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CONSTRUCTIVE APPROXIMATION

Remez-, Nikolskii-, and Markov-Type Inequalities for Generalized Nonnegative Polynomials with Restricted Zeros

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Abstract. Sharp Remez-, Nikolskii-, and Markov-type inequalities are proved for functions of the form

$$f(z) = |\omega| \prod_{j=1}^{m} |z - z_j|^{r_j} \qquad (\omega, z_j \in \mathbf{C};\ 0 < r_j \in \mathbf{R};\ j = 1, 2, \ldots, m)$$

under the assumptions

$$\sum_{j=1}^{m} r_j \le N \qquad \text{and} \qquad \sum_{\{j: |z_j| < 1\}} r_j \le K, \qquad 0 \le K \le N.$$

The Remez- and Nikolskii-type inequalities are new even for polynomials of degree at most n having at most k $(0 \le k \le n)$ zeros in the open unit disk.

1. Introduction

Generalized nonnegative polynomials were studied in a sequence of recent papers [4], [6], [7], [8], and [9]. A number of well-known inequalities in approximation theory were extended to them, by utilizing the generalized degree in place of the ordinary one. Our motivation was to find tools to examine systems of orthogonal polynomials simultaneously, associated with generalized Jacobi, or at least generalized nonnegative polynomial weight functions of degree at most $\Gamma > 0$. In a recent paper [10] we gave sharp estimates, in this spirit, for the Christoffel function on $[-1, 1]$ and for the distances of the consecutive zeros of orthogonal polynomials associated with generalized nonnegative polynomial weight functions of degree at most Γ. Typically, the extension of a polynomial inequality to generalized nonnegative polynomials is not trivial, and the proof is far from a simple density argument. However, there is an inequality, the less known Remez inequality, which can be extended quite simply to generalized nonnegative polynomials, preserving at least the best possible order of magnitude. Based on this

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extended Remez-type inequality, several other important inequalities were obtained for generalized nonnegative polynomials [4], [6], [8], [9].

In this paper we study Remez-, Nikolskii-, and Markov-type inequalities for generalized nonnegative polynomials of the form

$$f(z) = |\omega| \prod_{j=1}^{m} |z - z_j|^{r_j} \qquad (\omega, z_j \in \mathbf{C};\ 0 < r_j \in \mathbf{R};\ j = 1, 2, \dots, m)$$

under the assumptions

$$\sum_{j=1}^{m} r_j \le N \qquad \text{and} \qquad \sum_{\{j: |z_j| < 1\}} r_j \le K, \qquad 0 \le K \le N.$$

The Remez-type inequality will play a central role again. First, we establish it for ordinary polynomials of degree at most n having at most k $(0 \le k \le n)$ zeros in the open unit disk. This is the content of Theorem 3.1. In Theorem 3.2 we extend a numerical version of Theorem 3.1 to generalized nonnegative polynomials. Based on Theorem 3.2, we will prove our Nikolskii-type inequality (Theorem 3.3) and Markov-type inequality (Theorem 3.4) for generalized nonnegative polynomials with restricted zeros. Our theorems contain the earlier results (without any constraints on the zeros) as special cases.

2. Notations

Let Π_n be the set of all real algrebraic polynomials of degree at most n. The function

$$f(z) = \omega \prod_{j=1}^{m} (z - z_j)^{r_j} \qquad (0 \ne \omega;\ z_j \in \mathbf{R};\ j = 1, 2, \dots, m) \tag{2.1}$$

will be called a generalized complex algebraic polynomial of (generalized) degree

$$N = \sum_{j=1}^{m} r_j. \tag{2.2}$$

To be precise, in this paper we will use the definition

$$z^r = \exp(r \log|z| + ir \arg z) \qquad (z \in \mathbf{C},\ 0 < r \in \mathbf{R},\ -\pi \le \arg z < \pi). \tag{2.3}$$

When each $r_j > 0$ is an integer in (2.1), then f is an ordinary polynomial of degree N with complex coefficients. Obviously, for an f defined by (2.1), we have

$$|f(z)| = |\omega| \prod_{j=1}^{m} |z - z_j|^{r_j}. \tag{2.4}$$

Denote by $GCAP_N$ the set of all generalized complex algebraic polynomials of degree at most N. The set $\{|f|: f \in GCAP_N\}$ will be denoted by $|GCAP|_N$. It is easy to see that, restricted to the real line, every $f \in |GCAP|_N$ of the form (2.4) can be written as

$$f(z) = |\omega| \prod_{j=1}^{m} ((z - z_j)(z - \bar{z}_j))^{r_j/2} \qquad (z \in \mathbf{R}), \tag{2.5}$$

therefore, restricted to the real line, the elements of $|GCAP|_N$ can be considered as generalized nonnegative polynomials. Denote by $\Pi_{n,k}$ $(0 \le k \le n)$ the set of all $p \in \Pi_n$ which have at most k zeros (counting multiplicities) in the open unit disk. Let $|GCAP|_{N,K}$ $(0 \le K \le N)$ be the set of all $f \in |GCAP|_N$ of the form (2.4) for which

$$\sum_{\{j:|z_j|<1\}} r_j \le K. \tag{2.6}$$

We will denote by $m(A)$ the one-dimensional Lebesgue measure of a set $A \subset \mathbf{R}$. To formulate our Remez-type inequalities and their proofs briefly, we introduce the classes

$$\Pi_{n,k}(s) = \{p \in \Pi_{n,k}: m(\{x \in [-1, 1]: |p(x)| \le 1\}) \ge 2 - s\}$$

and

$$|GCAP|_{N,K}(s) = \{f \in |GCAP|_{N,K}: m(\{x \in [-1, 1]: f(x) \le 1) \ge 2 - s\},$$

where $0 \le k \le n$ are integers, $0 \le K \le N$, and $0 < s < 2$ are real numbers. We will use the standard notation

$$\|f\|_p = \left(\int_{-1}^{1} |f(x)|^p \, dx\right)^{1/p} \quad \text{if} \quad 0 < p < \infty,$$

and

$$\|f\|_\infty = \max_{-1 \le x \le 1} |f(x)|.$$

3. New Results

To establish a sharp Remez-type inequality for the classes $\Pi_{n,k}(s)$, we need the weighted Chebyshev polynomials $T_{n,k}$ of the form

$$T_{n,k}(x) = (x+1)^{n-k} Q(x), \qquad Q \in \Pi_k \quad (0 \le k \le n) \tag{3.1}$$

satisfying the properties

$$T_{n,k} \text{ equioscillates } k+1 \text{ times in } [-1, 1], \tag{3.2}$$

$$\max_{-1 \le x \le 1} |T_{n,k}(x)| = 1, \tag{3.3}$$

and

$$T_{n,k}(1) = 1. \tag{3.4}$$

More precisely, (3.2) and (3.3) mean that $T_{n,k}$ achieves the values

$$\pm \max_{-1 \le x \le 1} |T_{n,k}(x)| = \pm 1,$$

$k+1$ times in $[-1, 1]$ with alternating sign. The existence and uniqueness of such weighted Chebyshev polynomials $T_{n,k}$ are well known from the theory of weighted Chebyshev approximation. Explicit formulas for $T_{n,k}$ seem to be known only when $k = 0$, $k = n - 1$, or $k = n$.

