \varepsilon_1 = a_1 a \geqslant \alpha = \varepsilon_0$, while $|\varepsilon_N| < \frac{1}{2}$ for $N \geqslant 1$.) ▮

As a by-product, with $\{x\}$ denoting the fractional part of x:

PROPOSITION A2.3 (Best one sided approximation: if $n\alpha + \gamma - m$ is never zero).

N even: $\{m\alpha + \gamma\} < \{(n_N + q_N)\,\alpha + \gamma\}$ is impossible for $q_N < m < q_{N-1} + n_N$.

N odd: $\{m\alpha + \gamma\} > \{(n_N + q_N)\,\alpha + \gamma\}$ is similarly impossible.

In particular, $(n_N+q_N)\alpha+\gamma$ is the best one sided approximation to an integer in the range $]q_N, q_N+q_{N+1}+n_N[$. (Recall that each t_N^* is either n_N+q_N or $n_{N+1}+q_{N+1}$).

Proof. We check much as in the previous arguments that if $\bar{n} := m-q_N$ then $[\bar{n}\alpha+\gamma] \neq [\bar{n}\alpha+\gamma+\varepsilon_N]$. We apply Lemma A2.1 again. Hence either $m=q_N+n_N$ or $m=kq_{N+1}+hq_N+n_N$ for $h, k \geqslant 1$ and the conclusion follows. ∎

In particular, if $n_N \leqslant q_{N+1}$, then n_N+q_N gives the best one sided approximation to an integer for $q_N<m\leqslant q_N+q_{N+1}$.

APPENDIX 3: The Relationship between t_N^* (1.4) and t_N^{**} (2.9)

We proceed to study t_N^* and t_N^{**}. Similar results hold for s_N^* and s_N^{**}. We give these only where necessary.

Lemma A3.1. *For each k*

$$t_{2k} \geqslant q_{2k}, \qquad t_{2k+1} \geqslant q_{2k+2}$$

and these inequalities are strict if $k \geqslant m$ and if $c_{2m+1} < a_{2m+1}$.

Proof. Let $\bar{m}$ be the least integer with $c_{2m+1} < a_{2m+1}$ (Prop. A1.3). As therein, $t_{2n}=q_{2n}$, $t_{2n+1}=q_{2n+2}$ for $n \leqslant \bar{m}-1$. Then for $m \geqslant \bar{m}$

$$t_{2m} = t_{2m-1} + (a_{2m+1} - c_{2m+1})\, q_{2m} > q_{2m}$$

and

$$t_{2m+1} - q_{2m+2} \geqslant t_{2m} - q_{2m} > 0. \quad ∎$$

Proposition A3.2. *For all N one has $t_N^{**} \geqslant q_N$. Moreover, $t_N^{**} > q_N$ for $N \geqslant 2m+1$ if $c_{2m+1} < a_{2m+1}$.*

Proof. We establish this inductively, in pairs $2N$, $2N+1$.

(1) $c_{2N+2}>0$: $t_{2N}^{**}=t_{2N}-q_{2N}=t_{2N-1}-q_{2N}+(a_{2N+1}-c_{2N+1})\,q_{2N}$. But $c_{2N+2}>0$ implies $a_{2N+1}>c_{2N+1}$ ($a_{2N+1}=c_{2N+1}$ means $c_{2N+2}=0$ since $\alpha_{2N+1}-\gamma_{2N+1}=(1-\gamma_{2N})\,\alpha_{2N}^{-1}>0$ and so $\gamma_{2N+1}/\alpha_{2N+1}<1$). Hence

$$t_{2N}^{**} \geqslant t_{2N-1} - q_{2N} + q_{2N} \geqslant q_{2N},$$

and, for $N>M$, $t_{2N-1}>q_{2N}$ so

$$t^{**}_{2N} > q_{2N}.$$

Also,

$$t^{**}_{2N+1} = t^{**}_{2N} + c_{2N+2}q_{2N+1} > q_{2N+1}.$$

(2) $c_{2N+2}=0$ *and* $t_{2N} > q_{2N}+q_{2N+1}$: equivalently $t_{2N+1} > q_{2N+1}+q_{2N+2}$. So $t^{**}_{2N} = t_{2N} - q_{2N} > q_{2N}$ and $t^{**}_{2N+1} = t_{2N+1} - q_{2N+2} > q_{2N+1}$.

