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Some Restricted Partition Functions

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We show that the supremum norm on the unit disk, $\{|q| \leqslant 1\}$, of the nth partial product of $\prod_{k=1, p \nmid k}^{\infty} (1-q^k)$ is asymptotic to $p^{n/(p-1)}$ for $p = 2, 3, 5, 7, 11$, and 13 (but not for any $p > 15$). This, for these primes, is an asymptotically best possible result since if $\alpha_1, ..., \alpha_n$ are integers none of which are divisible by p then $\|\prod_{k=1}^{n} (1-q^{\alpha_k})\|_{\{|q|=1\}} \geqslant p^{n/(p-1)}$. © 1993 Academic Press, Inc.

1. Introduction

In 1959 Erdős and Szekeres [3] raised the problem of estimating

$$\eta(n) := \min_{\alpha_1, ..., \alpha_n \in \mathbb{N}} \left\| \prod_{k=1}^{n} (1-q^{\alpha_k}) \right\|_{\{|q|=1\}}.$$

Here $\|f(q)\|_A := \sup_{q \in A} |f(q)|$ denotes the supremum norm of f on A. They conjectured that, for any k,

$$\eta(n) \gg n^k$$

and showed that

$$\sqrt{2n} \leqslant \eta(n) = e^{o(n)}.$$

This was improved by Atkinson [1] to

$$\eta(n) = e^{O(n^{1/2} \log n)}$$

and by Odlyzko [6] to

$$\eta(n) = e^{O(n^{1/3} (\log n)^{4/3})}.$$

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We consider a related question. Namely we estimate

$$\eta(n, p) := \min_{\substack{\alpha_1, \ldots, \alpha_n \in \mathbb{N} \\ p \nmid \alpha_i}} \left\| \prod_{k=1}^{n} (1 - q^{\alpha_k}) \right\|_{\{|q| = 1\}}$$

so we are adding the condition that no exponent be divisible by p. We show that, for p a prime,

$$p^{n/(p-1)} \leqslant \eta(n, p)$$

while for $p = 2$, 3, 5, 7, 11, and 13

$$\eta(n, p) = O(p^{n/(p-1)}).$$

The key to the upper bounds is an analysis of the function

$$F_{p,n}(q) := \prod_{k=1}^{n} (1 - q^{pk-1})(1 - q^{pk-2}) \cdots (1 - q^{pk-(p-1)}),$$

which is the generating function for the number of even partitions of m minus the number of odd partitions of m, where the parts are of size less than pn and no part is divisible by p. Note that the function

$$F_{p,\infty}(q) := \prod_{k=1}^{\infty} (1 - q^{pk-1})(1 - q^{pk-2}) \cdots (1 - q^{pk-(p-1)}) = \prod_{\substack{k=1 \\ p \nmid k}}^{\infty} (1 - q^k)$$

has a similar interpretation.

We then prove the following theorems:

THEOREM 1. *For $p = 2$, 3, 5, 7, 11, 13*

$$\left\| F_{p,n}(q) \right\|_{\{|q| = 1\}} = p^n \left(1 + O\left(\frac{1}{n}\right)\right).$$

So $F_{p,n}$ asymptotically achieves the minimum for these p. Note that $F_{p,n}$ is a product of $n(p-1)$ terms. While for $p \geqslant 15$ we have

THEOREM 2. *For any fixed positive integer $p \geqslant 15$*

$$\|F_{p,n}(q)\|_{\{|q| = 1\}} \gtrsim (1.219\ldots)^{(p-1)n} > p^n.$$

Sudler [7] and Wright [8, 9] analysed the rate of growth of

$$\mu(n) := \left\| \prod_{k=1}^{n} (1 - q^k) \right\|_{\{|q| = 1\}}$$

and showed among other things that

$$(\mu(n))^{1/n} \to 1.219....$$

So for $p < 15$, the rate of growth for $F_{p,n}$ is worse than that for the above full product.

The lower bounds for $\eta(n, p)$ follow from the following proposition.

THEOREM 3. *Let $P(x)$ be any polynomial with integer coefficients and with a zero of order n at 1. Suppose that ζ_1 is a primitive mth root of unity and that $P(\zeta_1) \neq 0$. Then*

$$\max_{\zeta_k} |P(\zeta_k)|^{1/n} \geq |C_m(1)|^{1/\phi(m)}.$$

Here ζ_k are the conjugate roots to ζ_1 and C_m is the mth cylotomic polynomial. So $C_m(x) := \prod_{k=1}^{\phi(m)} (x - \zeta_k)$. Also, $\phi(m)$ is the Euler ϕ function.