Theorem 3.1. *Let $0 \le k \le n$ be arbitrary integers and let $0 < s < 2$ be an arbitrary real number. Then*

$$\max_{-1 \le x \le 1} |p(x)| \le T_{n,k}\left(\frac{2+s}{2-s}\right)$$

for every $p \in \Pi_{n,k}(s)$. Equality holds if and only if

$$p(x) = \pm T_{n,k}\left(\frac{\pm 2x}{2-s} + \frac{s}{2-s}\right).$$

The case $k = n$ (when there are no restrictions for the zeros) of Theorem 3.1 gives the Remez inequality [11, pp. 119–121] or [2] as a special case. When $k = 0$, Theorem 3.1 yields the corollary of the theorem from [3].

By estimating $T_{n,k}((2+s)/(2-s))$, we give a sharp numerical version of Theorem 3.1, and we extend this to the classes $|GCAP|_{N,K}$.

Theorem 3.2. *Let $0 \le K \le N$ and $0 < s \le 1$ be arbitrary real numbers. Then*

$$\max_{-1 \le x \le 1} f(x) \le \exp(c_1(\sqrt{NKs} + Ns))$$

for every $f \in |GCAP|_{N,K}(s)$ and for some $c_1 \le 9$.

We do not discuss what happens when $1 < s < 2$, since the case $0 < s \le 1$ is completely satisfactory for our applications.

As a consequence of Theorem 3.2 we will prove the following Nikolskii-type inequalities for $|GCAP|_{N,K}$.

Theorem 3.3. *Let χ be a nonnegative, nondecreasing function defined on $[0, \infty)$ such that $\chi(x)/x$ is nonincreasing, and let $0 \le K \le N$ and $0 < q < p \le \infty$ be arbitrary (extended) real numbers. There is an absolute constant $0 < c_2 \le 81e^2$ such that*

$$\|\chi(f)\|_p \le (c_2 \max\{1, q^2NK, qN\})^{1/q - 1/p}\|\chi(f)\|_q$$

for every $f \in |GCAP|_{N,K}$.

The case $K = N$ in Theorem 3.3 was proved in [9, Theorem 6]. If $qK \ge 1$, then the Nikolskii factor in the unrestricted case ($K = N$) is like $(\sqrt{c_2}qN)^{2/q - 2/p}$, while in our restricted cases it improves to $(\sqrt{c_2}q\sqrt{NK})^{2/q-2/p}$.

Theorem 3.2 will play a significant role in the proof of our Markov-type inequality for $|GCAP|_{N,K}$ as well. Throughout this paper f' denotes the derivative of f with respect to a real variable. Note that if $f \in |GCAP|_N$ is of the form (2.4) with each $r_j \ge 1$, then $f'(z_j)$ does not exist when $z_j \in \mathbf{R}$ and $r_j = 1$, but $|f(z_j)|$ (and $|f'(x)|$ for every real x) is well defined as the absolute value of the one-sided derivatives. This is our understanding, whenever we have $|f'(x)|$ ($x \in \mathbf{R}$) in the rest of the paper.

Theorem 3.4. *Let $0 \le K \le N$ be arbitrary real numbers. There is an absolute constant $c_3 > 0$ such that*

$$\max_{-1 \le x \le 1} |f'(x)| \le c_3 N(K+1) \max_{-1 \le x \le 1} f(x) \tag{3.5}$$

for every $f \in |GCAP|_{N,K}$ of the form (2.4) *with $1 \le r_j \in \mathbf{R}$, $j = 1, 2, \ldots, m$.*

The condition that each $r_j \ge 1$, $j = 1, 2, \ldots, m$, is needed to insure that $|f'(z_j)| < \infty$ if $z_j \in \mathbf{R}$. Theorem 3.4 is a generalization of a number of earlier results. Inequality (3.5) was proved in [1] for polynomials $f \in \Pi_{n,k}$ $(0 \le k \le n)$ having only real zeros. Another proof of (3.5) was given in [5] for all $f \in \Pi_{n,k}$ $(0 \le k \le n)$. Less general or less sharp results can be found in [12], [13], [14], [15], and [16]. The unrestricted generalized polynomial case $(K = N)$ of Theorem 3.4 was established in [4]. Up to the constant c_3, Theorem 3.4 is sharp even for the classes $\Pi_{n,k}$ [15, Example 1].

4. Proof of Theorems 3.1 and 3.2

To prove Theorem 3.1 we need a series of lemmas.

Lemma 4.1. *Let $0 \le k \le n$ be fixed integers and let $0 < s < 2$ and $1 \le A$ be fixed real numbers. If $|p(1)| \le A$ holds for every $p \in \Pi_{n,k}(s)$, then $\max_{-1 \le x \le 1} |p(x)| \le A$ holds for every $p \in \Pi_{n,k}(s)$ as well.*

Proof. The proof of the corresponding step in [2] works here as well, we present it for the sake of completeness. Let $p \in \Pi_{n,k}(s)$ be arbitrary. We choose a point $-1 \le y \le 1$ such that

$$|p(y)| = \max_{-1 \le x \le 1} |p(x)|. \tag{4.1}$$

It is easy to see that $p \in \Pi_{n,k}(s)$ implies either

$$m(\{-1 \le x \le y \colon |p(x)| \le 1\}) \ge \tfrac{1}{2}(1+y)(2-s) \tag{4.2}$$

or

$$m(\{y \le x \le 1 \colon |p(x)| \le 1\}) \ge \tfrac{1}{2}(1-y)(2-s). \tag{4.3}$$

Without loss of generality we may assume that (4.2) holds, otherwise we study $p(x) = p(-x) \in \Pi_{n,k}(s)$. Note that (4.2) implies

$$q(x) := p((y+1)x/2 + (y-1)/2) \in \Pi_{n,k}(s), \tag{4.4}$$

which, together with (4.1) and the assumption of the lemma, yields

$$\max_{-1 \le x \le 1} |p(x)| = |p(y)| = |q(1)| \le A, \tag{4.5}$$

and the lemma is proved. ∎

Lemma 4.2. *Let $0 \le k \le n$ be fixed integers and let $0 < s < 2$ and $1 \le A$ be fixed real numbers. If $|p(1)| \le A$ holds for every $p \in \Pi_{n,k}(s)$ of the form $p(x) = (1+x)^{n-k}Q(x)$ with $Q \in \Pi_k$, then $|p(1)| \le A$ holds for every $p \in \Pi_{n,k}(s)$ as well.*

Proof. Let $p \in \Pi_{n,k}(s)$ be arbitrary. Then there are polynomials $w \in \Pi_{n-k,0}$ and $Q \in \Pi_k$ such that $p = wQ$ and $w(x) \ge 0$ for every $-1 \le x \le 1$. By an observation of G. G. Lorentz [14] every $w \in \Pi_{n-k,0}$, nonnegative in $[-1, 1]$, can be written as

$$w(x) = \sum_{j=0}^{n-k} a_j(1+x)^j(1-x)^{n-k-j} \qquad \text{with all} \quad a_j \ge 0. \tag{4.6}$$