(3) $c_{2N+2}=0$ *and* $t_{2N} \leqslant q_{2N}+q_{2N+1}$: now

$$t^{**}_{2N} = t_{2N} = t_{2N-1} + (a_{2N+1} - c_{2N+1})\, q_{2N} \geqslant q_{2N}$$

(with strict inequality for $N > m$)

$$\begin{aligned} t^{**}_{2N+1} = t_{2N+1} &= t_{2N} + (a_{2N+2} + c_{2N+2})\, q_{2N+1} \\ &= t_{2N} - q_{2N} + q_{2N+2} \\ &= t^{**}_{2N} + q_{2N+2} - q_{2N} > q_{2N+2}. \end{aligned}$$

This covers all the cases. ∎

PROPOSITION A3.3. $t^{**}_N \leqslant q_N + q_{N+1}$ $(N \geqslant -1)$.

Proof. Again we proceed by cases inductively (in pairs $2N$, $2N+1$). First note that $t_{-1} = 1 = q_{-1}$ so that $t^{**}_{-1} \leqslant q_{-1} + q_0$. We will show that $t^{**}_{2N-1} \leqslant q_{2N-1} + q_{2N}$ implies the result for $2N$ and $2N+1$.

(1) $c_{2N+2} > 0$: we have

$$\begin{aligned} t^{**}_{2N} = t_{2N} - q_{2N} &= t_{2N-1} - q_{2N} + (a_{2N+1} - c_{2N+1})\, q_{2N} \\ &\leqslant t_{2N-1} - q_{2N} + q_{2N+1} - q_{2N-1} \\ &= t^{**}_{2N-1} + q_{2N+1} - q_{2N-1} \\ &\leqslant q_{2N} + q_{2N+1}. \end{aligned}$$

Similarly

$$\begin{aligned} t^{**}_{2N+1} = t_{2N+1} - q_{2N+2} &= t_{2N} - q_{2N} + c_{2N+2}q_{2N+1} \\ &\leqslant t^{**}_{2N} + a_{2N+2}q_{2N+1} \\ &\leqslant q_{2N+1} + q_{2N} + q_{2N+2} - q_{2N} \\ &= q_{2N+1} + q_{2N+2}. \end{aligned}$$

(2) $c_{2N+2} = 0$ *and* $t_{2N} \leqslant q_{2N} + q_{2N+1}$: then $t_{2N+1} \leqslant q_{2N+1} + q_{2N+2}$ so $t^{**}_{2N} = t_{2N}$, $t^{**}_{2N+1} = t_{2N+1}$ are as required.

(3) $c_{2N+2}=0$ *and* $t_{2N}>q_{2N}+q_{2N+1}$: then $t^{**}_{2N}=t_{2N}-q_{2N}=t_{2N+1}-q_{2N+2}=t^{**}_{2N+1}$ and

$$\begin{aligned} t^{**}_{2N+1}=t^{**}_{2N}&=t^{**}_{2N-1}-q_{2N}+(a_{2N+1}-c_{2N+1})\,q_{2N}\\ &\leqslant t^{**}_{2N-1}+a_{2N+1}q_{2N}\leqslant q_{2N-1}+q_{2N}+q_{2N+1}-q_{2N-1}\\ &=q_{2N}+q_{2N+1}. \quad \blacksquare \end{aligned}$$

