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CONSTRUCTIVE APPROXIMATION


# Hypergeometric Analogues of the Arithmetic–Geometric Mean Iteration

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**Abstract.** The arithmetic–geometric mean iteration of Gauss and Legendre is the two-term iteration $a_{n+1} = (a_n + b_n)/2$ and $b_{n+1} = \sqrt{a_n b_n}$ with $a_0 := 1$ and $b_0 := x$. The common limit is ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1 - x^2)^{-1}$ and the convergence is quadratic.

This is a rare object with very few close relatives. There are however three other hypergeometric functions for which we expect similar iterations to exist, namely: ${}_2F_1(\frac{1}{2} - s_1, \frac{1}{2} + s; 1; \cdot)$ with $s = \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$.

Our intention is to exhibit explicitly these iterations and some of their generalizations. These iterations exist because of underlying quadratic or cubic transformations of certain hypergeometric functions, and thus the problem may be approached via searching for invariances of the corresponding second-order differential equations. It may also be approached by searching for various quadratic and cubic modular equations for the modular forms that arise on inverting the ratios of the solutions of these differential equations. In either case, the problem is intrinsically computational. Indeed, the discovery of the identities and their proofs can be effected almost entirely computationally with the aid of a symbolic manipulation package, and we intend to emphasize this computational approach.

## 1. Introduction

The arithmetic–geometric mean iteration of Gauss and Legendre is an example of a two-term mean iteration. Here the means are

$$M_1(a, b) := \frac{a + b}{2} \qquad \text{and} \qquad M_2(a, b) := \sqrt{ab}. \tag{1.1}$$

We iterate the means by

$$a_{n+1} := M_1(a_n, b_n) \tag{1.2}$$

and

$$b_{n+1} := M_2(a_n, b_n) \tag{1.3}$$

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commencing with $a_0 := a$ and $b_0 := b$ and denote by

(1.4) $$M_1 \otimes M_2(a, b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

the common limit when it exists. (We will use the notation generally for the limit of any mean iteration.) Then the key fact about the arithmetic–geometric mean iteration is the identification of the limit. That is

(1.5) $$M_1 \otimes M_2(1, x) = 1/{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1 - x^2)$$

and this holds at least for $0 < x \leq 1$. The convergence is quadratic in the sense that

$$|a_{n+1} - b_{n+1}| = O(|a_n - b_n|^2)$$

and hence this process provides a remarkably rapid algorithm for computing the above hypergeometric function (roughly $2^n$ digits for $n$ iterations). The fact of convergence and the speed of convergence is an easy exercise; the finding of the limit is much more complicated. See [3]. Proving (1.5) is equivalent to proving that $F(x) := {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1 - x^2)$ satisfies the functional equation

(1.6) $$F(x) = \frac{2}{1+x} F\left(\frac{2\sqrt{x}}{1+x}\right),$$

and we may verify this by checking that both sides of (1.6) satisfy the right second-order hypergeometric differential equation with rational coefficients (and have the right behavior at 0 so the two solutions are the same). We discuss this further in later sections but here wish only to observe that if we can guess the functional equation (1.6) we can easily prove it.

An entirely different approach to this problem is to start with the Jacobian theta functions

(1.7) $$\theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \theta_4(q) := \theta_3(-q).$$

These are modular forms, a fact which will be important later. (In the notation of Whittaker and Watson $\theta_i(q) = \vartheta_i(0, q)$.) Currently we wish to observe that we may, by reasonably elementary means [3], prove that

(1.8a) $$\theta_3^2(q^2) = \frac{\theta_3^2(q) + \theta_4^2(q)}{2}$$

and

(1.8b) $$\theta_4^2(q^2) = \sqrt{\theta_3^2(q)\theta_4^2(q)}.$$

So $a := \theta_3^2$ and $b := \theta_4^2$ uniformize the arithmetic–geometric mean iteration quadratically in the sense that

(1.9a) $$a(q^2) = M_1(a(q), b(q))$$

and

$$b(q^2) = M_2(a(q), b(q)). \tag{1.9b}$$

(The uniformization would be cubic if the change of variables in $q$ is $q \to q^3$.) There is a third relation which is

$$_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1 - \frac{\theta_4^4(q)}{\theta_3^4(q)}\right) = \theta_3^2(q). \tag{1.10}$$