Using the nonnegativity of the coefficients a_j, we obtain

$$|p(x)| \ge |a_0(1+x)^{n-k}Q(x)| \tag{4.7}$$

for every $-1 \le x \le 1$. Hence, $p \in \Pi_{n,k}(s)$ implies that either $p(1) = 0$ or

$$q(x) := a_0(1+x)^{n-k}Q(x) \in \Pi_{n,k}(s), \tag{4.8}$$

and the assumption of the lemma yields $|p(1)| = |q(1)| \le A$, thus the lemma is proved. ■

Denote by $\Omega_{n,k}$ $(0 \le k \le n)$ the set of all polynomials p of the form $p(x) = (1+x)^{n-k}Q(x)$ with $Q \in \Pi_k$. We introduce the classes $\Omega_{n,k}(s) = \Omega_{n,k} \cap \Pi_{n,k}(s)$ $(0 \le k \le n,\ 0 < s < 2)$. It is routine to check that $\Omega_{n,k}(s)$ is a closed and bounded, hence, compact subset of Π_n in the uniform (and hence in any) norm on $[-1, 1]$, so we omit the details. Hence, for every $0 \le k \le n$ integers and $0 < s < 2$ real number there exists a $p^* = p^*_{n,k,s} \in \Omega_{n,k}(s)$ such that

$$|p^*(1)| = \sup_{p \in \Omega_{n,k}(s)} |p(1)|. \tag{4.9}$$

In our next propositions we examine the properties of p^*, following the method used in [2].

Proposition 4.3. *Every polynomial $p^* \in \Omega_{n,k}(s)$ satisfying* (4.9) *has all its zeros in $[-1, 1]$.*

Proof. Let $p^* \in \Omega_{n,k}(s)$ satisfy (4.9). If $p^*(z) = 0$ for a nonreal z, then the polynomial

$$p(x) := (1+\eta)p^*(x)\left(1 - \frac{\varepsilon(x-1)^2}{(x-z)(x-\bar{z})}\right)$$

with sufficiently small $\eta > 0$ and $\varepsilon > 0$ is in $\Omega_{n,k}(s)$, and contradicts the maximality of p^*. If $p^*(z) = 0$ for a real z outside $[-1, 1]$, then the polynomial

$$p(x) = (1+\eta)p^*(x)\left(1 - \varepsilon \operatorname{sgn}(z)\frac{1-x}{z-x}\right)$$

with sufficiently small $\eta > 0$ and $\varepsilon > 0$ is in $\Omega_{n,k}(s)$ and contradicts the maximality of p^*. Thus the proposition is proved. ■

Associated with a $p^* \in \Omega_{n,k}(s)$ satisfying (4.9) we will study the set

(4.10) $$M(p^*) := \{x \in [-1, 1]: |p^*(x)| \le 1\}.$$

Obviously, $M(p^*)$ is a disjoint union of at most n closed subintervals I_j of $[-1, 1]$. These intervals I_j will be called the components of $M(p^*)$.

Proposition 4.4. *If $p^* \in \Omega_{n,k}(s)$ satisfies* (4.9), *then each component of $M(p^*)$ contains at least one zero of p^*.*

Proof. By Lemma 4.3, p^* has all its zeros in $[-1, 1]$. Let $v = \deg p^*$. If p^* does not have any zero in a component of $M(p^*)$, then applications of Rolle's Theorem would show that the derivative of p^* has at least v zeros (counting multiplicities) in $[-1, 1]$, a contradiction. ∎

Proposition 4.5. *If $p^* \in \Omega_{n,k}(s)$ satisfies* (4.9), *then the set $M(p^*)$ has only one component.*

Proof. Assume that $M(p^*)$ has at least two components, and let I_1 be the closest component to 1. Then $p^* \in \Omega_{n,k}(s)$ is of the form

(4.11) $$p^*(x) = c \prod_{i=1}^{m_1} (x - x_i) \prod_{j=1}^{m_2} (x - y_j) \qquad (c \in \mathbf{R},\ m_1 + m_2 \le n),$$

where x_i, $i = 1, 2, \ldots, m_1$, are the zeros of p^* lying in I_1 and y_j, $j = 1, 2, \ldots, m_2$, are the remaining zeros of p^* (every zero is listed as many times as its multiplicity). Let η and η' be the left-hand endpoint of I_1 and the right-hand endpoint of the component next to I_1, respectively. Then it is easy to see that

(4.12) $$p(x) := c \prod_{i=1}^{m_1} (x - x_i + h) \prod_{j=1}^{m_2} (x - y_j)$$

with $0 < h \le \eta - \eta'$ is in $\Omega_{n,k}(s)$, and $|p(1)| > |p^*(1)|$, a contradiction. ∎

Proposition 4.6. *If $p^* \in \Omega_{n,k}(s)$ satisfies* (4.9), *then $|p^*(x)| \le 1$ for every $-1 \le x \le 1 - s$.*

Proof. If $1 \le k \le n$, then $p(-1) = 0$ and the lemma follows immediately from Proposition 4.5 and the relation $p^* \in \Omega_{n,k}(s)$. When $k = 0$, denote the only component of $M(p^*)$ by $[\alpha, \beta]$. Assume that $-1 < \alpha$. Then $\deg(p^*) \ge 1$, and by Propositions 4.3 and 4.5, p^* has all its zeros in $[\alpha, \beta]$, therefore $|p^*|$ is strictly increasing in $[\beta, \infty)$. Now it is easy to see that the polynomial $p(x) = p^*(x + 1 + \alpha)$ is in $\Omega_{n,k}(s)$ and $|p(1)| = |p^*(2 + \alpha)| > |p^*(1)|$, a contradiction. Thus the proposition is proved. ∎

Lemma 4.7. *If $p^* \in \Omega_{n,k}(s)$ satisfies* (4.9), *then*

$$p^*(x) = \pm T_{n,k}\left(\frac{2x}{2-s} + \frac{s}{2-s}\right) \qquad (x \in \mathbf{C}),$$

where the weighted Chebyshev polynomials $T_{n,k}$ $(0 \le k \le n)$ are defined by (3.1)–(3.4).

