Constr. Approx. (1992) 8: 381–399

CONSTRUCTIVE APPROXIMATION

Rational Approximation to Lipschitz and Zygmund Classes

P. B. Borwein and S. P. Zhou

Abstract. In this paper, characterizations for $\lim_{n\to\infty}(R_n(f)/\omega(n^{-1}))=0$ in H^ω and for $\lim_{n\to\infty} n^{r+\alpha}R_n(f)=0$ in W^r Lip α, $r\ge 1$, are given, while, for Z, a generalization to a related result of Newman is established.

1. Introduction

Let $C^r_{[-1,1]}$ be the class of continuous functions which have r continuous derivatives on $[-1,1]$, for notational convenience, let $C_{[-1,1]}=C^0_{[-1,1]}$,

$$\operatorname{Lip}\alpha=\{f\in C_{[-1,1]}\colon \omega(f,t)\le t^\alpha\},\qquad 0<\alpha\le 1,$$
$$Z=\{f\in C_{[-1,1]}\colon \omega_2(f,t)\le t\},$$

where

$$\omega(f,t):=\max_{0\le h\le t}\|f(x+h)-f(x)\|_{[-1,1-h]},$$
$$\omega_m(f,t):=\max_{0\le h\le t}\left\|\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(x+jh)\right\|_{[-1,1-mh]},$$
$$\|f\|_A:=\sup_{x\in A}|f(x)|,\qquad \|f\|:=\|f\|_{[-1,1]}.$$

Also, for any set H,

$$W^rH=\{f\in C^r_{[-1,1]}\colon f^{(r)}\in H\}.$$

Rational functions are a classical tool in approximation theory that turn out to be more convenient in many cases than polynomials. After the classical results of Zolotarjov, substantial progress was made in 1964 when Newman (see [3]) showed that $|x|$ is uniformly approximated by rationals much better than by polynomials. Newman's result stimulated the appearance of many substantial results in this field.

Date received: January 2, 1991. Date revised: March 24, 1992. Communicated by Ronald A. DeVore.
AMS classification: 41A20.
Key words and phrases: Rational approximation, Lipschitz class, Zygmund class, Modulus of continuity.

In the case of polynomial approximation to Lip 1 functions, the error bound is well known to be $O(1/n)$, and this is unimprovable. For rational approximation to Lip 1 functions the story is different. Newman conjectured that

$$\lim_{n\to\infty} nR_n(f) = 0$$

for any fixed $f \in \text{Lip } 1$. Here, for $f \in C_{[-1,1]}$,

$$R_n(f) = \inf_{r\in R_{n,n}} \|f - r\|,$$

where

$$R_{n,n} = \left\{\frac{p}{q}: p\in \Pi_n,\ q\in \Pi_n\right\},$$

and Π_n is the class of all polynomials of degree $\leq n$.

Newman's conjecture was proved by Popov [7]. Further interesting results of Popov, Petrushev, and others concern rational approximation for the class of absolutely continuous functions, bounded variation functions, and convex functions (see [6]). For example, they established the following results:

1. If f is defined on $[-1, 1]$ and $f^{(r)}$ is absolutely continuous for $r \geq 1$, then

$$R_n(f) = O\left(n^{-r-1} \max_{0<h\leq n^{-1}} \int_{-1}^{1-h} |f^{(r+1)}(x+h) - f^{(r+1)}(x)|\, dx\right).$$

2. If f is defined on $[-1, 1]$ and $f^{(r)}$ is convex and bounded for $r \geq 1$, then

$$R_n(f) = O(n^{-r-2}).$$

On the other hand, Newman [4] showed if $f \in Z$ and $f'(x)$ exists almost everywhere on $[-1, 1]$, then

$$\lim_{n\to\infty} nR_n(f) = 0,$$

and thus gave another proof to his own conjecture.

A natural question now arises: If dropping the assumption that $f'(x)$ exists almost everywhere (which is guaranteed by such conditions as absolute continuity, bounded variation, convexity, etc.), can we still get some benefit from rational approximation?