Finally

PROPOSITION A3.4. *For* $N\geqslant 2m+1$ *if* $a_{2m+1}>c_{2m+1}$

$$t^{**}_N=t^*_N, \qquad s^{**}_N=s^*_N. \tag{A3.1}$$

Proof. Apply Proposition A1.2 and observe that

(i) $t_{2N}\alpha+\gamma-s_2N=(1+\gamma_{2N+1}-\alpha_{2N+1})\,\varepsilon_{2N}$

(ii) $(t_{2N}-q_{2N})\,\alpha+\gamma-(s_{2N}-p_{2N})=(\gamma_{2N+1}-\alpha_{2N+1})\,\varepsilon_{2N}$

(iii) $(t_{2N}-2q_{2N})\,\alpha+\gamma-(s_{2N}-2p_{2N})=(\gamma_{2N+1}-\alpha_{2N+1}-1)\,\varepsilon_{2N}<0$

(iv) $(t_{2N+1}-q_{2N+2})\,\alpha+\gamma-(s_{2N+1}-p_{2N+2})=(1-\gamma_{2N+2})\,\varepsilon_{2N+1}<0$

(v) $(t_{2N+1}-q_{2N+1}-q_{2N+2})\,\alpha+\gamma-(s_{2N+1}-p_{2N+1}-p_{2N+2})=-\gamma_{2N+2}\varepsilon_{2N+1}>0.$

We first show (A3.1) for N even.

(1) *Suppose* $c_{2N+2}>0$: then $\alpha_{2N+1}\leqslant\gamma_{2N+1}$ and so (ii) is non-negative. This, with (iii), shows that

$$[(t_{2N}-2q_{2N})\,\alpha+\gamma]=s_{2N}-2p_{2N}-1$$
$$[(t_{2N}-2q_{2N})+\alpha+\gamma+\varepsilon_{2N}]=s_{2N}-2p_{2N}.$$

Since $t_{2N}-q_{2N}=t^{**}_{2N}\in\,]q_{2N},q_{2N}+q_{2N+1}]$, $n:=t_{2N}-2q_{2N}$ lies in $[1,q_{2N+1}]$ and so by Lemma A2.1, $n=n_N$ and

$$t^{**}_{2N}=t_{2N}-q_{2N}=n_{2N}+q_{2N}=t^*_{2N}$$
$$s^{**}_{2N}=s_{2N}-p_{2N}=m_{2N}+p_{2N}=s^*_{2N}.$$

(2) *Suppose* $c_{2N+2}=0$ *and* $t_{2N}\leqslant q_{2N}+q_{2N+1}$: now $\alpha_{2N+1}<\gamma_{2N+1}$ and (i) and (ii) show

$$[(t_{2N}-q_{2N})\,\alpha+\gamma]=s_{2N}-p_{2N}-1$$
$$[(t_{2N}-q_{2N})\,\alpha+\gamma+\varepsilon_{2N}]=s_{2N}-p_{2N}.$$

Again $q_{2N} < t_{2N} - q_{2N} + q_{2N+1}$ so $t_{2N} - q_{2N}$ is the only candidate for n_{2N} and

$$t_{2N}^{**} = t_{2N} = n_{2N} + q_{2N} = t_{2N}^{*}$$

$$s_{2N}^{**} = s_{2N} = m_{2N} + p_{2N} = s_{2N}^{*}.$$

(3) *Suppose* $c_{2N+2} = 0$ *and* $t_{2n} \geqslant q_{2N} + q_{2N+1}$: in this case

$$t_{2N+1}^{**} = t_{2N+1} - q_{2N+2} = t_{2N} - q_{2N} = t_{2N}^{**} = q_{2N} + q_{2N+1}.$$

Then (iv) and (v) yield

$$[(t_{2N+1} - q_{2N+1} - q_{2N+2})\alpha + \gamma] = s_{2N+1} - p_{2N+1} - p_{2N+2}$$

$$[(t_{2N+1} - q_{2N+1} - q_{2N+2})\alpha + \gamma + \varepsilon_{2N+1}] = s_{2N+1} - p_{2N+1} - p_{2N+2} - 1.$$