Proof. Let $T_{n,k,s}$ be the polynomials $T_{n,k}$, transformed linearly from $[-1, 1]$ to $[-1, 1-s]$, namely

$$T_{n,k,s}(x) := T_{n,k}\left(\frac{2x}{2-s} + \frac{s}{2-s}\right).$$

Then $T_{n,k,s}$ is of the form

$$T_{n,k,s}(x) = (x+1)^{n-k}Q(x), \qquad Q \in \Pi_k, \tag{4.13}$$

satisfying the properties

$$T_{n,k,s} \text{ equioscillates } k+1 \text{ times in } [-1, 1-s], \tag{4.14}$$

$$\max_{-1 \le x \le 1-s} |T_{n,k,s}(x)| = 1, \tag{4.15}$$

and

$$T_{n,k,s}(1-s) = 1. \tag{4.16}$$

Without loss of generality we may assume that $p^*(1) > 0$, otherwise we study the polynomial $-p^*$. It follows from (4.9), $T_{n,k,s} \in \Omega_{n,k}(s)$, $T_{n,k,s}(1) > 0$, and $p^*(1) > 0$ that

$$0 < \eta := \frac{T_{n,k,s}(1)}{p^*(1)} \le 1. \tag{4.17}$$

We claim that $T_{n,k,s} - \eta p^* \in \Pi_n$ has at least $n+1$ zeros (counting multiplicities) in $\mathbf{R}$, and hence $p^* \equiv \eta T_{n,k,s}$. To see this note that $T_{n,k,s} - \eta p^*$ has $n-k$ zeros at -1; it has at least k zeros (counting multiplicities) in $(-1, 1-s]$ (if $0 \le k \le n-1$) or in $[-1, 1-s]$ (if $k = n$) because of (4.14), (4.17), and Proposition 4.6; finally, it has a zero at 1 because of the choice of η. Hence $\eta p^* \equiv T_{n,k,s}$. If $0 < \eta < 1$, then the polynomial $T_{n,k,s} \in \Omega_{n,k}(s)$ would contradict (4.9). Therefore $\eta = 1$, and the lemma is proved. ∎

Using Lemmas 4.1, 4.2, and 4.7, we can prove Theorem 3.1 easily.

Proof of Theorem 3.1. The inequality of the theorem follows immediately from Lemmas 4.1, 4.2, and 4.7 and the extremal property (4.9) of p^*. To find all the extremal polynomials $p \in \Pi_{n,k}(s)$ for which

$$\max_{-1 \le x \le 1} |p(x)| = T_{n,k}\left(\frac{2+s}{2-s}\right), \tag{4.18}$$

we choose a point $y \in [-1, 1]$ such that

$$|p(y)| = \max_{-1 \le x \le 1} |p(x)| = T_{n,k}\left(\frac{2+s}{2-s}\right). \tag{4.19}$$

First we show that $-1 < y < 1$ contradicts (4.19). Indeed, $p \in \Pi_{n,k}(s)$ implies either (4.2) or (4.3). Without loss of generality we may assume that (4.2) holds, otherwise we study $P(x) = p(-x) \in \Pi_{n,k}(s)$. Then

$$q(x) := p((y+1)x/2 + (y-1)/2) \in \Pi_{n,k}(s) \tag{4.20}$$

and

$$q(1) = T_{n,k}\left(\frac{2+s}{2-s}\right). \tag{4.21}$$

Now $q \in \Omega_{n,k}(s)$, otherwise it would be possible to find a $\tilde{q} \in \Omega_{n,k}(s)$ such that $|\tilde{q}(1)| > |q(1)|$, by utilizing the Lorentz representation of a polynomial from $\Pi_{n-k,0}$, similarly to the proof of Lemma 4.2, and this, together with (4.21), would contradict the already proved part of the theorem. Hence, Lemma 4.7 yields that

$$q(x) \equiv \pm T_{n,k}\left(\frac{2x}{2-s} + \frac{s}{2-s}\right), \tag{4.22}$$

thus $m(\{x \in [-1, 1]: |p(x)| \le 1\}) = \frac{1}{2}(1 + y)(2 - s) < 2 - s$, e.g., $p \notin \Pi_{n,k}(s)$, a contradiction. Therefore $y = \pm 1$, and Lemma 4.7 implies that

$$p(\pm x) \equiv \pm T_{n,k}\left(\frac{2x}{2-s} + \frac{s}{2-s}\right), \tag{4.23}$$

which finishes the proof of the theorem. ■

To prove Theorem 3.2 we will need to estimate the zeros of the constrained Chebyshev polynomials $T_{n,k}$. To formulate our next lemma we introduce some notations. Let $\alpha_1 > \alpha_2 > \cdots > \alpha_{2k}$ be the zeros of the shifted Chebyshev polynomial $T_{2k}(x - 1) = \cos(2k \arccos(x - 1))$ $(0 \le x \le 2)$ of degree $2k$. Let $1 \le k$, $2k < n$, $m = n - 2k$,

$$t_k(x) := \prod_{j=1}^{k} (x - \alpha_j), \tag{4.24}$$

$$q_n(x) := (x + 2m/k)^{m+k} t_k(x), \tag{4.25}$$

and

$$h_n(x) := q_n((1 + m/k)x + 1 - m/k) \tag{4.26}$$

(we have shifted from $[-2m/k, 2]$ to $[-1, 1]$). The following lemma was proved in [1].

Lemma 4.8. *Let $\gamma_1 < \gamma_2 < \cdots < \gamma_k$ and $\rho_1 < \rho_2 < \cdots < \rho_k$ be the zeros of h_n (defined by (4.26)) lying in $(-1, 1)$, and the zeros of $T_{n,k}$ (defined by (3.1)–(3.4)) lying in $(-1, 1)$, respectively. Then $\rho_j \le \gamma_j$ for every $j = 1, 2, \ldots, k$.*

From Lemma 4.8 we will easily obtain the following.

Lemma 4.9. *Let $T_{n,k}$ $(0 \le k \le n)$ be the weighted Chebyshev polynomials defined by (3.1)–(3.4). We have*

$$T_{n,k}\left(\frac{2+s}{2-s}\right) \le \exp(c_1(\sqrt{nks} + ns))$$

for every $0 \le k \le n$ integers and $0 < s \le 1$ real numbers, where $c_1 = 9$ is a suitable choice.

Proof. First let $1 \le k \le n/2$. Using $0 < s \le 1$, Lemma 4.8, (4.24), (4.25), and (4.26), we obtain

$$\begin{aligned}(4.27)\qquad T_{n,k}\left(\frac{2+s}{2-s}\right) &\le T_{n,k}(1+2s) = \frac{T_{n,k}(1+2s)}{T_{n,k}(1)} \\ &= \left(\frac{2+2s}{2}\right)^{n-k} \prod_{j=1}^{k} \frac{1+2s-\rho_j}{1-\rho_j} \\ &\le \left(\frac{2+2s}{2}\right)^{n-k} \prod_{j=1}^{k} \frac{1+2s-\gamma_j}{1-\gamma_j} \\ &= (1+s)^{n-k}\, \frac{t_k(2+2(n-k)s/k)}{t_k(2)} \\ &\le (1+s)^{n-k}\, \frac{T_{2k}(1+2(n-k)s/k)}{T_{2k}(1)}.\end{aligned}$$

From the well-known explicit formula

$$(4.28)\qquad T_{2k}(x) = \tfrac{1}{2}((x+\sqrt{x^2-1})^{2k} + (x-\sqrt{x^2-1})^{2k}) \qquad (x \in \mathbf{R}\backslash(-1,1)),$$

we can deduce that

$$\begin{aligned}(4.29)\qquad T_{2k}(1+2(n-k)s/k) &\le \exp(2k(\sqrt{4(n-k)s/k} + 4(n-k)s/k)) \\ &\le \exp(8(\sqrt{nks} + ns)).\end{aligned}$$