Proof. By assumption

$$P(x) = (x-1)^n \, Q(x),$$

where Q has integer coefficients. Thus

$$\left| \prod_{k=1}^{\phi(m)} P(\zeta_k) \right| = \left| \prod_{k=1}^{\phi(m)} (1-\zeta_k)^n \, Q(\zeta_k) \right| = |C_m(1)|^n \, I,$$

where I is a non-zero integer. It follows that

$$\max_{\zeta_k} |P(\zeta_k)|^{\phi(m)} \geq |C_m(1)|^n. \quad \blacksquare$$

This proof method was suggested by B. Richmond and L. Szekely.

INEQUALITY 1. *Let $P(x)$ be a polynomial with integer coefficients and with a zero of order n at 1. Suppose that ζ_1 is a primitive p^αth root of unity for some prime p and $P(\zeta_1) \neq 0$. Then*

$$\max_{\zeta_k} |P(\zeta_k)|^{1/n} \geq p^{1/(p^\alpha - p^{\alpha-1})}.$$

Proof. The value of $C_m(1)$ is given by

$$C_m(1) = \begin{cases} p & \text{if } \; m = p^k, \, p \text{ prime} \\ 1 & \text{otherwise} \end{cases}$$

and the result follows. $\blacksquare$

2. Estimates of $|\prod_{k=1}^{n} \sin(k\theta+\gamma)|$

We proceed to estimate various sin products. For this we use the Farey decomposition. Namely for fixed n, if $\theta \in [0, 1)$ then for some ε, $|\varepsilon| \leqslant 1$,

$$\theta = \frac{s}{q} + \frac{\varepsilon}{q(n+1)}, \qquad \text{where } \begin{cases} q = 1 \text{ and } s = 0 \text{ or } 1 \\ \text{or} \\ 2 \leqslant q \leqslant n,\ 1 \leqslant s < q,\ (s, q) = 1. \end{cases}$$

The estimates, as in [7], are different for small q and large q and break into three parts which comprise the first few lemmas. The sin product we wish to estimate is

$$\prod_{k=1}^{n} |\sin(k\theta + \gamma)|.$$

Lemma 1 provides an estimate for $q \geqslant 50$.

Lemma 1. *Suppose* $\theta = s/q + \varepsilon/q(n+1)$, *where* $0 < s < q \leqslant n$, $-1 \leqslant \varepsilon \leqslant 1$, *and* $(s, q) = 1$. *Then for* $q \geqslant 50$ and any real δ, $\prod_{k=1}^{n} |\sin(k\theta + \delta)| \leqslant (0.6)^n$.

Proof. Since $|\sin(\alpha + \beta)| \leqslant |\sin \alpha| + |\beta|$

$$\prod_{k=1}^{n} |\sin(k\theta\pi + \delta)| \leqslant \prod_{k=1}^{n} \min \left\{ \left| \sin\left(\frac{s}{q} k\pi + \delta\right) \right| + \frac{\pi}{q}, 1 \right\}. \tag{1}$$

Now observe that for $0 \leqslant m \leqslant q-1$

$$I_m := \left\{ \left(k\pi \frac{s}{q} + \delta \right) \bmod \pi \mid k = 1, 2, ..., n \right\} \cap \left(\frac{m\pi}{q}, \frac{(m+1)\pi}{q} \right]$$

has cardinality either $\lfloor n/q \rfloor$ or $\lfloor n/q \rfloor + 1$, because if

$$\left(k_1 \pi \frac{s}{q} + \delta \right) \bmod \pi \qquad \text{and} \qquad \left(k_2 \pi \frac{s}{q} + \delta \right) \bmod \pi$$

both lie in $(m\pi/q, (m+1)\pi/q]$ then

$$\left| (k_1 - k_2)\pi \frac{s}{q} + h\pi \right| < \frac{\pi}{q} \qquad \text{for some } h$$

and

$$|(k_1 - k_2) + hq| < 1.$$

Thus $k_1 \equiv k_2 \bmod q$ and the rest follows from the pigeon hole principle. From (1) and the above we have

$$\prod_{k=1}^{n} |\sin(k\theta\pi + \delta)| \leqslant \prod_{k=1}^{n} \min\left\{\left|\sin\left(\frac{s}{q}k\pi + \delta\right)\right| + \frac{\pi}{q}, 1\right\}$$

$$\leqslant \prod_{k=1}^{q-1}\left(\left|\sin\left(\frac{k\pi}{q}\right)\right| + \frac{\pi}{q}\right)^{\lfloor n/q \rfloor} \tag{2}$$

