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An Interesting Infinite Product*

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In a note published in the Monthly some time ago, Z. A. Melzak [4] proved that

$$\frac{\pi}{2e}=\prod_{n=1}^{\infty}\left(1+\frac{2}{n}\right)^{(-1)^{n+1}n}$$

and

$$\frac{6}{\pi e}=\prod_{n=2}^{\infty}\left(1+\frac{2}{n}\right)^{(-1)^{n}n}.$$

Great care must be taken with these formulae given that the general term in the products behaves like $e^{\pm 2}$ and so convergence is quite problematic. An unambiguous way to rewrite this result is as

$$\lim_{n\to\infty}\prod_{k=1}^{2n+1}\left(1+\frac{2}{k}\right)^{(-1)^{k+1}k}=e^{2}\lim_{n\to\infty}\prod_{k=1}^{2n}\left(1+\frac{2}{k}\right)^{(-1)^{k+1}k}=\frac{\pi e}{2}.$$

We generalize this surprising product by considering the function

$$D(x):=\lim_{n\to\infty}\prod_{k=1}^{2n+1}\left(1+\frac{x}{k}\right)^{(-1)^{k+1}k} \tag{1}$$

and provide explicit evaluations for all rational numbers of the form p/q with $q=1$, 2, or 3. For example,

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$$D\left(\frac{1}{2}\right)=\frac{2^{1/6}\sqrt{\pi}\,A_1^3}{\Gamma(1/4)}\,e^{G/\pi}$$

$$D\left(\frac{3\pm 1}{6}\right)=\left[\frac{\pi}{\Gamma(1/6)\,\Gamma(1/3)}\right]^{(3\pm 1)/6}\left(\frac{2^{10(3\pm 1)}}{3^{3(2\pm 1)}}\right)^{1/72}A_1^{3\pm 1}e^{5\pi\,\mathrm{Cl}_2(\pi/3)/6}$$

$$D(1)=\frac{A_1^6}{2^{1/6}\sqrt{\pi}}$$

and more generally for positive integral n

$$D(2n)=\left[\frac{(2n-1)!!}{(2n-2)!!}\right]^{2n}\frac{1}{(2n-1)!!}\prod_{k=1}^{n-1}\left(\frac{2k}{2k+1}\right)^{2k}\times\left(\frac{e\pi}{2}\right)^{n}$$

$$D(2n+1)=\left[\frac{(2n-2)!!}{(2n-1)!!}\right]^{2n}(2n)!!\prod_{k=1}^{n-1}\left(\frac{2k+1}{2k}\right)^{2k}\times\left(\frac{2e}{\pi}\right)^{n}\frac{A_1^6}{2^{1/6}\sqrt{\pi}}.$$

Here as usual $(2n)!!=2n(2n-2)\cdots 4\cdot 2$ and $(2n-1)!!=(2n-1)\times(2n-3)\cdots 3\cdot 1$. Also we have introduced Glaisher's and Catalan's constants A_1 and G, respectively, and Clausen's integral

$$\mathrm{Cl}_2(\theta):=-\int_0^{\theta}\log\left|2\sin\frac{\phi}{2}\right|\,d\phi.$$

The constants A_1 and G are defined by the relations

$$A_1:=\exp\left\{\frac{1}{4}-\int_0^{\infty}\frac{e^{-s}}{s^3}\left[1-\frac{s}{2}+\frac{s^2}{12}-\frac{s}{e^s-1}\right]ds\right\}=1.28243\cdots$$

$$G:=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=0.91597\cdots.$$

The key to these evaluations is to find a closed form expression for the product in Eq. (1). This is the content of our main theorem.

THEOREM. *For* $x>0$

$$D(x)=\left[\frac{\Gamma(x/2+1/2)}{\Gamma(x/2)}\right]^{x}\left[\frac{\Gamma_1(x/2)\,\Gamma_1(1/2)}{\Gamma_1(x/2+1/2)}\right]e^{x/2}. \tag{2}$$

Proof. Before we proceed, we introduce the generalized gamma function [2] which is defined by the relation

$$\log\Gamma_1(x):=\int_0^{x}\log\Gamma(t)\,dt+\frac{x^2}{2}-\frac{x}{2}-\frac{x}{2}\log 2\pi \tag{3}$$

and which enjoys the properties

$$\Gamma_1(x+1) = x^x \Gamma_1(x) \tag{4}$$

$$\frac{1}{n} \log \Gamma_1(nx) = \sum_{k=0}^{n-1} \log \Gamma_1\left(x + \frac{k}{n}\right) - \frac{n^2-1}{n}\left(\log A_1 - \frac{1}{12}\right) + \left(\frac{nx^2}{2} - \frac{x}{2} + \frac{1}{12n}\right) \log n \tag{5}$$

and

$$\frac{\Gamma_1(1-x)}{\Gamma_1(x)} = \exp\left(-\frac{1}{2\pi} \operatorname{Cl}_2(2\pi x)\right). \tag{6}$$

Note that Eq. (5) is the analogue of Gauss' multiplication formula for the gamma function. The derivation of Eq. (2) follows from the observation that

$$\frac{d}{dz} \int_z^{z+1} \log \Gamma(x+t)\, dt = \log(x+z)$$

(this and other well known formulae for Γ may be found in [1]). On integrating this equation we find that

$$\int_z^{z+1} \log \Gamma(x+t)\, dt = (x+z) \log(x+z) - x - z + \int_0^1 \log \Gamma(t)\, dt$$