By Propositions A3.2 and A3.3, $t_{2N+1}^{**} > q_{2N+2}$ and so

$$t_{2N+1}^{**} - q_{2N+1} = n_{2N+1} \leqslant q_N.$$

Hence, by Proposition A2.2,

$$n_{2N} = n_{2N+1} + q_{2N+1} = t_{2N+1}^{**} = t_{2N+1} - q_{2N+2} = t_{2N} - q_{2N}$$

and

$$t_{2N}^{**} = t_{2N} - q_{2N}.$$

In particular $t_{2N}^{*} = n_{2N} = t_{2N}^{**}$. Similarly

$$s_{2N}^{*} = m_{2N} = m_{2N+1} + p_{2N+1} = s_{2N+1} - p_{2N+2} = s_{2N} = p_{2N} = s_{2N}^{**}.$$

Finally, we treat the case in which N is odd. We have (iv), (v), and

(vi) $(t_{2N+1} - q_{2N+1})\alpha + \gamma - (s_{2N+1} - p_{2N+1})$
$= -\varepsilon_{2N+1}(\gamma_{2N+2} + \alpha_{2N+2}) > 0$

(vii) $t_{2N+1}\alpha + \gamma - s_{2N+1} = (1 - \gamma_{2N+2} - \alpha_{2N+2})\,\varepsilon_{2N+1}$

(viii) $(t_{2N+1} + q_{2N+1})\alpha + \gamma - (s_{2N+1} + p_{2N+1})$
$= (2 - \gamma_{2N+2} - \alpha_{2N+2})\,\varepsilon_{2N+1} < 0.$

We consider three more cases:

Case 1. $t_{2N+1} \leqslant q_{2N+1} + q_{2N+2}$ and $\alpha_{2N+2} + \gamma_{2N+2} > 1$. Then $c_{2N+2} = 0$ (else $t_{2N+1}^{**} = t_{2N+1} - q_{2N+2} > q_{2N+1}$). Now t_{2N+1} is a "violator," hence either $t_{2N+1} = n_{2N+1}$ or $t_{2N+1} = n_{2N+1} + kq_{2N+2} + sq_{2N+1}$ $(s \geqslant 0,\ k \geqslant 1)$. Thus $s = 0$ and $k = 1$, whence

$$t_{2N+1} - q_{2N+2} = n_{2N+1}.$$

Since $c_{2N+2}=0, t_{2N} \leqslant q_{2N}+q_{2N+1}$ and the even analysis (2) yields $t_{2N}-q_{2N}=n_{2N}$. Thus

$$n_{2N}=t_{2N}-q_{2N}=t_{2N+1}-q_{2N+2}=n_{2N+1}.$$

But the argument in Proposition A2.2 rules this out. Hence $n_{2N+1}=t_{2N+1}$.

Case 2. $t_{2N+1} \leqslant q_{2N+1}+q_{2N+2}$ and $\alpha_{2N+2}+\gamma_{2N+2}<1$. Then again $c_{2N+2}=0$ and $t_{2N+1}-q_{2N+1}$ is a "violator." Since $q_{2N+1}<t_{2N+1} \leqslant q_{2N+1}+q_{2N+2}$ we see that $n_{2N+1}=t_{2N+1}-q_{2N+1}$.

Case 3. $t_{2N+1}>q_{2N+1}+q_{2N+2}$. Then as before using (iv) and (v), $t_{2N+1}-q_{2N+1}-q_{2N+2}$ is a "violator" which lies in the range $[1, t^{**}_{2N+1}-q_{2N+1}]$ inside $[1, q_{2N+2}]$. Thus $n_{2N+1}=t_{2N+1}-q_{2N+1}-q_{2N+2}$.