(Here we have estimated sin on all the partition points of I_m by the value at an endpoint except for those in the interval around $\pi/2$, where we have used 1. The extra term in some of the I_m is also estimated by 1.) For large q (odd) one uses

$$\prod_{k=1}^{q-1}\left(\left|\sin\left(\frac{k\pi}{q}\right)\right| + \frac{\pi}{q}\right) \leqslant \prod_{k=1}^{(q-1)/2}\left(\left|\sin\left(\frac{k\pi}{q}\right)\right| + \frac{\pi}{q}\right)^2 \leqslant \prod_{k=1}^{(q-1)/2}\left(\frac{(k+1)\pi}{q}\right)^2$$

$$= \frac{\left(\left(\frac{q-1}{2}+1\right)!\right)^2 \pi^{q-1}}{q^{q-1}}$$

and

$$\left(\frac{\left(\left(\frac{q-1}{2}+1\right)!\right)^2 \pi^{q-1}}{q^{q-1}}\right)^{1/q} \sim \frac{\pi}{2e} = 0.5778....$$

The asymptotic is the same for even q. This, with some care over initial estimates, gives the result. ∎

The next three lemmas give estimates for $2 \leqslant q \leqslant 50$. Let

$$I(\alpha, \beta, \gamma) := \int_0^1 |\sin(\alpha\pi t + \beta)|^{\gamma}\, dt$$

and

$$S_M(\alpha, \beta, \gamma) := \frac{1}{M}\sum_{k=1}^{M}\left|\sin\left(\alpha\pi\frac{k}{M} + \beta\right)\right|^{\gamma} dt.$$

LEMMA 2. *For $1 \geqslant \gamma > 0$ there exist a constant c_γ independent of α and β and M so that*

$$|I(\alpha, \beta, \gamma) - S_M(\alpha, \beta, \gamma)| \leqslant c_\gamma\left(\frac{\alpha}{M}\right)^{\gamma}.$$

Proof.

$$|I(\alpha,\beta,\gamma)-S_M(\alpha,\beta,\gamma)| \leqslant \sup_{|t_1-t_2|\leqslant 1/M} \left| |\sin(\alpha\tau_1\pi+\beta)|^{\gamma}-|\sin(\alpha\tau_2\pi+\beta)|^{\gamma}\right|$$

$$\leqslant c_{\gamma}\left(\frac{\alpha}{M}\right)^{\gamma}. \quad \blacksquare$$

LEMMA 3. *For* $50\geqslant q\geqslant 2$, $(s,q)=1$, *and any real* ζ

$$\left(\frac{1}{q}\sum_{j=1}^{q}\left|\sin\left(\zeta+\frac{js\pi}{q}\right)\right|^{1/1000}\right)^{1000}\leqslant\begin{cases}0.71, & q=2\\ 0.635, & q=3\\ 0.6, & q\geqslant 4.\end{cases}$$

Proof. Extensive but straightforward numerical calculation. ∎

LEMMA 4. *Suppose* $\theta=s/q+\varepsilon/q(n+1)$, *where* $q\geqslant 2$, $0<s<q\leqslant n$, $(s,q)=1$, *and* $-1\leqslant\varepsilon\leqslant 1$. *Then independently of* θ *and* δ,

$$\left(\prod_{k=1}^{n}|\sin(k\theta\pi+\delta)|\right)^{1/n}\lesssim\begin{cases}0.6, & q\geqslant 4\\ 0.635, & q=3\\ 0.71, & q=2.\end{cases}$$

Proof. We may, by Lemma 1, assume that $q\leqslant 50$. By the extended arithmetic–geometric mean inequality

$$\left(\prod_{k=1}^{n}|\sin(k\theta\pi+\delta)|\right)^{\gamma/n}\leqslant\frac{1}{n}\sum_{k=1}^{n}|\sin(k\theta\pi+\delta)|^{\gamma}$$

$$=\frac{1}{n}\sum_{k=1}^{n}\left|\sin\left(\frac{ks\pi}{q}+\frac{k\varepsilon\pi}{q(n+1)}+\delta\right)\right|^{\gamma}.$$