Newman's work [5] shows that if $f \in \text{Lip } \alpha$, $0 < \alpha < 1$, and for almost all $x \in [-1, 1]$,

$$\lim_{h\to 0+} h^{-\alpha}(f(x+h) - f(x)) = 0,$$

then

$$\lim_{n\to\infty} n^{\alpha} R_n(f) = 0.$$

Let $\omega(t)$ be a nondecreasing continuous function on $[0, \infty)$ with

(1) $$\omega(t) > 0 \quad \text{for} \quad t > 0,$$

(2) $$\omega(0) = 0.$$

Define

$$H^{\omega} = \{f \in C_{[-1,1]}: \omega(f, t) \le \omega(t)\}.$$

In trigonometric approximation, the sufficient condition for which[1]

(3) $$E_n(f) = o(\omega(n^{-1})), \qquad n \to \infty,$$

is equivalent to

(4) $$\omega(f, t) = o(\omega(t)), \qquad t \to 0+,$$

is exactly the following condition:

(5) $$t \int_t^1 \frac{\omega(u)}{u^2}\, du = O(\omega(t)),$$

where $E_n(f)$ is the best uniform approximation to f by nth-degree trigonometric polynomials (see [1]).

Moreover, we can show that if we want (4) to hold for all functions satisfying (3), then condition (5) is also necessary.

It should also be noted that H^{ω} is not the class for which the rational approximation is better than the polynomial one. Szabados [9] proved that under the condition

$$\lim_{t \to 0+} \frac{\omega(t)}{t} = +\infty,$$

there exists a function $f \in H^{\omega}$ such that

$$\limsup_{n \to \infty} \frac{R_n(f)}{\omega(n^{-1})} > 0.$$

Following Newman, with some careful calculation, we prove the following theorem which gives a characterization for $\lim_{n \to \infty} n^{r+\alpha} R_n(f) = 0$ in W^r Lip α for $0 < \alpha < 1$.

Theorem 1. *Let r be a nonnegative integer, $0 < \alpha < 1$, $f(x) \in W^r$ Lip α. Then*

(6) $$\lim_{n \to \infty} n^{r+\alpha} R_n(f) = 0$$

if and only if

(7) $$\lim_{h \to 0+} h^{-\alpha} \int_{-1}^{1-h} |f^{(r)}(x+h) - f^{(r)}(x)|\, dx = 0.$$

[1] In (3) and (4) "o" can also be replaced by "O" or "$\sim$".

Remark 1. Recall that (see [6]) $R_n(x^\alpha) = O(e^{-\pi\sqrt{\alpha n}})$ for any positive noninteger α, so we see that the condition (7) for $r = 0$ cannot guarantee that $f \in \mathrm{Lip}\ \alpha$ without additional assumptions.

In the particular case $r = 0$, we have the following theorem.

Theorem 2. *Let $\omega(t)$ be a nondecreasing continuous function on $[0, \infty)$ satisfying (1), (2), (5), and*[2]

$$\int_0^t \frac{\omega(u)}{u}\, du = O(\omega(t)). \tag{5'}$$

Assume $f(x) \in H^\omega$. Then

$$\lim_{n\to\infty} \frac{R_n(f)}{\omega(n^{-1})} = 0 \tag{8}$$

if and only if

$$\lim_{h\to 0+} \frac{1}{\omega(h)} \int_{-1}^{1-h} |f(x+h) - f(x)|\, dx = 0. \tag{9}$$

Remark 2. From an example given by the referee, we see that condition (5′) is required for Theorem 2 to hold for all $f \in H^\omega$. This is another difference between rational approximation and polynomial approximation.

In the Zygmund class we have the following theorem.

Theorem 3. *Let r be a nonnegative integer, $f(x) \in W^rZ$. If*

(10) *$F_h^r(x)$ converges to zero in measure as $h \to 0+$,*

then

$$\lim_{n\to\infty} n^{r+1} R_n(f) = 0, \tag{11}$$

where

$$F_h^r(x) = \max\left\{ \left| \frac{f^{(r)}(x+h_1) - f^{(r)}(x)}{h_1} - \frac{f^{(r)}(x) - f^{(r)}(x-h_2)}{h_2} \right| : 0 < h_1, h_2 \le h, \quad x + h_1, x - h_2 \in [-1, 1] \right\}.$$

Remark 3. Since $f^{(r+1)}(x)$ exists almost everywhere on $[-1, 1]$ implies (10), we see that Theorem 3 improves the result of Newman [4] mentioned above.