This, together with (4.27), (4.28), $(1+s)^{n-k} \le \exp(ns)$, and $T_{2k}(1) = 1$, yields

$$(4.30)\qquad T_{n,k}\left(\frac{2+s}{2-s}\right) \le \exp(9(\sqrt{nks} + ns))$$

for every $1 \le k \le n/2$ and $0 < s \le 1$. If $k = 0$, then $T_{n,k}(s) = 2^{-n}(1+x)^n$, and the desired inequality follows immediately. If $n/2 < k \le n$, then Theorem 3.1, (4.28), and $T_{n,n} = T_n$ yield

$$\begin{aligned}(4.31)\qquad T_{n,k}\left(\frac{2+s}{2-s}\right) &\le T_{n,n}\left(\frac{2+s}{2-s}\right) \le T_n(1+2s) \\ &\le (1+4s+2\sqrt{s})^n \le \exp(n(4s+2\sqrt{s}) \\ &\le \exp(4(\sqrt{nks} + ns)) \qquad (0 < s \le 1),\end{aligned}$$

hence the lemma is completely proved. ■

Combining Theorem 3.1 and Lemma 4.9 we obtain

$$(4.32)\qquad \max_{-1 \le x \le 1} |p(x)| \le \exp(c_1(\sqrt{nks} + ns))$$

for every $p \in \Pi_{n,k}(s)$ and $0 < s \le 1$. To prove Theorem 3.2 we extend (4.32) for every $f \in |GCAP|_{N,K}(s)$, when $0 \le K \le N$ and $0 < s \le 1$ are arbitrary real numbers.

Proof of Theorem 3.2. Let $f \in |GCAP|_{N,K}(s)$ be of the form (2.4) and first assume that each $r_j, j = 1, 2, \ldots, m$, is rational, thus $r_j = v_j/v$ with suitable positive integers v_j and v. Note that

$$p(z) := |\omega|^{2v} \prod_{j=1}^{m} (z - z_j)^{v_j}(z - \bar{z}_j)^{v_j} \in \Pi_{2Nv, 2Kv}(s) \tag{4.33}$$

and

$$f(z) = p(z)^{1/(2v)}, \qquad z \in \mathbf{R}. \tag{4.34}$$

Hence, applying Theorem 3.1 to p defined by (4.33) and taking its $2v$th root, we get the theorem. The case when each $r_j, j = 1, 2, \ldots, m$, is arbitrary positive real follows from the already proved rational case by a limiting argument, and the theorem is proved. ∎

Now we prove Theorem 3.3 as a straightforward application of Theorem 3.2.

Proof of Theorem 3.3. It is suffcient to prove the theorem when $p = \infty$, and then a simple argument gives the desired result for arbitrary $0 < q < p < \infty$. To see this, assume that

$$\|\chi(f)\|_\infty \leq M^{1/q} \|\chi(f)\|_q$$

for every $f \in |GCAP|_{N,K}$ and $0 < q < \infty$ with some factor M. Then we easily obtain that

$$\begin{aligned} \|\chi(f)\|_p^p = \|\chi(f)^{p-q+q}\|_1 &\leq \|\chi(f)\|_\infty^{p-q} \|\chi(f)\|_q^q \\ &\leq M^{p/q-1} \|\chi(f)\|_q^{p-q} \|\chi(f)\|_q^q, \end{aligned}$$

and therefore

$$\|\chi(f)\|_p \leq M^{1/q - 1/p} \|\chi(f)\|_q$$

for every $f \in |GCAP|_{N,K}$ and $0 < q < p < \infty$. Thus, in the sequel let $0 < q < p = \infty$. Using Theorem 3.2 with

$$s = \min\{1, (c_1^2 q^2 NK)^{-1}, (c_1 qN)^{-1}\}, \tag{4.35}$$

and recalling the conditions prescribed for χ, we can easily conclude that

$$\begin{aligned} &m(\{x \in [-1, 1]: (\chi(f(x)))^q \geq e^{-2} \|\chi(f)\|_\infty^q\}) \\ &\quad \geq m(\{x \in [-1, 1]: f(x) \geq e^{-2/q} \|(f)\|_\infty\}) \\ &\quad \geq m(\{x \in [-1, 1]: f(x) \geq \exp(-c_1(\sqrt{NKs} + Ns)\|(f)\|_\infty\}) \geq s \end{aligned} \tag{4.36}$$

for every $f \in |GCAP|_{N,K}$. Now integrating only on the subset E of $[-1, 1]$, where

$$(\chi(f(x)))^q \geq e^{-2} \|\chi(f)\|_\infty^q, \tag{4.37}$$

and using (4.35) and (4.36), we obtain that

$$\|\chi(f)\|_\infty^q \leq \frac{e^2}{m(E)} \int_E (\chi(f(x)))^q \, dx \leq c_2 \max\{1, q^2 NK, qN\} \|\chi(f)\|_q^q$$

for every $f \in |GCAP|_{N,K}$ where $c_2 = c_1^2 e^2$. Since $c_1 = 9$ is a suitable choice in Theorem 3.2, so is $c_2 = 81e^2$ in this theorem. ■

5. Proof of Theorem 3.4

To prove Theorem 3.4 we need a series of lemmas. Our first lemma guarantees the existence of an extremal function.

Lemma 5.1. *Given $r_j \ge 1$, $j = 1, 2, \ldots, m$, $\sum_{j=1}^m r_j \le N$, and $-1 < \delta < 1$ real numbers, there exists a $0 \not\equiv \tilde{f} \in |GCAP|_{N,K}$ of the form*

$$(5.1) \qquad \tilde{f} = \prod_{j=1}^{m} |\tilde{P}_j|^{r_j} \qquad (\tilde{P}_j(z) = A_j z + B_j;\ A_j,\ B_j \in \mathbf{C};\ j = 1, 2, \ldots, m)$$

such that

$$(5.2) \qquad \frac{|\tilde{f}'(1)|}{\max\limits_{-1 \le x \le \delta} \tilde{f}(x)} = \sup_f \frac{|f'(1)|}{\max\limits_{-1 \le x \le \delta} f(x)} = L,$$

where the supremum in (5.2) *is taken for all $f \in |GCAP|_{N,K}$ of the form*

$$(5.3) \qquad f = \prod_{j=1}^{m} |P_j|^{r_j} \qquad (P_j(z) = c_j(z - z_j);\ c_j,\ z_j \in \mathbf{C};\ j = 1, 2, \ldots, m).$$

Proof. Let

$$(5.4) \qquad 0 \not\equiv f_i = \prod_{j=1}^{m} |P_{j,i}|^{r_j} \in |GCAP|_{N,K}$$

$$(P_{j,i}(z) = c_{j,i}(z - z_{j,i});\ c_{j,i},\ z_{j,i} \in \mathbf{C};\ j = 1, 2, \ldots, m;\ i = 1, 2, \ldots)$$

such that

$$(5.5) \qquad \frac{|f_i'(1)|}{\max\limits_{-1 \le x \le \delta} f_i(x)} \ge \min\{L - 1/i,\ i\} \qquad (i = 1, 2, \ldots).$$

Without loss of generality we may assume that

$$(5.6) \qquad \max_{-1 \le x \le 1} |P_{j,i}(x)| = 1 \qquad (j = 1, 2, \ldots, m;\ i = 1, 2, \ldots).$$