(the value of the definite integral of the right hand member can be easily shown to be $\frac{1}{2} \log 2\pi$). We now allow z to be an integer variable and sum from 0 to n. After some algebra we find an expression of the form

$$\begin{aligned} Di_n(x) &:= \sum_{k=1}^{n} k \log\left(1 + \frac{x}{k}\right) \\ &= \int_0^{n+1} \log \frac{\Gamma(x+t)}{\Gamma(t)}\, dt - x \log \frac{\Gamma(x+n+1)}{\Gamma(x)} + (n+1)x. \end{aligned}$$

Therefore the logarithm of the required finite product is

$$\log D_n(x) := \log \frac{\prod_{k=0}^{n} (1 + x/(2k+1))^{2k+1}}{\prod_{k=1}^{n} (1 + x/2k)^{2k}} = Di_{2n+1}(x) - 4Di_{2n}\left(\frac{x}{2}\right)$$

which after some manipulation leads us to

$$D_n(x) = \frac{[\Gamma(x/2+1/2)/\Gamma(x/2)]^x \exp\{-2 \int_0^{x/2} \log(\Gamma(t+1/2)/\Gamma(t))\, dt\}}{[\Gamma(n+x/2+3/2)/\Gamma(n+x/2+1)]^x \exp\{-2 \int_{n+1}^{n+1+x/2} \log(\Gamma(t+1/2)/\Gamma(t))\, dt\}}.$$

We can readily integrate the logarithmic integrals using Eq. (3) and we find

$$D_n(x)=\left[\frac{\Gamma(x/2+1/2)}{\Gamma(x/2)}\right]^x\left[\frac{\Gamma_1(x/2)\,\Gamma_1(1/2)}{\Gamma_1(x/2+1/2)}\right]\left[\frac{\Gamma(n+x/2+1)}{\Gamma(n+x/2+3/2)}\right]^x$$
$$\times\left[\frac{\Gamma_1(n+x/2+3/2)\,\Gamma_1(n+1)}{\Gamma(n+x/2+1)\,\Gamma_1(n+3/2)}\right].$$

The entire term involving n approaches $e^{x/2}$ in the limit as n tends to infinity. This follows from the asymptotic forms of the gamma functions:

$$\log\frac{\Gamma(x+a)}{\Gamma(x+b)}\sim(a-b)\log x+O\left(\frac{1}{x}\right)$$

and

$$\log\frac{\Gamma_1(x+a)}{\Gamma_1(x+b)}\sim(a-b)\left[\left(x+\frac{a+b-1}{2}\right)\log x+\frac{a+b-1}{2}\right]+O\left(\frac{1}{x}\right).$$

This concludes the proof. ∎

To see the evaluations of $D(1/2)$ and $D((3\pm 1)/2)$ of the introduction we use the Theorem and Eqs. (5) and (6). From Eqs. (5) and (6) we deduce that

$$\log\Gamma_1\left(\frac{3\pm 2}{6}\right)=\frac{5}{6}\log A_1-\frac{5}{72}+\frac{1}{144}\log 3+\frac{1}{72}\log 2\mp\frac{1}{4\pi}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)$$
$$\log\Gamma_1\left(\frac{3\pm 1}{6}\right)=\frac{4}{3}\log A_1-\frac{1}{9}-\frac{1}{72}\log 3\mp\frac{1}{6\pi}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)$$
$$\log\Gamma_1\left(\frac{2\pm 1}{4}\right)=\frac{9}{8}\log A_1-\frac{3}{32}\mp\frac{G}{\pi}$$
$$\log\Gamma_1\left(\frac{1}{2}\right)=\frac{3}{2}\log A_1-\frac{1}{8}-\frac{1}{24}\log 2.$$

These couple with Eq. (4) and the formula $\Gamma(x+1)=x\Gamma(x)$ to allow us to evaluate $D(p/q)$, $q=2$ or 3. Explicitly, at integer values we use the following relations. The first two are well known properties of the gamma function, the latter two are deduced from Eq. (4).

$$\Gamma(n+1)=n!$$
$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n-1)!!\,\sqrt{\pi}}{2^n}$$

$$\Gamma_1(n+1) = 1^1.2^2.3^3 \ldots n^n$$

$$\Gamma_1^2\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n} \prod_{k=1}^{n-1} \left(\frac{2k+1}{2}\right)^{2k} \Gamma_1^2\left(\frac{1}{2}\right).$$

It is interesting to note that Eq. (2) satisfies a functional equation of the form

$$D(x)\, D(x+1) = \frac{\Gamma(x/2+1)}{\Gamma(x/2+1/2)} \Gamma_1^4\left(\frac{1}{2}\right) e^{x+1/2} \tag{7}$$

and a reflection formula

$$D(x)\, D(1-x) = \frac{\pi}{\sin(\pi x/2)} \frac{\Gamma_1^4(1/2)}{\Gamma(x/2)\, \Gamma(1/2-x/2)} e^{(2/\pi)\, Ti_2(\tan(\pi x/2)) + 1/2}, \tag{8}$$

here as in [3]

$$Ti_2(x) := \int_0^x \frac{\tan^{-1} t}{t}\, dt.$$

Using Eq. (7) we can analytically continue our infinite product for all negative values. We find

$$D(-2n+x) = \frac{(2n-1)!!}{(2e)^n} \prod_{k=1}^{n-1} \left(\frac{2k+1}{2k}\right)^{2k} \times D_{n-1}(-2n+x)\, D(x) \left[\frac{\Gamma(x/2)}{\Gamma(x/2+1/2)}\right]^{2n}$$

which reveals a pole structure of order $2n$ at the negative even integers. We can also deduce $D(-2n-1)=0$ which of course agrees with Eq. (1).

We conclude by remarking that Eq. (1) conspires only to give us the constants π and e when we evaluate the product at positive even integers. Other evaluations lead to the introduction of other transcendental constants.

References

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