[2] Condition (5′) is one of the "equivalence conditions" in [1].

2. Proof of Theorem 2

First we deduce (8) from condition (9). Fix an ε, $0 < \varepsilon < 1$. Note that convergence in mean implies convergence in measure, so there is an $h_\varepsilon > 0$ such that, for every h, $0 < |h| \le h_\varepsilon$,

$$\operatorname{mes}\{x\colon |f(x+h) - f(x)| \ge \varepsilon\omega(|h|)\} < \varepsilon^2.$$

Write

$$S_h := \{x\colon |f(x+h) - f(x)| \ge \varepsilon\omega(|h|)\},$$

we then have, for every h, $0 < |h| \le h_\varepsilon$, and $x \in [-1, 1]\backslash S_h$, $x + h \in [-1, 1]$,

$$|f(x+h) - f(x)| \le \varepsilon\omega(|h|). \tag{12}$$

Let

$$S_h(\varepsilon) = \bigcup_{j=1}^{[\varepsilon^{-1}]} S_{jh/[\varepsilon^{-1}]},$$

where $[x]$ is the greatest integer not exceeding x, then evidently,

$$\operatorname{mes}(S_h(\varepsilon)) \le \varepsilon.$$

Given n such that

$$N = \left[\frac{\sqrt{\varepsilon n}}{30}\right] \ge h_\varepsilon^{-1}.$$

Set $S := S_{N^{-1}}(\varepsilon)$. Denote

$$I_j = \left[-1 + \frac{j}{N}, -1 + \frac{j+1}{N}\right], \qquad j = 0, 1, \ldots, 2N-1,$$

$$A = \{I_j\colon I_j \not\subseteq S, j = 0, 1, \ldots, 2N-1\}.$$

Lemma 1. *Let* $I = [a, b] \in A$,

$$l(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a),$$

then, for all $x \in I$,

$$|f(x) - l(x)| \le 6\omega(\varepsilon N^{-1}).$$

Proof. Since $I \not\subseteq S$, there is a $y \in I\backslash S$. For any given $x \in I$, we have a $y'_x \in I$ such that

$$|y - y'_x| = \frac{jN^{-1}}{[\varepsilon^{-1}]} \qquad \text{for some } j, \quad 0 \le j \le [\varepsilon^{-1}],$$

and

$$|y'_x - x| \le \frac{N^{-1}}{[\varepsilon^{-1}]}.$$

Hence, by (12),

$$\begin{aligned}|f(x)-f(y)| &\le |f(x)-f(y'_x)|+|f(y'_x)-f(y)|\\ &\le 2\omega(\varepsilon N^{-1})+\varepsilon\omega(N^{-1})\le 3\omega(\varepsilon N^{-1}).\end{aligned}$$

Thus it follows that

$$|f(y)-l(x)|\le\left|\frac{x-a}{b-a}(f(y)-f(b))+\frac{b-x}{b-a}(f(y)-f(a))\right|\le 3\omega(\varepsilon N^{-1}),$$

and we get

$$|f(x)-l(x)|\le|f(x)-f(y)|+|f(y)-l(x)|\le 6\omega(\varepsilon N^{-1}). \qquad ■$$

Now break up each I of $\{I_j\colon I_j\subset S,\ 0\le j\le 2N-1\}$ into $K=[1/\varepsilon]$ equal subintervals. Denote the collection of all such subintervals by B. Note that $\text{mes}(S)<\varepsilon$, so B contains at most $N\varepsilon$ of the $\{I_j\}$, consequently B has at most $N\varepsilon K\le N$ subintervals.