Since the set $\{p: p(z) = Az + B,\ A,\ B \in \mathbf{C},\ \max_{-1 \le x \le 1} |p(x)| = 1\}$ is compact in the uniform norm, for every $j = 1, 2, \ldots, m$ there is a subsequence of $\{P_{j,i}\}_{i=1}^{\infty}$, without loss of generality we may assume that this is $\{P_{j,i}\}_{i=1}^{\infty}$ itself, which converges to a $\tilde{P}_j$ of the form $\tilde{P}_j(z) = \tilde{A}_j z + \tilde{B}_j$ with $\tilde{A}_j$, $\tilde{B}_j \in \mathbf{C}$ in the uniform (and hence in any) norm on $[-1, 1]$. Now it is easy to check that $\tilde{f} = \prod_{j=1}^{m} \tilde{P}_j$ satisfies the requirements. The fact $\tilde{f} \in |GCAP|_{N,K}$ comes from $f_i \in |GCAP|_{N,K}$, $i = 1, 2, \ldots$, and Hurwitz's Theorem. ■

Lemma 5.2. *Assume that* $0 \not\equiv \tilde{f} \in |GCAP|_{N,K}$ *satisfies* (5.1) *and* (5.2). *Then* $\tilde{f}$ *has only real zeros.*

Proof. Assume that $\tilde{f}(\alpha) = 0$ for $\alpha \notin \mathbf{R}$. Without loss of generality we may assume that $\tilde{P}_1(\alpha) = 0$. Vieta's formula for the product of the zeros of a polynomial shows that if $|\alpha| \geq 1$, $0 < \varepsilon < 1$, and

$$(\beta - \alpha)(\beta - \bar{\alpha}) - \varepsilon(\beta - 1)^2 = 0, \tag{5.7}$$

then $|\beta| \geq 1$. We study the function

$$f_\varepsilon(z) := (1 - \varepsilon)\tilde{f}(z)\left|\frac{z - \beta}{z - \alpha}\right|^{r_1} \in |GCAP|_{N,K} \tag{5.8}$$

which is obviously of the form (5.1). The relation $f_\varepsilon \in |GCAP|_{N,K}$ is straightforward from $\tilde{f} \in |GCAP|_{N,K}$ if $|\alpha| < 1$, while it follows from (5.7) and $\tilde{f} \in |GCAP|_{N,K}$ if $|\alpha| \geq 1$. By (5.7), restricted to the real line, we have

$$f_\varepsilon(z) = \tilde{f}(z)\left|1 - \varepsilon\frac{(z-1)^2}{(z-\alpha)(z-\bar{\alpha})}\right|^{r_1/2} \qquad (z \in \mathbf{R}), \tag{5.9}$$

hence $-1 < \delta < 1$ and $\alpha \notin \mathbf{R}$ imply

$$\max_{-1 \leq x \leq \delta} f_\varepsilon(x) < \max_{-1 \leq x \leq \delta} f(x) \tag{5.10}$$

if $\varepsilon > 0$ is sufficiently small. This, together with (5.8) and

$$|f_\varepsilon'(1)| = \tilde{f}'(1), \tag{5.11}$$

contradicts (5.2), and the lemma is proved. ■

We remark that if $\delta = 1$, then (5.10) does not necessarily hold any more, and the above proof fails to work. This explains the technical assumption $-1 < \delta < 1$ in Lemma 5.1.

If $f \in |GCAP|_N$ is of the form (2.4) with pairwise different $z_j \in \mathbf{C}$, $j = 1, 2, \ldots, m$, then we say that the multiplicity of z_j in f is r_j. Our next lemma is a Rolle-type theorem for the classes $|GCAP|_{N,K}$.

Lemma 5.3. *If* $0 \not\equiv f \in |GCAP|_{N,K}$ $(0 \leq K \leq N)$ *is of the form* (2.4) *with all* $r_j \geq 1$, $j = 1, 2, \ldots, m$, *and* f *has only real zeros, then* $|f'| \in |GCAP|_{N,K+1}$ *has only real zeros as well, and the multiplicity of at least one of any two adjacent zeros of* $|f'|$ *is* 1.

We remark that $|GCAP|_{N,K+1}$ is defined to be $|GCAP|_{N,N} = |GCAP|_N$ if $K + 1 \geq N$.

Proof. Every $f \in |GCAP|_{N,K}$ having only real zeros is of the form

$$f(z) = |\omega| \prod_{j=1}^{m} Q_j(z)^{r_j/2} \qquad (z \in \mathbf{R}) \tag{5.12}$$

with

(5.13) $$Q_j(z) = (z - z_j)^2 \qquad (z_j \in \mathbf{R}, j = 1, 2, \dots, m),$$

where $z_1 < z_2 < \cdots < z_m$,

(5.14) $$\sum_{j=1}^{m} r_j \le N \qquad \text{and} \qquad \sum_{\{j: |z_j| < 1\}} r_j \le K.$$

Using the product rule to differentiate $f(z)$ in (5.12), we obtain

(5.15) $$|f'(z)| = \prod_{j=1}^{m} |z - z_j|^{r_j - 1} |P(z)| \qquad \text{with a} \quad P \in \Pi_{m-1}.$$

By applications of Rolle's Theorem, $|f'|$, and hence P has a zero in each of the intervals $(z_1, z_2), (z_2, z_3), \dots, (z_{m-1}, z_m)$. Since $P \in \Pi_{m-1}$, this, together with (5.14) and (5.15), shows that $|f'| \in |GCAP|_{N,K+1}$ has only real zeros, and the multiplicity of at least one of any two adjacent zeros of $|f'|$ is 1, indeed. Thus the lemma is proved. ∎

Our next result is a Chebyshev-type inequality on the size of $g(y)$ when $g \in |GCAP|_{N,K}$ has all its zeros in $(-\infty, 1]$.

Corollary 5.4. *There is an absolute constant $c_4 \ge 1$ such that*

$$g(y) \le c_4 \max_{-1 \le x \le 1} g(x) \qquad (-1 \le y \le 1 + (N(K+1))^{-1})$$

for every $f \in |GCAP|_{N,K}$ $(0 \le K \le N)$ which has all its zeros in $(-\infty, 1]$.

Proof. This follows from the Remez-type inequality of Theorem 3.2 for the classes $|GCAP|_{N,K}$, transformed linearly to the interval $[-1, y]$ if

$$1 \le y \le 1 + (N(K+1))^{-1}.$$ ∎

Our next lemma is a Markov-type inequality at 1 for the classes $|GCAP|_{N,K}$ (under some additional assumptions which will be dropped later).

Lemma 5.5. *Assume that $0 \le K \le N$,*

(5.16) $f \in |GCAP|_{N,K}$ *is of the form* (2.4) *with all $r_j \ge 1$,* $j = 1, 2, \dots, m$,

(5.17) *f has only real zeros*

and

(5.18) $$\lim_{h \to 0^+} \frac{f(1) - f(1-h)}{h} \ge 0.$$

Then

$$|f'(1)| \le c_4 N(K+1) \max_{-1 \le x \le 1} f(x),$$

where c_4 is the same as in Corollary 5.4.