Now any two of (1.6), (1.8a, b), and (1.10) imply the third, essentially because of the uniqueness of solution of the analytic functional equations involved. Also (1.6) lends itself to computational proof as already observed and so does (1.8a, b) because these are identities on entire modular forms and can be verified by checking the vanishing of a predetermined number of the coefficients of the series expansion. The third equation (1.10) lends itself to being uncovered computationally. We wish to "discover" the corresponding formula for the three other cases where the underlying behavior is based on modular forms and thus derive algorithms for the three hypergeometric functions

$$F_s(x) := {_2F_1}\left(\frac{1}{2} - \frac{1}{s}, \frac{1}{2} + \frac{1}{s}; 1; x\right), \qquad s = 3, 4, 6. \tag{1.11}$$

These various identities package into the following four theorems: We will need the following modular forms and functions:

(1.12)
$$\theta_2(q) := q^{1/4} \sum_{n=-\infty}^{\infty} q^{n^2+n},$$

$$\theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2},$$

$$\theta_4(q) := \sum_{n=-\infty}^{\infty} (-q)^{n^2} = \theta_3(-q),$$

$$a_6(q) := \sum_{n,m=-\infty}^{\infty} q^{n^2+nm+m^2} = \theta_3(q)\theta_3(q^3) + \theta_2(q)\theta_2(q^3),$$

$$b_6(q) := \frac{3a_6(q^3) - a_6(q)}{2},$$

$$c_6(q) := \frac{a_6(q^{1/3}) - a_6(q)}{2},$$

$$J(q^2) := \frac{4(\theta_3^8(q) - \theta_3^4(q)\theta_4^4(q) + \theta_4^8(q))^3}{27(\theta_2(q)\theta_3(q)\theta_4(q))^8}.$$

Here $\theta_2$, $\theta_3$, and $\theta_4$ are the Jacobian theta functions. The function $a_6$ is a modular form of weight 1 on $\Gamma(3)$ and $J$ is Klein's absolute invariant and is a modular function on $\Gamma$. Note also that $\theta_2^2$, $\theta_3^2$, and $\theta_4^2$ are all modular forms of weight one on at least $\Gamma(8)$. From a computational point of view it makes sense to use the sparse theta functions as the building blocks for the other modular forms and so we have defined all our forms rationally in $\theta_2$, $\theta_3$, and $\theta_4$.

**Theorem 1** ($s = \infty$, $1/s = 0$). *The underlying modular forms are*

$$a_\infty = (\theta_3(q))^2 \qquad \text{and} \qquad b_\infty = (\theta_4(q))^2.$$

*They satisfy*

$$F_\infty\left(1 - \frac{b_\infty(q)}{a_\infty(q)}\right) = a_\infty(q).$$

(a) *Quadratic Iteration. Let*

$$M_1(a, b) := \frac{a+b}{2}$$

*and*

$$M_2(a, b) = \sqrt{ab}.$$

*Then*

$$M_1 \otimes M_2(1, x) = \frac{1}{F_\infty(1 - x^2)}.$$

*The uniformizing variables (in $q \to q^2$) are $a_\infty$ and $b_\infty$.*

(b) *Cubic Iteration. Let*

$$U = \sqrt{a^2 - \sqrt[3]{4a^2b^2(a^2 - b^2)}},$$

$$M_1(a, b) := \frac{U + \sqrt{(3a^2 - U^2 + (4ab^2 - 2a^3)/U)}}{3} =: A,$$

$$M_2(a, b) := \frac{bA + ab}{3A - a} = M_1(b, a).$$

*Then*

$$M_1 \otimes M_2(1, x) = \frac{1}{F_\infty(1 - x^2)}.$$

*The uniformizing variables are the same as in part(a) only $q \to q^3$.*

**Theorem 2** ($s = 6$). *The underlying modular forms are*

$$a_6(q) := \sum_{n,m=-\infty}^{\infty} q^{n^2+nm+m^2} \quad \text{and} \quad b_6(q) := \frac{3a_6(q^3) - a_6(q)}{2}.$$