Lemma 2. *Let* $I=[a,b]\in B$,

$$l(x)=\frac{f(b)-f(a)}{b-a}(x-a)+f(a),$$

then, for all $x\in I$,

$$|f(x)-l(x)|\le 2\omega(\varepsilon N^{-1}).$$

Proof. Write

$$g(x)=f(x)-l(x)$$

and

$$|g(y)|=\max_{x\in[a,b]}|g(x)|,$$

then

$$|g(x)|\le|g(y)-g(a)|\le|f(y)-f(a)|+\left|\frac{f(b)-f(a)}{b-a}\right|(y-a).$$

Since $f\in H^\omega$,

$$|f(y)-f(a)|\le\omega(|y-a|)\le\omega(\varepsilon N^{-1}),$$

$$\left|\frac{f(b)-f(a)}{b-a}\right|(y-a)\le\omega(\varepsilon N^{-1}),$$

so

$$|f(x)-l(x)|\le 2\omega(\varepsilon N^{-1}). \qquad ■$$

Define $L(f, x)$ as follows: For each $I \in A \cup B$, let $L(f, x)$ be equal to $f(x)$ at the endpoints of the interval I, and be linear in between. Combining Lemmas 1 and 2, we obtain, for all $x \in [-1, 1]$,

$$|f(x) - L(f, x)| \leq 6\omega(\varepsilon N^{-1}). \tag{13}$$

Lemma 3 [2, p. 175]. *Let $f \in C_{[-1,1]}$ and $r \geq 1$. Then there exists an* nth-*degree polynomial $p_n(f, x)$ such that*

$$\|f - p_n(f)\| \leq C(r)\omega_r(f, n^{-1}),$$
$$\|p_n^{(r)}(f)\| \leq C(r)n^r\omega_r(f, n^{-1}),$$

where here and throughout the whole paper, $C(x)$ always indicates a constant depending upon x only ($C(x)$ may have different values in different places).

Let $I = [a, b]$ be a subinterval in $A \cup B$. With $x \in I$, by direct calculation,

$$|p_M(f, x) - L(p_M, x)| \leq \left|\frac{p_M(f, b) - p_M(f, a)}{b - a}\right|(x - a) + |p_M(f, x) - p_M(f, a)|$$
$$\leq 2\|p_M'(f)\|N^{-1},$$

and with Lemma 3, we have

$$|p_M(f, x) - L(p_M, x)| \leq CM\omega(M^{-1})N^{-1}. \tag{14}$$

Moreover,

$$|L(f, x) - L(p_M, x)| \leq C\|f - p_M\| \leq C\omega(M^{-1}) \tag{15}$$

also follows from Lemma 3.

Lemma 4 [4]. *Let $T(x) = |x - 1| + |x + 1| - 2|x|$, then*

$$L(f, x) - L(p_M, x) = \sum_{j=1}^{N^*} y_j T(\lambda_j(x - x_j)),$$

where $N^* \leq 5N$, $|y_j| \leq C\omega(M^{-1})$ (see (15)), $-1 \leq x_1 < x_2 < \cdots < x_{N^*} \leq 1$, and each λ_j is either N or NK.

Lemma 5 [4]. *There exists a $t(x) \in R_{2m+1,2m+1}$ such that*

$$|T(x) - t(x)| \leq \frac{C}{1 + x^2} e^{-\sqrt{m/3}}$$

on the whole line.

Write

$$R(x) = \sum_{j=1}^{N^*} y_j t(\lambda_j(x - x_j)),$$

from Lemmas 4 and 5,

$$|L(f, x) - L(p_M, x) - R(x)| \le C\omega(M^{-1})e^{-\sqrt{m/3}} \sum_{j=1}^{N^*} \frac{1}{1 + \lambda_j^2(x - x_j)^2},$$

let $x \in [x_{i-1}, x_i]$, and, say $j \ge i$, from $\lambda_j \ge N$ and $|x - x_j| \ge |i - j| N^{-1}K^{-1}$, we have

$$\frac{1}{1 + \lambda_j^2(x - x_j)^2} \le \frac{K^2}{1 + |i - j|^2},$$

a similar conclusion also holds for $j < i$. Summing over all j we get

$$\sum_{j=1}^{N^*} \frac{1}{1 + \lambda_j^2(x - x_j)^2} \le K^2 \sum_{j=0}^{\infty} \frac{1}{1 + j^2},$$