Now we divide the sum into q parts by

$$\left\{\frac{ks\pi}{q}+\delta+\frac{k\varepsilon\pi}{q(n+1)}\right\}_{k=1}^{n}:=I_1\cup I_2\cup\cdots\cup I_q,$$

where

$$I_j:=\left\{\frac{(kqs+js)\,\pi}{q}+\delta+\frac{(kq+j)\,\varepsilon\pi}{q(n+1)}\right\}_{k=0}^{\lfloor (n-j)/q\rfloor}$$

and

$$I_j \bmod \pi = =\left\{\frac{js\pi}{q}+\delta+\frac{(kq+j)\,\varepsilon\pi}{q(n+1)}\right\}_{k=0}^{\lfloor (n-j)/q\rfloor}.$$

so

$$\frac{1}{n}\sum_{k=1}^{n}\left|\sin\left(\frac{ks\pi}{q}+\frac{k\varepsilon\pi}{q(n+1)}+\delta\right)\right|^{\gamma}$$

$$=\frac{1}{n}\left(\sum_{j=1}^{q}\left\lfloor\frac{n-j}{q}\right\rfloor\cdot\left(\frac{1}{\lfloor(n-j)/q\rfloor}\sum_{k=0}^{\lfloor(n-j)/q\rfloor}\left|\sin\left(\frac{js\pi}{q}+\delta+\frac{(kq+j)\,\varepsilon\pi}{q(n+1)}\right)\right|^{\gamma}\right)$$

$$\lesssim\frac{1}{n}\sum_{j=1}^{q}\left\lfloor\frac{n}{q}\right\rfloor\left(\int_0^1\left|\sin\frac{\varepsilon\pi t}{q}+\frac{js\pi}{q}+\delta\right|^{\gamma}dt\right)$$

$$\lesssim\frac{1}{q}\sum_{j=1}^{q}\int_0^1\left|\sin\left(\frac{\varepsilon\pi t}{q}+\frac{js\pi}{q}+\delta\right)\right|^{\gamma}dt,$$

where the penultimate inequality is essentially Lemma 2. Thus for some $\rho'\in[0,1]$ from the above we have, by replacing the integral by its maximum,

$$\left(\prod_{k=1}^{n}|\sin(k\theta\pi+\delta)|\right)^{\gamma/n}\lesssim\frac{1}{q}\sum_{j=1}^{q}\left|\sin\left(\frac{\rho\pi}{q}+\delta+\frac{js\pi}{q}\right)\right|^{\gamma}$$

$$=\frac{1}{q}\sum_{j=1}^{q}\left|\sin\left(\rho'+\frac{js\pi}{q}\right)\right|^{\gamma}.$$

Now with $\gamma=1/1000$ and Lemma 3, we obtain the result. ∎

3. Some Additional Lemmas

As before let

$$F_{p,n}(q):=\frac{\prod_{k=1}^{pn}(1-q^k)}{\prod_{k=1}^{n}(1-q^{pk})}.$$

Note that

$$\|F_{p,n}(q)\|_{\{|q|=1\}}=\left\|2^{(p-1)n}\frac{\prod_{k=1}^{pn}\sin(k\theta\pi)}{\prod_{k=1}^{n}\sin(kp\theta\pi)}\right\|_{[0,1]}.$$

Lemma 5. *If p is a positive integer then*

$$\|F_{p,n}(q)\|_{\{|q|=1\}}\geqslant p^n.$$

Proof. This is immediate from evaluation at a primitive pth root of unity. ∎

This is a special (easy) case of Inequality 1 when p is prime.

LEMMA 6 (Sudler [7]).

$$\lim_{n\to\infty} \left\| \prod_{k=1}^{n} (1-q^k) \right\|_{\{|q|=1\}}^{1/n} = 1.2197....$$

LEMMA 7. For fixed $p \geqslant 2, 3$ ... $\|F_{p,n}(q)\|_{\{|q|=1\}}^{1/(p-1)n} \gtrsim 1.2197....$

Note that $p^{1/(p-1)} < 1.2197$ for $p \geqslant 15$ so Theorem 1 cannot hold for $p \geqslant 15$.