*They satisfy*

$$F_6\left(1 - \left(\frac{b_6(q)}{a_6(q)}\right)^3\right) = a_6(q).$$

(a) *Quadratic Iteration.* *Let*

$$M_1(a, b) := \frac{\sqrt[3]{2b^3 - a^3 + 2\sqrt{b^6 - a^3b^3}} + \sqrt[3]{2b^3 - a^3 - 2\sqrt{b^6 - a^3b^3}}}{2}$$

*and*

$$M_2(a, b) := \frac{\sqrt[3]{b^3 + \sqrt{b^6 - a^3b^3}} + \sqrt[3]{b^3 - \sqrt{b^6 - a^3b^3}}}{2},$$

*then*

$$M_2 \otimes M_2(1, x) = \frac{1}{F_6(1 - x^3)}.$$

*The uniformizing variables (in $q \to q^2$) are $a_6$ and $b_6$.*

(b) *Cubic Iteration.* *Let*

$$M_1(a, b) := \frac{a + 2b}{3},$$

$$M_2(a, b) := \sqrt[3]{b\left(\frac{a^2 + ab + b^2}{3}\right)}.$$

*Then*

$$M_1 \otimes M_2(1, x) = \frac{1}{F_6(1 - x^3)}.$$

*The uniformizing variables are the same as in part* (a) *only $q \to q^3$.*

**Theorem 3** ($s = 4$). *The underlying modular forms are*

$$a_4(q) := \sqrt{\theta_3^4(q) + \theta_2^4(q)} \quad \text{and} \quad b_4(q) := \theta_4^2(q).$$

*They satisfy*

$$F_4\left(1 - \left(\frac{b_4(q)}{a_4(q)}\right)^4\right) = a_4(q).$$

(a) *Quadratic Iteration.* *Let*

$$M_1(a, b) := \frac{a + 3b}{4}$$

*and*

$$M_2(a, b) := \sqrt{b\left(\frac{a + b}{2}\right)}.$$

*Then*

$$M_1 \otimes M_2(1, x) = \frac{1}{F_4^2(1 - x^2)}.$$

*The uniformizing variables (in $q \to q^2$) are $a_4^2$ and $b_4^2$.*

(b) *Cubic Iteration.* *Let*

$$M_1(a, b) := \sqrt{\frac{a^2b + 3B(b^2 - B^2)}{b}},$$

$$B := M_2(a, b) := \frac{U + \sqrt{3b^2 - U^2 - (2a^2b)/U}}{3},$$

*where*

$$U = \sqrt{b^2 + \sqrt[3]{b^2(a^4 - b^4)}}.$$

*Then*

$$M_1 \otimes M_2(1, x) = \frac{1}{F_4(1 - x^4)}.$$

*The uniformizing variables in $(q \to q^3)$ are $a_4$ and $b_4$.*

**Theorem 4** ($s = 3$). *The underlying modular forms are*

$$a_3(q) = \sqrt[4]{a_6^4(q) + 8a_6(q)c_6^3(q)} \qquad \text{and} \qquad b_3(q) = \sqrt[6]{\frac{1 + \sqrt{1 - 1/J(q)}}{2}}\, a_3(q).$$

*They satisfy*

$$F_3\left(1 - \left(\frac{b_3(q)}{a_3(q)}\right)^6\right) = a_3(q).$$

(a) *Quadratic Iteration.* *Let*

$$\begin{aligned} A := M_1(a, b) := (\tfrac{5}{16}(a^6 + 8b^6 - 8a^3b^3 + 4b^{3/2}(b^3 - a^3)^{1/2}a^3 - 8b^{9/2}(b^3 - a^3)^{1/2})^{1/3} \\ + \tfrac{5}{16}(a^6 + 8b^6 - 8a^3b^3 - 4b^{3/2}(b^3 - a^3)^{1/2}a^3 \\ + 8b^{9/2}(b^3 - a^3)^{1/2})^{1/3} + \tfrac{3}{8}a^2)^{1/2} \end{aligned}$$

*and*

$$M_2(a,b) := \left(\frac{22b^3A^2 - 2a^2b^3 - 11a^3A^2 - 22a^2A^3 + a^5 + 32A^5}{4(16A^2 - 11a^2)}\right)^{1/3},$$