so we obtain for all $x \in [-1, 1]$ that

$$\begin{aligned} (16) \qquad |L(f, x) - L(p_M, x) - R(x)| &\le C\varepsilon^{-2}\omega(M^{-1})e^{-\sqrt{m/3}} \sum_{j=0}^{\infty} \frac{1}{1 + j^2} \\ &\le C\varepsilon^{-2}\omega(M^{-1})e^{-\sqrt{m/3}}, \end{aligned}$$

and combining (13), (14), and (16), we get

$$\begin{aligned} (17) \quad \|f - p_M(f) - R\| &\le \|f - L(f)\| + \|L(p_M) - p_M(f)\| + \|L(f) - L(p_M) - R\| \\ &\le 6\omega(\varepsilon N^{-1}) + CM\omega(M^{-1})N^{-1} + C\varepsilon^{-2}\omega(M^{-1})e^{-\sqrt{m/3}}. \end{aligned}$$

Lemma 6.1. *Let $\omega(t)$ be a nondecreasing continuous function on $[0, \infty)$ satisfying (1), (2), and (5). Then there exists a constant $M_0 > 1$ such that, for all t, $0 < t \le M_0^{-1}$,*

$$\frac{\omega(M_0 t)}{\omega(t)} \le \tfrac{1}{2}M_0.$$

Proof. Let

$$\int_t^1 \frac{\omega(u)}{u^2}\, du \le M_1 t^{-1}\omega(t).$$

Suppose $0 < t < t_1 < 1$. Since $\omega(t)$ is nondecreasing,

$$\begin{aligned} \int_t^1 \frac{\omega(u)}{u^2}\, du &\ge \int_{t_1}^1 \frac{\omega(u)}{u^2}\, du \\ &\ge \omega(t_1) \int_{t_1}^1 \frac{du}{u^2} \\ &= \omega(t_1)(t_1^{-1} - 1), \end{aligned}$$

hence

$$\omega(t_1)(t_1^{-1} - 1) \le M_1 t^{-1}\omega(t),$$

thus there is a constant M_2 for $0 < t < t_1 < 1$ such that

$$t_1^{-1}\omega(t_1) < M_2 t^{-1}\omega(t),$$

and in the case $0 < t_1 < t_2 < 1$,

$$\frac{\omega(t_2)}{t_2}\int_{t_1}^{t_2}\frac{du}{u} \le M_2\int_{t_1}^{t_2}\frac{\omega(u)}{u^2}\,du \le M_1M_2\frac{\omega(t_1)}{t_1},$$

or

$$\frac{\omega(t_2)}{t_2}\log\frac{t_2}{t_1} \le M_1M_2\frac{\omega(t_1)}{t_1}.$$

In particular when $t_2 = e^{2M_1M_2}t_1$,

$$\frac{\omega(t_2)}{t_2} \le \frac{1}{2}\frac{\omega(t_1)}{t_1}.$$

Take $M_0 = e^{2M_1M_2}$, then, from above discussion for $M_0 t < 1$,

$$\frac{\omega(M_0 t)}{M_0 t} \le \frac{1}{2}\frac{\omega(t)}{t},$$

that is,

$$\frac{\omega(M_0 t)}{\omega(t)} \le \tfrac{1}{2}M_0.$$

■

Lemma 6.2. *Let $\omega(t)$ be a nondecreasing continuous function on $[0, \infty)$ satisfying* (1), (2), *and* (5′). *Then there exists a constant $M_1 > 1$ such that, for all t, $0 < t \le 1$,*

$$\frac{\omega(M_1^{-1}t)}{\omega(t)} \le \tfrac{1}{2}.$$

Proof. The proof is somewhat similar to that of Lemma 6.1. Let

$$\int_0^t \frac{\omega(u)}{u}\,du \le A\omega(t),$$

then, since $\omega(t)$ is monotone for $0 < t_1 < t$,

$$\omega(t_1)\log\left(\frac{t}{t_1}\right) = \omega(t_1)\int_{t_1}^{t}\frac{du}{u} \le \int_{t_1}^{t}\frac{\omega(u)}{u}\,du \le A\omega(t),$$