Proof.

$$\|F_{p,n}(q)\|_{\{|q|=1\}} \geqslant \frac{\|\prod_{k=1}^{pn} (1-q^k)\|_{\{|q|=1\}}}{\|\prod_{k=1}^{n} (1-q^{pk})\|_{\{|q|=1\}}},$$

so with Lemma 6

$$\|F_{p,n}(q)\|_{\{|q|=1\}}^{1/(p-1)n} \gtrsim \frac{(1.2197...)^{p/(p-1)}}{(1.2197...)^{1/(p-1)}} = (1.2197...). \quad \blacksquare$$

LEMMA 8. *Let* α *and* β *be integers*

$$\left\| \prod_{k=1}^{n} (1-q^{(\alpha k+\beta)}) \right\|_{\{|q|=1\}}^{1/n} \gtrsim (1.2197...).$$

Proof. As for Lemma 7. ▮

4. THE ANALYSIS OF $F_{p,n}$ AWAY FROM pTH ROOTS OF UNITY

Let

$$B_{p,n}(\theta) := 2^{(p-1)n} \prod_{k=1}^{n} \prod_{j=1}^{p-1} \left(\sin k\theta\pi + \frac{j\theta\pi}{p} \right).$$

Then

$$\|F_{p,n}(q)\|_{\{|q|=1\}} = \|B_{p,n}(\theta)\|_{[0,p]}.$$

LEMMA 9. *Fix* p (*not necessarily prime*). *For* $\theta = s/q + \varepsilon/q(n+1)$, $(s,q)=1$, $|\varepsilon|<1$.

$$|B_{p,n}(\theta)|^{1/(p-1)n} \lesssim \begin{cases} 1.2, & q \geqslant 4 \\ 1.27, & q=3 \\ 1.42, & q=2 \end{cases}$$

so

$$|B_{p,n}(\theta)|^{1/(p-1)n} \lesssim p^{1/(p-1)} \qquad \textit{if} \quad \begin{cases} p \leqslant 16 & \textit{and} \quad q \geqslant 4 \\ p \leqslant 11 & \textit{and} \quad q \geqslant 3 \\ p < 6 & \textit{and} \quad q \geqslant 2. \end{cases}$$

Proof.

$$|B_{p,n}(\theta)|^{1/(p-1)n} \leqslant 2\left(\prod_{j=1}^{p-1} |S_{n,j}(\theta)|\right)^{1/(p-1)n},$$

where

$$S_{n,j}(\theta) = \prod_{k=1}^{n} \sin\left(k\theta\pi + \frac{j\theta\pi}{p}\right).$$

Now apply Lemma 4 to $S_{n,j}(\theta)$. ∎

We are reduced to analyzing $B_{p,n}(\theta)$ for $\theta = s/q + \varepsilon/(n+1)\,q$ for $q = 1, 2,$ and 3.

LEMMA 10. *Let $\gamma > 0$. Let $\theta = s/q + \varepsilon/(n+1)\,q$, $0 < s < pq - 1$ and $(s, q) = 1$. Then if $q \leqslant Q$ for some fixed Q*

(a) $|B_{p,n}(\theta)|^{1/(p-1)n}$

$$\lesssim 2\left(\frac{1}{p-1} \sum_{j=1}^{p-1} \frac{1}{q} \sum_{i=1}^{q} \int_0^1 \left|\sin\left(\frac{\varepsilon\pi\tau}{q} + \frac{is\pi}{q} + \frac{js\pi}{pq}\right)\right|^{\gamma} dt\right)^{1/\gamma}$$

$$\lesssim 2\left(\max_{\tau \in [-1,1]} \frac{1}{q(p-1)} \sum_{j=1}^{p-1} \sum_{i=1}^{q} \left|\sin\left(\frac{\pi}{q}\left(\tau + is + \frac{js}{p}\right)\right)\right|^{\gamma}\right)^{1/\gamma}.$$

For $q \geqslant 2$,

$$\text{(b)} \qquad |B_{p,n}(\theta)|^{1/(p-1)n} \lesssim \begin{cases} 1.42, & p = 3 \\ 1.37, & p = 4 \\ 1.25, & p = 5 \\ 1.2, & 50 \geqslant p \geqslant 6. \end{cases}$$

Proof. Part (a) is estimating exactly as in the proof Lemma 4 (except now $\delta := (j\theta\pi)/p$ and we have $p-1$ terms in the sum).