*then*

$$M_1 \otimes M_2(1,x) = \frac{1}{F_3^2(1-x^3)}.$$

*The uniformizing variables (in $q \to q^2$) are $a_3^2$ and $b_3^2$.*

The quadratic iteration of Theorem 1 is classical and is due to Gauss and Legendre and others. This is discussed at length in [3]. The cubic iteration is new. Theorem 2(b) is derived in [4] but part (a) is new. Theorem 3(a) is discussed in [6], part (b) is new. Theorem 4 is new. The complicated nature of this iteration seems inevitable since the underlying modular function is

$$\frac{1}{J(q)} = 4\left(1 - \left(\frac{b_3(q)}{a_3(q)}\right)^6\right)\left(\frac{b_3(q)}{a_3(q)}\right)^6$$

whose quadratic modular equation is moderately complex. The iterations all converge either quadratically or cubically, so they essentially double or triple the number of correct digits. They all converge uniformly in some complex neighborhood of 1. Some relatives of these iterations are presented in the following theorems.

**Theorem 5.** *Let*

$$M_1(a,b) := \frac{a+3b}{4}$$

*and*

$$M_2(a,b) := \frac{\sqrt{ab}+b}{2}.$$

*Then*

$$M_1 \otimes M_2(1,x) = {}_{1/3}F_2\left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{2}; 1, 1; \frac{27x^2(1-x)}{4}\right)$$
$$= {}_{1/2}F_1^2\left(\tfrac{1}{6}, \tfrac{1}{3}; 1; \frac{27x^2(1-x)}{4}\right).$$

*The uniformizing variables (in $q \to q^2$) are $a_6^2(q)$ and $(\theta_4(q)\theta_4(q^3))^2$.*

**Theorem 6.** *Let*

$$M_1(a, b) := \frac{a + 15b}{16}$$

*and*

$$M_2(a, b) := \frac{b + \sqrt{b((a + 3b)/4)}}{2}.$$

*Then*

$$M_1 \otimes M_2(1, x) = 1/{}_3F_2^2\left(\tfrac{1}{6}, \tfrac{5}{6}, \tfrac{1}{2}; 1, 1; \frac{27x^2(1 - x)}{4}\right)$$
$$= 1/{}_2F_1^4\left(\tfrac{1}{12}, \tfrac{5}{12}; 1; \frac{27x^2(1 - x)}{4}\right).$$

*The uniformizing variables (in $q \to q^2$) are $\theta_3^8(q) + \theta_4^8(q) - \theta_3^4(q)\theta_4^4(q)$ and $\theta_3^4(q)\theta_4^4(q)$.*

**Theorem 7.** *Let*

$$M_1(a, b) := \frac{a + b}{2}$$

*and*

$$M_2(a, b) := \frac{3\sqrt{b((b + 2a)/3)} - b}{2}.$$

*Then*

$$M_1 \otimes M_2(1, x) = 1/{}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; (1 - x)\left(1 + \frac{x}{2}\right)^2\right).$$

*The uniformizing variables (as $q \to q^2$) are*

$$a = \theta_3\theta_3(q^3) + \theta_2\theta_2(q^3) = a_6$$

*and*

$$b = \frac{b_6^2(q)}{b_6(q^2)}, \qquad b_6 = \frac{3a_6(q^3) - a_6(q)}{2}.$$

*Moreover, if $R$ is this limit mean and $B$ denotes the limit mean in Theorem 5, then*

$$R(1, x)^2 = B\left(1, x\left(\frac{x + 2}{3}\right)\right).$$

**Theorem 8.** *Let*

$$M_1(a, b) := A = \sqrt{\frac{a^2 + 8\sqrt{ab}b}{9}}.$$

*and*

$$M_2(a, b) := \frac{((\sqrt{ab} + 2b)/3)^{2/3}((a + \sqrt{ab} + b)/3)^{2/3}}{A^{1/3}}.$$

*Then*

$$M_1 \otimes M_2(1, x^{2/3}) = 1 \Big/ \left\{ {}_3F_2\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; 1, 1; \frac{(4x)^3(1-x)}{(1+8x)}\right) \right\}^{1/2}.$$

*The uniformizing variables (as $q \to q^3$) are*

$$\theta_3^2(q) \quad \text{and} \quad \left(\frac{3\theta_3(q^9) - \theta_3(q)}{2}\right)^2.$$

Note that this is slightly cleaner as a tree term iteration with $c = \sqrt{ab}$.