therefore when $t/t_1 \ge e^{2A}$ we have

$$\omega(t_1) \le \tfrac{1}{2}\omega(t).$$

Let

$$M_1 = e^{2A},$$

it follows that, for all t, $0 < t \le 1$,

$$\omega(M_1^{-1}t) \le \tfrac{1}{2}\omega(t). \qquad \blacksquare$$

Fix a natural number l satisfying

$$\varepsilon \le 2^{-l} < 2\varepsilon,$$

set

$$\varepsilon_0^{-1} = M_0^l,$$

then

$$\varepsilon_0^{-1} \le M_0^{-(\log\varepsilon)/(\log 2)} = \varepsilon^{-(\log M_0)/(\log 2)}. \tag{18}$$

On applying Lemma 6.1, we have

$$\begin{aligned}\omega(\varepsilon_0^{-1}n^{-1}) &\le \tfrac{1}{2}M_0\,\omega(M_0^{l-1}n^{-1}) \\ &\le 2^{-l}\varepsilon_0^{-1}\omega(n^{-1}) \\ &\le 2\varepsilon\varepsilon_0^{-1}\omega(n^{-1}).\end{aligned} \tag{19}$$

Take

$$N = \left[\frac{n}{30}\,\varepsilon^{1/2}\right],$$

$$m = [\varepsilon^{-1/2}],$$

and

$$M = \left[\frac{n}{2}\,\varepsilon_0\right].$$

From (17)–(19),

$$\begin{aligned}&\|f - p_M(f) - R\| \\ &\quad \le C\{\omega(\varepsilon^{1/2}n^{-1}) + \varepsilon^{1/2}\omega(n^{-1}) + \varepsilon^{-1-(\log M_0)/(\log 2)}e^{-3^{-1/2}\varepsilon^{-1/4}}\omega(n^{-1})\},\end{aligned}$$

thus, for small enough ε,

$$\|f - p_M(f) - R\| \le C\omega(\varepsilon^{1/2}n^{-1}),$$

while

$$\deg(p_M(f) + R) = 15Nm + M \le n.$$

Taking Lemma 6.2 into account, we obtain

$$\|f - p_M(f) - R\| \le C2^{\log\sqrt{\varepsilon}/\log M_1}\omega(n^{-1})$$

and we have proved the sufficiency part.

Now we turn to the necessity part of Theorem 2.

Lemma 7 [6, p. 162]. *Let* $r(x) \in R_{n,n}$, *then*

$$\int_{-1}^{1} |r'(x)|\, dx \leq 2n\|r\|.$$

Now applying Lemma 7, in a way similar to the proof of Petrushev and Popov [6, p. 163], we have

$$\int_{-1}^{1-h} |f(x+h) - f(x)|\, dx \leq Ch \sum_{n=1}^{[h^{-1}]} R_n(f).$$

Because

$$\lim_{n\to\infty} \frac{R_n(f)}{\omega(n^{-1})} = 0,$$

there is a sequence ε_n, $\lim_{n\to\infty} \varepsilon_n = 0$, such that

$$R_n(f) \leq \varepsilon_n \omega(n^{-1}).$$

By standard method,

$$\sum_{n=1}^{[h^{-1/2}]} R_n(f) \leq C \sum_{n=1}^{[h^{-1/2}]} \omega\left(\frac{1}{n+1}\right) \leq C \sum_{n=1}^{[h^{-1/2}]} \int_{1/(n+1)}^{1/n} \frac{\omega(u)}{u^2}\, du \leq C \int_{h^{1/2}}^{1} \frac{\omega(u)}{u^2}\, du,$$

similarly,

$$\sum_{n=[h^{-1/2}]+1}^{[h^{-1}]} R_n(f) \leq C \max_{h^{-1/2}\leq k\leq h^{-1}} \varepsilon_k \int_{h}^{1} \frac{\omega(u)}{u^2}\, du.$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{[h^{-1}]} R_n(f) &= \sum_{n=1}^{[h^{-1/2}]} R_n(f) + \sum_{n=[h^{-1/2}]+1}^{[h^{-1}]} R_n(f)\\ &\leq \int_{h^{1/2}}^{1} \frac{\omega(u)}{u^2}\, du + \max_{h^{-1/2}\leq k\leq h^{-1}} \varepsilon_k \int_{h}^{1} \frac{\omega(u)}{u^2}\, du.\end{aligned}$$