For $p \leqslant 50$, part (b) is now just Lemma 9 and an extensive numerical check involving finding the max over τ in (a) with $\gamma = 1/1000$ for the q values not covered by the lemma. ∎

We are reduced to considering the case $q := 1$, which breaks into two subcases corresponding to whether $s = 0$ or $s \neq 0$.

LEMMA 11 ($s = 0$). *If* $|\theta| \leqslant 1/(n+1)$ *then* $|B_{p,n}(\theta)|^{1/n(p-1)} \lesssim 1.2197\ldots$.

Proof. For some ε, $|\varepsilon| \leqslant 1$, $\theta = \varepsilon/(n+1)$ and

$$\begin{aligned}|B_{p,n}(\theta)| &= 2^{(p-1)n} \prod_{k=1}^{n} \prod_{j=1}^{p-1} \left| \sin\left(\frac{k\varepsilon\pi}{n+1} + \frac{\pi j \varepsilon}{p(n+1)}\right)\right| \\ &\leqslant 2^{(p-1)n} \prod_{k=1}^{n} \prod_{j=1}^{p-1} \left(\left|\sin\left(\frac{k\varepsilon\pi}{n+1}\right)\right| + \frac{\pi}{n+1}\right) \\ &\leqslant 2^{(p-1)n} \left(\prod_{k=1}^{n} \left(\left|\sin\frac{k\varepsilon\pi}{n+1}\right| + \frac{\pi}{n+1}\right)\right)^{p-1} \\ &\leqslant 2^{(p-1)n} \left(\prod_{k=1}^{n} \left|\sin\left(\frac{k\varepsilon\pi}{n+1}\right)\right| \left(1 + \frac{\pi}{(n+1)\,|\sin k\varepsilon\pi/(n+1)|}\right)\right)^{p-1}.\end{aligned}$$

Now observe (as in [7]) that the max of $|B_{p,n}(\theta)|$ for $|\theta| \leqslant 1/(n+1)$ occurs for $\frac{1}{4} \leqslant |\varepsilon| \leqslant 1$ since otherwise all terms in the product are increasing. Thus, for some c,

$$|B_{p,n}(\theta)| \leqslant 2^{(p-1)n} \left(\prod_{k=1}^{n} \left|\sin\left(\frac{k\varepsilon\pi}{n+1}\right)\right| \left(1 + \frac{c\pi}{k}\right)\right)^{p-1}.$$

So with Lemma 6

$$|B_{p,n}(\theta)|^{1/(p-1)n} \lesssim 1.2197\ldots. \quad \blacksquare$$

It remains to analyze $B_{p,n}(\theta)$ in neighbourhoods of the integers (this corresponds to analyzing $F_{p,n}$ is neighbourhoods of the non-trivial pth roots of unity).

5. $F_{p,n}$ AT NON-TRIVIAL pTH ROOTS

We need the following known lemma.

LEMMA 12.

(a) $$\sum_{m=1}^{p-1} \cot\left(\frac{\pi m}{p}\right) = 0$$

(b) $$\sum_{m=1}^{p-1} \frac{1}{\sin^2(m\pi/p)} = \frac{p^2-1}{3}$$

$$\text{(c)} \qquad \sum_{m=1}^{p-1} \frac{m}{\sin^2(m\pi/p)} = \frac{(p^2-1)p}{6}$$

$$\text{(d)} \qquad \sum_{m=1}^{p-1} \frac{2\cos(\pi m/p)}{\sin^3(\pi m/p)} = 0.$$

Now let

$$z_h := e^{2\pi hi/p + 2i\varepsilon}$$

be a small perturbation of a pth root of unity. Then

$$F_{p,n}(z_h) = 2^{(p-1)n} \exp\left(i\left(\pi \frac{n(p-1)h}{2} + \frac{n^2(p)(p-1)}{2}\varepsilon\right)\right)$$
$$\times \prod_{k=0}^{n-1} \prod_{m=1}^{p-1} \sin\left(\frac{\pi mh}{p} + (pk+m)\varepsilon\right).$$

Let h be an integer and

$$G_h(\varepsilon) := \log\left(\prod_{k=0}^{n-1} \prod_{m=1}^{p-1} \sin\left(\frac{\pi mh}{p} + (pk+m)\varepsilon\right)\right).$$