**Theorem 9.** *Let*

$$M_1(a, b) := \frac{a + 2b}{3}$$

*and*

$$M_2(a, b) := 1/3\, \frac{b^{1/3}(b^{1/3}2^{1/3}(a+b)^{2/3} + 2^{2/3}(a+b)^{1/3}a^{2/3} - b^{2/3}a^{1/3})}{a^{1/3}}.$$

*Then, for $x > \frac{1}{2}\sqrt{3 + 2\sqrt{3}}$,*

$$M_1 \otimes M_2(1, x) := 1/{}_2F_1\left(\tfrac{1}{4}, \tfrac{1}{4}; 1; \frac{4x^3(1+x)^3(2-x-x^2)}{(1+2x)^2}\right)$$

*The uniformizing variables (in $q \to q^3$) are $\theta_3^2(q)$ and $[3\theta_3^2(q^3) - \theta_3^2(q)]/2$.*

**Theorem 10.** *Let*

$$M_1(a, b) := \frac{a + 80b}{81}$$

*and*

$$M_2(a, b) := \frac{b^{1/3}(3^{2/3}c^{2/3}(a + 26b) + 9cb^{1/3}(3^{1/3}c^{1/3} + 3b^{1/3}))}{81c},$$

*where $c := 8b + a$.*

*Then*

$$M_1 \otimes M_2(1, x) := 1/_2F_1^4\left(\tfrac{1}{12}, \tfrac{5}{12}; 1; \frac{64x^3(1-x)}{1+8x}\right).$$

*The uniformizing parameters (as* $q \to q^3$*) are* $a_6(a_6^3 + 8c_6^3)$ *and* $a_6 b_6^3$.

The remainder of this paper is an exposition of how these iterations can be found and proved. The necessary theory of hypergeometric function may be accessed in [1], [3], and [9]. The modular function and form theory is in [3], [14], and [15]. Various aspects of mean iterations are discussed in [3], [4], [5], and [6]. The relationship between such iterations and fast algorithms for pi and the elementary functions is pursued in [3], [8], and [13]. In [10] the theory of basic hypergeometric series is presented. Many of the underlying transformations for hypergeometric functions have basic analogues and so much of the theory of this paper can be explored from this point of view. Finally, the ghost of Ramanujan walks these pages. It was through work of his that we uncovered much of Theorem 2 initially. See [2], [11], and [12].

## 2. Derivations

The route we follow is very classical and begins with the inversion of ratios of hypergeometric functions. The difference for us is that the calculations involved can now be done with great ease and at various points it makes sense to substitute exact machine computation for thought. Let

$$u_1(x) := {}_2F_1\left(\frac{1}{2} - \frac{1}{s}, \frac{1}{2} + \frac{1}{s}; 1; x\right) \tag{2.1}$$

and

$$\begin{aligned} u_2(x) := {} & \left(-\frac{\cos(\pi/s)}{\pi}\right)(\ln x)u_1(x) \\ & + \frac{\cos(\pi/s)}{\pi} \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - 1/s)_n(\frac{1}{2} + 1/s)_n}{(n!)^2}\left(2\Psi(n+1) - \Psi\left(\frac{1}{2} + \frac{1}{s} + n\right)\right. \\ & \left. - \Psi\left(\frac{1}{2} - \frac{1}{s} + n\right)\right)x^n \\ = {} & {}_2F_1\left(\frac{1}{2} - \frac{1}{s}, \frac{1}{2} + \frac{1}{s}; 1; 1 - x\right). \end{aligned} \tag{2.2}$$

Then $u_1$ and $u_2$ are conjugate solutions of

$$f'' + \left[\frac{1}{x} + \frac{1}{x-1}\right]f' - \left(\frac{1 - (2/s)^2}{4x(1-x)}\right)f = 0. \tag{2.3}$$

So

(2.4) $$q(x) := e^{-(\pi/\cos(\pi/s))(u_2/u_1)(x)} := xe^{G(x)},$$

where

(2.5)

$$G(x) := \frac{\sum_{n=0}^{\infty} \{[(\frac{1}{2} - 1/s)_n(\frac{1}{2} + 1/s)_n]/(n!)^2\}(2\Psi(n+1) - \Psi(\frac{1}{2} + 1/s + n) - \Psi(\frac{1}{2} - 1/s + n))x^n}{u_1(x)}$$

is analytic in a neighborhood of zero.