Combining all the above estimates and condition (5), we get

$$\int_{x,x+h\in[-1,1]} |f(x+h) - f(x)|\, dx \leq C\left\{h^{1/2}\omega(h^{1/2}) + \max_{h^{-1/2}\leq k\leq h^{-1}} \{\varepsilon_k\}\omega(h)\right\},$$

$$\lim_{h\to 0} \max_{h^{-1/2}\leq k\leq h^{-1}} \{\varepsilon_k\} = 0,$$

and, with Lemma 6.1,

$$\frac{h^{1/2}\omega(h^{1/2})}{\omega(h)} \leq 2^{-m_0+1},$$

where

$$m_0 = -\frac{\log h}{2\log M_0}.$$

Altogether, we have now obtained (9) and finished the proof of Theorem 2. ∎

3. Proof of Theorem 1

It is easy to deduce (6) from condition (7) by using Theorem 2 and the following lemma:

Lemma 8 [6, p. 244]. *If $f \in C^r_{[-1,1]}$, then the estimate*

$$R_n(f) \le C(r)n^{-r-2}\sum_{j=0}^{n} jR_j(f^{(r)}), \qquad n \ge r,$$

holds.

Now we turn to the necessity part of Theorem 1.

Lemma 9 [3]. *Let $R(x) \in R_{n,n}$, $A \subset (-\infty, \infty)$. Then, for each given $\delta > 0$, there exists a set $e \subset (-\infty, \infty)$ with* mes$(e) \le \delta$ *such that, for all $x, x' \in A \backslash e$,*

$$|R(x) - R(x')| < \frac{2n}{\delta}\|R\|_A|x - x'|.$$

Lemma 10 [3]. *Let $r \ge 1$. Assume $f \in C_{[-1,1]}$ and*

$$\sum_{j=0}^{\infty}\left(\frac{R_j(f)}{j+1}\right)^{1/(r+1)} < \infty.$$

Then, for each given $\delta > 0$, there exists a set $E_1^n \subset (-\infty, \infty)$ with mes$(E_1^n) \le \delta$ *such that $f(x)$ has r continuous derivatives on $A_n := [-1, 1]\backslash E_1^n$ and*

$$\begin{aligned} I_n(f^{(r)}, A_n) &:= \|f^{(r)}(x) - R_n^{(r)}(f, x)\|_{A_n} \\ &\le \frac{C(r)}{\delta^r}\left\{\sum_{k \ge [n/2]}\left(\frac{R_k(f)}{k+1}\right)^{1/(r+1)}\right\}^{r+1}, \end{aligned}$$

where $R_n(f, x)$ is the best rational approximant to f from $R_{n,n}$.

Let $r \ge 1$. In our case, from Lemma 10, for each given $\delta > 0$, there exists a set $E_1^n \subset (-\infty, \infty)$ with mes$(E_1^n) \le \delta/2$ such that

$$\begin{aligned} I_n(f^{(r)}, A_n) &\le \frac{C(r)}{\delta^r}\left\{\sum_{k \ge [n/2]}\left(\frac{R_k(f)}{k+1}\right)^{1/(r+1)}\right\}^{r+1} \\ &= o\left(\frac{C(r)}{\delta^r}\left\{\sum_{k \ge [n/2]} k^{-1-\alpha/(r+1)}\right\}^{r+1}\right) \\ &= o(\delta^{-r}n^{-\alpha}), \qquad n \to \infty. \end{aligned}$$