In Case 1: $c_{2N+2}=0$ so $t^{**}_{2N+1}=t_{2N+1}=n_{2N+1}$ and $n_{2N+1}+q_{2N+1}=t_{2N+1}+q_{2N+1}>q_{2N+2}+q_{2N+1}$ by Lemma A3.1. Thus

$$t^{**}_{2N+1}=n_{2N+1}=t^{*}_{2N+1}.$$

In Case 2: $n_{2N+1}=t_{2N+1}-q_{2N+1}=t^{**}_{2N+1}-q_{2N+1}$ (again since $c_{2N+2}=0$). Hence

$$q_{2N+1}+q_{2N+2} \geqslant t^{**}_{2N+1}=n_{2N+1}+q_{2N+1}=t^{*}_{2N+1}.$$

In Case 3: $t^{**}_{2N+1}=t_{2N+1}-q_{2N+2}=n_{2N+1}+q_{2N+1}=t^{*}_{2N+1}$.

Similarly $s^{**}_{2N+1}=s^{*}_{2N+1}$ in each case.

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References

1. P. E. Böhmer, Über die Transzendenz gewisser dyadischer Brüche, *Math. Ann.* **96** (1927), 367–377.
2. J. M. Borwein and P. B. Borwein, "Pi and the AGM—A Study in Analytic Number Theory and Computational Complexity," Wiley, New York, 1987.
3. J. M. Borwein and P. B. Borwein, Strange series and high precision fraud, *MAA Monthly* **99** (1992), 622–640.
4. A. J. Brentjes, "Multi-dimensional Continued Fraction Algorithms," Mathematical Center Tracts, Vol. 145, Math. Center, Amsterdam, 1981.
5. J. W. S. Cassels, Über $\lim_{x \to +\infty} x|\theta x+\alpha-y|$, *Math. Ann.* **127** (1954), 288–304.
6. J. L. Davison, A series and its associated continued fraction, *Proc. Amer. Math. Soc.* **63** (1977), 26–32.

7. A. S. FRAENKEL, The bracket function and complementary sets of integers, *Canad. J. Math.* **21** (1969), 6–27.
8. R. L. GRAHAM, D. E. KNUTH, AND O. PATASHNIK, "Concrete Mathematics," Addison-Wesley, Reading, MA, 1989.
9. G. H. HARDY AND E. M. WRIGHT, "An Introduction to the Theory of Numbers," 4th ed., Oxford Univ. Press, London, 1960.
10. G. H. HARDY AND J. E LITTLEWOOD, Notes on the Theory of Series (XXIV): A curious power series, *Proc. Cambridge Philos. Soc.* **42** (1946), 85–90.
11. J. H. LOXTON AND A. J. VAN DER POORTEN, Arithmetic properties of certain functions in several variables, III, *Bull. Austral. Math. Soc.* **16** (1977), 15–67.
12. J. H. LOXTON AND A. J. VAN DER POORTEN, Transcendence and algebraic independence by a method of Mahler, *in* "Transcendence Theory—Advances and Applications" (A. Baker and D. W. Masser, Eds.), pp. 211–226, Academic Press, San Diego, CA, 1977.
13. K. MAHLER, Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen, *Math. Ann.* **101** (1929), 342–366.
14. L. J. MORDELL, Irrational power series, II, *Acta Arith.* **11** (1965), 181–188.
15. M. NEWMAN, Irrational power series, *Proc. Amer. Math. Soc.* **11** (1960), 699–702.
16. I. J. SCHOENBERG, "Mathematical time exposures," Math. Assoc. Amer., Washington, DC, 1982.
17. M. WALDSCHMIDT, Nombres transcendents, *in* "Lecture Notes in Mathematics, Vol. 402," Springer-Verlag, New York, 1974.
18. H. S. WALL, "Analytic Theory of Continued Fractions," Van Nostrand, Toronto/New York/London, 1948.
19. R. WALLISSER, Eine Bemerkung über irrationale Werte und Nichfortsetzbarkeit von Potenzreihen mit ganzzahligen Koeffizienten, *Colloq. Math.* **23** (1971), 141–144.