We can now solve for $x$ as a function of $q$. That is, we invert the series in (2.4) to get a function $X_S(q)$ which is analytic in a neighborhood of zero. If we set

(2.6) $$q = e^{i\pi t}$$

and consider $X_S$ as a function of $t$ we observe that

(2.7) $$X_S(t+2) = X_S(t)$$

and

(2.8) $$X_s\left(-\frac{[\cos(\pi/s)]^{-2}}{t}\right) = 1 - X_s(t).$$

So the function $g(t) := X_S(t)(1 - X_S(t))$ is invariant under the two linear fractional transformations

(2.9) $$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -[\cos(\pi/s)]^{-2} \\ 1 & 0 \end{pmatrix}.$$

We hope now that $g(t)$ will be a modular function with respect to some finite index subgroup of $\Gamma$, in which case there will be a quadratic modular equation

(2.10) $$X_S(q^2) = A(X_S(q)),$$

where $A$ is an algebraic function. We can then (essentially) invert (2.10) to get a quadratic iteration for the appropriate hypergeometric function. The cases where this happens have been much studied and give rise to the Schwartz functions [9]. This will happen when $s = n$ and $[\cos(\pi/s)]^2$ is rational. So for $1/s \in [0, \frac{1}{2}]$ we are limited to $1/s = 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, and $\frac{1}{6}$. The case $1/s = \frac{1}{2}$ is trivial because $F_S \equiv 1$. The other four cases correspond to Theorems 1 to 4.

We need to know very little of this theory to proceed (provided we have decided to look at the right cases, which is unlikely without prior knowledge).

*For fixed s.* We first symbolically generate the power series expansion for $X_S(q)$ by inverting (2.4). We then let

(2.11) $$a_s(q) := F_s(X_s(q))$$

and also symbolically generate its series expansion. In each case in Theorems 1 through 4 this is the underlying modular form $a_S$. The underlying modular form $b_S$ is given by

$$b_s(q) := [1 - X_s(q)]^{(1/2 - 1/s)} a_s(q). \tag{2.12}$$

So

$$1 - X_s(q) = \left(\frac{b_s(q)}{a_s(q)}\right)^{1/(1/2 - 1/s)}. \tag{2.13}$$

The first point is that it is relatively simple to find initial terms of power series expansions for $a_s$ and $b_s$. The second point is that it is relatively simple to find a superset of the homogeneous polynomial relations (modular equations) of the form

$$P(a_s(q), b_s(q), a_s(q^2), b_s(q^2)) \equiv 0. \tag{2.14}$$

These give rise to the mean iterations of the examples, which can now be proved via the following proposition.

**Proposition 1** (Mechanical Checking). *Suppose we wish to prove that*

$$M_1 \otimes M_2(1, x) := 1 \Big/ \left({}_2F_1\left(\frac{1}{2} - \frac{1}{s}, \frac{1}{2} + \frac{1}{s}; 1; U(x)\right)\right)^v,$$

*where $M_1(1, z)$, $M_2(1, z)$ are both analytic in a neighborhood of 1, $U(z)$ is analytic in a neighborhood of 0 and $v$ is a rational number. It suffices to check that:*

$$\text{(a)} \qquad M'' + aM'' + bM = (b \circ t)M(t')^2,$$

$$\text{(b)} \qquad t''M + 2M't' + at'M = (a \circ t)M(t')^2,$$

*where*

$$a := a(x) = \frac{1 - 2x}{x(1 - x)},$$

$$b := b(x) = \frac{-1 + (2/s)^2}{4x(1 - x)},$$

$$T(x) := \frac{M_2(1, x)}{M_1(1, x)},$$

$$m(x) := M_1(1, x),$$

$$t(x) := U(T(U^{-1}(x))),$$

*and*

$$M(x) := m(U^{-1}(x))^{-1/v}.$$

Note that if all of $M_1$, $M_2$, and $U$ are algebraic functions and $v$ is a rational number then (a) and (b) amount to verifying an algebraic equation. This is always a finite exact computation, certainly theoretically, but very often practically (in, for example, a symbolic manipulation package like Maple).