Proof. In addition to (5.16), (5.17), and (5.18), first assume that

(5.19) $$f \text{ has all its zeros in } [-1, 1].$$

Then $|f'|$ is increasing in $[1, \infty)$. Using Corollary 5.4 with $y = 1 + (N(K+1))^{-1}$, the Mean Value Theorem, and the monotonicity of $|f'|$ in $[1, \infty)$, we obtain

$$|f'(1)| \le |f'(\xi)| = \frac{f(y) - f(1)}{y - 1} \le \frac{c_4}{y-1} \max_{-1 \le x \le 1} f(x)$$
$$\le c_4 N(K+1) \max_{-1 \le x \le 1} f(x),$$

where $\xi \in (1, y)$ is a suitable number. Now we get rid of assumption (5.19). Let $f \in |GCAP|_{N,K}$ be of the form (2.4), where each z_j, $j = 1, 2, \ldots, m$, is real. By an observation of G. G. Lorentz, for every $z_j \in \mathbf{R} \setminus [-1, 1]$ there are numbers $A_j > 0$ and $B_j > 0$ such that

(5.20) $$|z - z_j| = A_j(1 + z) + B_j(1 - z) \qquad (-1 \le z \le 1).$$

Now (5.20) implies that

(5.21) $$\prod_{\{j : |z_j| > 1\}} |z - z_j|^{r_j} \ge \prod_{\{j : |z_j| > 1\}} (A_j(1 + z))^{r_j} \qquad (-1 \le z \le 1).$$

We define

(5.22) $$F(z) := |\omega| \prod_{\{j : |z_j| \le 1\}} |z - z_j|^{r_j} \prod_{\{j : |z_j| > 1\}} (A_j |1 + z|)^{r_j}.$$

Then from (5.20), (5.21), (5.22), (5.16), (5.17), and (5.18) we can deduce that

(5.23) $$F(z) \le f(z) \qquad (-1 \le z \le 1),$$

(5.24) $$F(1) = f(1),$$

(5.25) $$|F'(1)| \ge \lim_{h \to 0^+} \frac{f(1) - f(1 - h)}{h},$$

(5.26) $F \in |GCAP|_{N,K}$ is of the form (2.4) with all $r_j \ge 1$, $j = 1, 2, \ldots, m$,

and

(5.27) $$F \text{ has all its zeros in } [-1, 1].$$

Since F satisfies the assumptions of the already proved case (see (5.26) and (5.27)), by (5.23), (5.24), and (5.25) we obtain

$$|f'(1)| \le |F'(1)| \le c_4 N(K+1) \max_{-1 \le x \le 1} F(x)$$
$$\le c_4 N(K+1) \max_{-1 \le x \le 1} f(x),$$

thus the lemma is proved. ∎

The next lemma is needed for technical reasons, to prove our other important tool, Lemma 5.7.

Lemma 5.6. *Assume that*

(5.28) $$g \in |GCAP|_{N,K+1} \qquad (0 \le K \le N) \text{ has all its zeros in } [-1, 1],$$

(5.29) $$g(1) = \max_{-1 \le x \le 1} g(x),$$

(5.30) $$|g'| \text{ is increasing on } [\gamma, \infty),$$

where γ is the largest value from $[-1, 1)$ for which

(5.31) $$g(\gamma) = \tfrac{1}{2} \max_{-1 \le x \le 1} g(x).$$

Then we have

(5.32) $$|g'(y)| \le c_4 N(K+2) \max_{-1 \le x \le 1} g(x) \qquad (\gamma \le y \le 1)$$

and

(5.33) $$\gamma \le 1 - \frac{c_5}{N(K+2)} \qquad \text{with} \quad c_5 = \frac{1}{2c_4}.$$

As before, $|GCAP|_{N,K+1}$ is defined to be $|GCAP|_N$ if $K + 1 \ge N$.

Proof. Let

(5.34) $$\delta = \frac{1}{N(K+2)} \qquad \text{and} \qquad \gamma \le y \le 1.$$

Using the Mean Value Theorem and (5.30), we can find a $\xi \in (y, y + \delta)$ such that

(5.35) $$\frac{g(y+\delta) - g(y)}{\delta} = |g'(\xi)| \ge |g'(y)|.$$

Now Lemma 5.4 and (5.34) yield

(5.36) $$g(y + \delta) \le c_4 \max_{-1 \le x \le 1} g(x),$$

which, together with (5.35) and (5.34), implies (5.32). Using (5.29), (5.31), the Mean Value Theorem, and (5.32) we can find a $\xi \in (\gamma, 1)$ such that

$$\begin{aligned} \tfrac{1}{2} \max_{-1 \le x \le 1} g(x) &= g(1) - g(\gamma) = (1 - \gamma)|g'(\xi)| \\ &\le (1 - \gamma) c_4 N(K+2) \max_{-1 \le x \le 1} g(x), \end{aligned}$$

whence (5.33) follows. Thus the lemma is proved. ■

Our next lemma examines what happens if assumption (5.18) fails to hold in Lemma 5.5.

Lemma 5.7. *Assume that* (5.16) *and* (5.17) *hold, but instead of* (5.18) *we have*

$$\lim_{h\to 0^+} \frac{f(1)-f(1-h)}{h} < 0. \tag{5.37}$$

Further, let $g = |f'|$ *and*

$$g(1) = \max_{-1\le x\le 1} g(x). \tag{5.38}$$

Then

$$g(y) \ge \tfrac{1}{2} \max_{-1\le x\le 1} g(x)$$

for every $y \in [1 - c_5(N(K+2))^{-1}, 1]$, *where* c_5 *is the same as in Lemma* 5.6.

Proof. From (5.16), (5.17), (5.37), and Corollary 5.4 we can deduce that

(5.39) $g = |f'| \in |GCAP|_{N,K+1}$ has only real zeros,

(5.40) at least one of any two adjacent zeros of g has multiplicity 1,

(5.41) the largest zero of g in $(-\infty, 1)$ has multiplicity 1.

Let $g \in |GCAP|_{N,K+1}$ be of the form

$$g(z) = |\tilde{\omega}| \prod_{j=1}^{\tilde{m}} |z - \tilde{z}_j|^{\tilde{r}_j} \qquad (0 \ne \tilde{\omega} \in \mathbf{C};\ \tilde{z}_j \in \mathbf{R};\ \tilde{r}_j > 0;\ j = 1, 2, \ldots, \tilde{m}). \tag{5.42}$$

Note that for every $\tilde{z}_j \in \mathbf{R}\setminus[-1, 1]$, there are numbers $\tilde{A}_j > 0$ and $\tilde{B}_j > 0$ such that

$$|z - \tilde{z}_j| = \tilde{A}_j(1+z) + \tilde{B}_j(1-z) \qquad (-1 \le z \le 1), \tag{5.43}$$

and

$$\prod_{\{j:|\tilde{z}_j|>1\}} |z - \tilde{z}_j|^{\tilde{r}_j} \ge \prod_{\{j:|\tilde{z}_j|>1\}} (\tilde{A}_j(1+z))^{\tilde{r}_j} \qquad (-1 \le z \le 1). \tag{5.44}$$