Then

$$G_h'(0) = \sum_{k=0}^{n-1} \sum_{m=1}^{p-1} (pk+m)\cot\left(\frac{\pi mh}{p}\right)$$
$$G_h''(0) = \sum_{k=0}^{n-1} \sum_{m=1}^{p-1} (pk+m)^2 \frac{-1}{\sin^2(\pi mh/p)}$$
$$G_h'''(0) = \sum_{k=0}^{n-1} \sum_{m=1}^{p-1} (pk+m)^3 \frac{2\cos(\pi mh/p)}{\sin^3(\pi mh/p)}.$$

We now deduce

LEMMA 13. *For* $G := G_1$

$$G'(0) = nS_p$$
$$G''(0) = -\tfrac{1}{12} n(2n+1)(n-1)p^2(p^2-1) - nT_p$$
$$|G'''(0)| = O(n^3)$$

and for any $c > 0$ *there exists* $c_0 > 0$ *so that,*

$$|G'''(x)| \leqslant c_0(n^3|x|) \qquad \textit{for} \quad |x| \leqslant \frac{c}{n^2},$$

where

$$S_p := \sum_{m=1}^{p-1} m \cot\left(\frac{\pi m}{p}\right) < 0$$

and

$$T_p = \sum_{m=1}^{p-1} \frac{m^2}{\sin^2(2\pi/m)} > 0.$$

It follows now that

$$G(z) = G(0) + G'(0)\, z + \frac{G''}{2}(0)\, z^2 + \cdots$$

has a max at (approximately) the solution of

$$2G'(0) = -zG''(0)$$

or

$$z \sim \frac{C_p}{n^2}.$$

Furthermore, $F_{p,n}(z_n)$ has a unique max or min in the interval $|\varepsilon| \leqslant 1/(n+1)\,p$ because $F_{p,n}$ is a trig polynomial with the number of roots equal to its degree so in each interval between zeros there is exactly one critical point. This gives us

LEMMA 14. *Suppose now that p is prime. If $z = e^{2\pi i h/p + 2i\varepsilon}$, where $(h, p) = 1$, $h < p$, and $|\varepsilon| \leqslant 1/(n+1)\,p$, then*

$$|F_{p,n}(z)| \leqslant p^n \left(1 + O\left(\frac{1}{n}\right)\right).$$

We only argued for $h = 1$, but $(h, p) = 1$ is entirely analogous (sums (a), (b), and (d) of Lemma 12 do not change with $m \rightarrow mh$). We have now deduced Theorem 1 for p prime, $p \leqslant 15$. Theorem 2 follows from Lemmas 7, 9, and 14. The max now occurs in a neighbourhood of 0 not at a primitive pth root of unity. One might observe that we have actually shown that

$$\|F_{p,n}(q)\|_{\{|q|=1\}}^{1/n} \rightarrow (1.219...)^{p-1}$$

for $15 < p \leqslant 50$ (and for each additional p one can establish this by numerically checking (b) of Lemma 10 for this p).

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References

1. F. V. Atkinson, On a problem of Erdős and Szekeres, *Canad. Math. Bull.* **1** (1961), 7–12.
2. P. Erdős, Problems and results on Diophantine approximations, *in* "Asymptotic Distribution Modulo 1" (J. F. Koksma and L. Kuipers, Eds.), Noordhoff, Groningen, 1962.
3. P. Erdős and G. Szekeres, On the product $\prod_{k=1}^{n}(1-z^{a_k})$, *Acad. Serve. Sci. Publ. Inst. Math.* **13** (1959), 29–34.
4. G. H. Hardy and E. M. Wright, "An Introduction to the Theory of Numbers," 5th ed., Oxford Univ. Press, London/New York, 1979.
5. Z. A. Melzak, A note on the Tarry-Escott problem, *Canad. Math. Bull.* **4** (1961), 233–237.
6. A. M. Odlyzko, Minima of cosine sums and maxima of polynomials on the unit circle, *J. London Math. Soc.* **26** (1982), 412–420.
7. C. Sudler, An estimate for a restricted partition function, *Quart. J. Math.* **15** (1964), 1–10.
8. E. M. Wright, Proof of a conjecture of Sudler, *Quart. J. Math.* **15** (1964), 11–15.
9. E. M. Wright, A closer estimation for a restricted partition function, *Quart. J. Math.* **15** (1964), 283–287.

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