**Proof.** The proof is based on the fact that if

$$f''(x) + \alpha(x)f'(x) + \beta(x)f(x) = 0$$

and if for suitable differentiable $N$, $\alpha$, $\beta$, and $t$

$$\text{(a1)} \qquad N'' + \alpha N' + \beta N = (\beta \circ t)M(t')^2,$$

and

$$\text{(a2)} \quad t''N + 2N't' + \alpha t'N = (\alpha \circ t)N(t')^2,$$

then

$$f^*(x) = N(x)f(t(x))$$

is also a solution of the above differential equation (see [3] and [5]). Now in the case we are in there is a unique solution that is analytic at zero. So in this case, $f^*(x)$ is a multiple of $f$ (if $f$ is the original analytic solution). Since $f^*(0) = f(0) = 1$ they are the same. The proof amounts to checking that

$$\frac{1}{(F_s(U(x))^v} := H(x)$$

satisfies the right functional equation, namely

$$H(U(x)) = m(x)H(U(t(x)),$$

or equivalently that, with $y := U(x)$, that

$$F_S(y) = M(x)F_S(t(y)). \qquad ■$$

## 3. More Computational Details

Suppose we have generated say 100 terms of $a_\infty(q) := (\theta_3(q))^2$ and $b_\infty(q) := (\theta_4(q))^2$ by inverting (2.4) and using (2.11) and (2.12). Then if $P$ is a homogeneous polynomial of degree $N$ in four variables and

$$P(a_\infty(q), b_\infty(q), a_\infty(q^2), b_\infty(q^2)) \equiv 0,$$

then $P := P(q)$ must vanish through 100 terms of its power series expansion. The converse is not true. An identity to 100 terms might not be an identity. (However, since $\theta_3^2(q)$, $\theta_4^2(q)$, $\theta_4^2(q^2)$ are all entire modular forms of weight one on $\Gamma(8)$ a subgroup of $\Gamma$ of index $12 \cdot 32$, it follows that $P(q)$ is an entire modular form of weight $N$ on $\Gamma(8)$ and can hence have a zero of order at most $N \cdot 32$ at $q = 0$. So checking to 100 terms actually constitutes a proof for homogeneous identities of degree less than four). It is a straightforward computational exercise to generate a basis for the homogeneous relations of fixed degree (to a fixed number of terms).

This is a linear problem and just amounts to finding the kernel of a matrix. So, for example, for $N = 2$ there is a basis of five homogeneous relations on $a_\infty(q)$, $b_\infty(q)$, $a_\infty(q^2)$, $b_\infty(q^2)$. Here we have let $a = a_\infty(q)$, $b = b_\infty(q)$, $A = a_\infty(q^2)$, $B = b_\infty(q^2)$.

(1) $A(a + b - 2A)$;
(2) $B(a + b - 2A)$;
(3) $(2A + a - b)(a + b - 2A)$;
(4) $b^2 - 2bA + B^2$; and
(5) $b(a + b - 2A)$.

Note that (1) is the identity

$$a_\infty(q^2) := \frac{a_\infty(q) + b_\infty(q)}{2},$$

while (4) and (1) give the identity

$$b_\infty(q^2) := \sqrt{a_\infty(q) b_\infty(q)}.$$

These two are the arithmetic geometric mean identities.

If we look for a homogeneous relation of degree 4 in $a_\infty(q)$, $b_\infty(q)$ and $\alpha := a_\infty(q^3)$ we find just one

$$16ab^2\alpha - 27\alpha^4 + a^4 - 8a^3\alpha + 18a^2\alpha^2$$

from which we deduce (b) of Theorem 1. The other underlying identities may all be discovered similarly. Once discovered, the related mean iterations may then be proved by Proposition 1. However, in order to prove the modular identities that identify the uniformizing parameters we need a little modular form theory of the type discussed above parenthetically. The proofs are then entirely a matter of expanding the appropriate identity to a prerequisite number of terms

$$1 + \frac{N[\Gamma : G]}{12}$$

where $G$ is the group on which the corresponding functions are entire modular forms of weight 1 and $N$ is the degree of the homogeneous identity. More of this is illustrated for $a_6$ and $b_6$ in [7].

However, given that we can identify $G$, all other steps of this development can be and were carried out computationally in Maple.

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*Note Added in Proof.* In *Cubic modular identities of Ramanujan, hypergeometric functions and analogues of the arithmetic–geometric mean* (preprint) a survey of the results of this paper and of related recent work is given.

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