We define

$$G(z) := |\tilde{\omega}| \prod_{\{j:|\tilde{z}_j|\le 1\}} |z - \tilde{z}_j|^{\tilde{r}_j} \prod_{\{j:|\tilde{z}_j|>1\}} (\tilde{A}_j|1+z|)^{\tilde{r}_j}. \tag{5.45}$$

Then, from (5.39)–(5.43), we can easily deduce

(5.46) $G \in |GCAP|_{N,K+1}$ has all its zeros in $[-1, 1)$,

$$G(z) \le g(z) \qquad (-1 \le z \le 1), \tag{5.47}$$

$$g(1) = G(1) = \max_{-1\le x\le 1} G(x) = \max_{-1\le x\le 1} g(x) \tag{5.48}$$

and

(5.49) $|G'|$ is increasing in (γ, ∞),

where γ is the largest value from $[-1, 1)$ for which

$$G(\gamma) = \tfrac{1}{2} \max_{-1\le x\le 1} G(x). \tag{5.50}$$

Observation (5.49) needs some further explanation. From (5.39)–(5.42), (5.45), and (5.48) we conclude that $G \in |GCAP|_{N,K+1}$ is of the form

$$(5.51) \qquad G(z) = |\alpha| \prod_{j=1}^{\mu} |z - w_j|^{\rho_j},$$

where $0 \neq \alpha \in \mathbf{C}$, $(-1 \leq) w_1 < w_2 < \cdots < w_\mu (< 1)$, $\rho_j > 0$ for every $j = 1, 2, \ldots, \mu$, and $\rho_\mu = 1$. Now, similarly to the proof of Lemma 5.3, we obtain that $|G'|$ is of the form

$$(5.52) \qquad |G'(z)| = |\alpha| \prod_{j=1}^{\mu} |z - w_j|^{\rho_j - 1} |P(z)|,$$

where $P \in \Pi_{\mu-1}$ has a simple zero x_j on each of the intervals (w_j, w_{j+1}), $j = 1, 2, \ldots, \mu - 1$. This, together with $\rho_j > 0$ ($j = 1, 2, \ldots, \mu$) and $\rho_\mu = 1$, implies that $|G'|$ is of the form

$$(5.53) \qquad |G'(z)| = |\tilde{\alpha}| \prod_{j=1}^{\mu-1} \left| \frac{z - x_j}{z - w_j} \right|^{\gamma_j} |z - x_j|^{\delta_j}$$

with some $0 \neq \tilde{\alpha} \in \mathbf{C}$, $-1 \leq w_j < x_j < 1$, $\gamma_j \geq 0$, and $\delta_j \geq 0$, $j = 1, 2, \ldots, \mu - 1$. Since every factor in (5.53) is nondecreasing for $z \geq \max_{1 \leq j \leq \mu-1} x_j$, so is the product. Hence (5.49) holds, indeed.

Note that $G \in |GCAP|_{N,K+1}$ satisfies the assumptions of Lemma 5.6 (see (5.46), (5.48), and (5.49)), hence Lemma 5.6, together with (5.47) and (5.48), implies

$$g(y) \geq G(y) \geq \tfrac{1}{2} \max_{-1 \leq x \leq 1} G(x) = \tfrac{1}{2} \max_{-1 \leq x \leq 1} g(x)$$

for every $y \in [\gamma, 1]$ where $\gamma \leq 1 - c_5(N(K+2))^{-1}$. Thus the lemma is proved. ∎

Now we are ready to prove Theorem 3.4.

Proof of Theorem 3.4. First we prove that there is an absolute constant $c_6 > 0$ such that

$$(5.54) \qquad |f'(1)| \leq c_6 N(K+1) \max_{-1 \leq x \leq 1} f(x)$$

for every $f \in |GCAP|_{N,K}$ satisfying (5.16) and (5.17). To see this, without loss of generality, we may assume that

$$(5.55) \qquad |f'(1)| = \max_{-1 \leq x \leq 1} |f'(x)|.$$

(If (5.55) does not hold, then we take a point $x_0 \in [-1, 1)$ such that $|f'(x_0)| = \max_{-1 \leq x \leq 1} |f'(x)|$, and use a linear transformation such that $-1 \to -1$ and $x_0 \to 1$ if $x_0 \geq 0$, or $1 \to -1$ and $x_0 \to 1$ if $x_0 < 0$.) Now, if (5.18) holds in addition to (5.16), (5.17), and (5.55), then Lemma 5.5 gives (5.54). On the other hand, if (5.37) holds in addition to (5.16), (5.17), and (5.55), then Lemma 5.7 yields

$$(5.56)\qquad \begin{aligned} |f'(1)| = g(1) &\le \frac{1}{1-\gamma}\int_\gamma^1 2g(x)\,dx \\ &= \frac{2}{1-\gamma}\int_\gamma^1 |f'(x)|\,dx \le \frac{2N(K+2)}{c_5}\left|\int_\gamma^1 f'(x)\right| \\ &\le \frac{2N(K+2)}{c_5}|f(1)-f(\gamma)| \le \frac{2N(K+2)}{c_5}\max_{-1\le x\le 1} f(x). \end{aligned}$$

Hence (5.54) follows for every $f \in |GCAP|_{N,K}$ satisfying (5.16) and (5.17). Now we show the validity of (5.54) without the assumption (5.17). By the Remez-type inequality for $|GCAP|_{N,K}$ (Theorem 3.2), for every $\varepsilon > 0$ and $N > 0$ there is a $\delta = \delta(\varepsilon, N) \in (-1, 1)$ such that

$$(5.57)\qquad \max_{-1\le x\le 1} f(x) \le (1+\varepsilon)\max_{-1\le x\le \delta} f(x)$$

for every $f \in |GCAP|_{N,K}$ $(0 \le K \le N)$. Now, from (5.54) and (5.57) we obtain

$$(5.58)\qquad |f'(1)| \le (1+\varepsilon)c_6 N(K+1)\max_{-1\le x\le \delta} f(x)$$

for every $f \in |GCAP|_{N,K}$ satisfying (5.16) and (5.17), which, together with Lemmas 5.1 and 5.2, implies (5.58) for every $f \in |GCAP|_{N,K}$ satisfying only (5.16). Therefore

$$(5.59)\qquad |f'(1)| \le c_6 N(K+1)\max_{-1\le x\le 1} f(x)$$

for every $f \in |GCAP|_{N,K}$ satisfying (5.16). Now let $0 \le y \le 1$. It follows from (5.59) by a linear transformation that

$$(5.60)\qquad \begin{aligned} |f'(y)| &\le \frac{2c_6}{1+y} N(K+1)\max_{-1\le x\le y} f(x) \\ &\le 2c_6 N(K+1)\max_{-1\le x\le 1} f(x) \end{aligned}$$

for every $f \in |GCAP|_{N,K}$ satisfying (5.16). Finally, if $f \in |GCAP|_{N,K}$, then $\tilde{f}(x) = f(-x) \in |GCAP|_{N,K}$, hence (5.60) implies that

$$(5.61)\qquad |f'(-y)| \le 2c_6 N(K+1)\max_{-1\le x\le 1} f(-x) = 2c_6 N(K+1)\max_{-1\le x\le 1} f(x).$$

By (5.60) and (5.61) the theorem is completely proved. ■

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