Write

$$I_n(f^{(r)}, A_n) = \delta^{-r}\varepsilon_n^*(n+1)^{-\alpha}, \tag{20}$$

$$\varepsilon_n^* \to 0, \qquad n \to \infty,$$

$$t_n(x) = R_n^{(r)}(f, x).$$

For any given $h > 0$, fix k satisfying

$$2^k \le h^{-1} < 2^{k+1},$$

then

$$|f^{(r)}(x+h) - f^{(r)}(x)| \le 2I_{2^k}(f^{(r)}, A_{2^k}) + \sum_{j=1}^{k} |\Delta_h P_j(x)| + |\Delta_h P_0(x)|$$

for x, $x + h \in A_{2^k}$, where

$$P_0(x) = t_1(x) - t_0(x),$$

$$P_j(x) = t_{2^j}(x) - t_{2^{j-1}}(x), \qquad j = 1, 2, \ldots, k,$$

and

$$\Delta_h f(x) = f(x+h) - f(x).$$

By (20), we can find a sequence of sets e_j^* with $\operatorname{mes}(e_j^*) \le \delta_j/3$ such that, for $x \in [-1, 1]\backslash e_j^*$,

$$I_0(f^{(r)}, [-1,1]\backslash e_0^*) \le \delta_0^{-r}\varepsilon_0^*, \qquad j = 0,$$

$$I_{2^j}(f^{(r)}, [-1,1]\backslash e_j^*) \le \delta_j^{-r}\varepsilon_{2^j}^* 2^{-j\alpha}, \qquad j = 1, 2, \ldots, k.$$

At the same time there is another sequence of sets e_j' with $\operatorname{mes}(e_j') \le \delta_{j+1}/3$ such that, for $x \in [-1, 1]\backslash e_j'$,

$$I_{2^j}(f^{(r)}, [-1, 1]\backslash e_j') \le \delta_{j+1}^{-r}\varepsilon_{2^j}^* 2^{-j\alpha}, \qquad j = 0, 1, \ldots, k-1.$$

While, by applying Lemma 9, we have the third sequence of sets e_j'' with $\operatorname{mes}(e_j'') \le \delta_j/3$ such that, for x, $x + h \in [-1, 1]\backslash e_j''$,

$$|\Delta_h P_j(x)| \le 2n\delta_j^{-1} h \|P_j\|_{[-1,1]\backslash e_j''}, \qquad j = 0, 1, \ldots, k.$$

Altogether for x, $x + h \in [-1, 1]\backslash e_j$,

$$|\Delta_h P_0| \le \frac{2}{\delta_0} h(I_0(f^{(r)}, [-1, 1]\backslash e_0) + I_1(f^{(r)}, [-1, 1]\backslash e_0))$$

$$\le \frac{C(r)}{\delta_0^{r+1}} \varepsilon_0 h, \qquad j = 0,$$

$$|\Delta_h P_j| \le \frac{2}{\delta_j} 2^j h(I_{2^j}(f^{(r)}, [-1, 1]\backslash e_j) + I_{2^{j-1}}(f^{(r)}, [-1, 1]\backslash e_j))$$

$$\le \frac{C(r)}{\delta_j^{r+1}} \varepsilon_{2^j} h 2^{j(1-\alpha)}, \qquad j = 1, 2, \ldots, k,$$

where

$$e_j := e_j^* \cup e'_{j-1} \cup e''_j, \qquad \mathrm{mes}(e_j) \le \delta_j, \qquad e'_{-1} = \varnothing, \qquad j = 0, 1, \dots, k,$$
$$\varepsilon_0 = \max\{\varepsilon_0^*, \varepsilon_1^*\},$$
$$\varepsilon_{2^j} = \max\{\varepsilon_{2^{j-1}}^*, \varepsilon_{2^j}^*\}, \qquad j = 1, 2, \dots, k.$$

Choose

$$\delta_0 = \frac{\delta}{2}\left\{(\varepsilon_0)^{1/(r+2)} + \sum_{i=1}^{k}(\varepsilon_{2^i}2^{i(1-\alpha)})^{1/(r+2)}\right\}^{-1}(\varepsilon_0)^{1/(r+2)},$$
$$\delta_j = \frac{\delta}{2}\left\{(\varepsilon_0)^{1/(r+2)} + \sum_{i=1}^{k}(\varepsilon_{2^i}2^{i(1-\alpha)})^{1/(r+2)}\right\}^{-1}(\varepsilon_{2^j}2^{j(1-\alpha)})^{1/(r+2)}, \qquad j = 1, 2, \dots, k,$$

then, for x, $x + h \in A_{2^k}\backslash e := A^*$, $e = \bigcup_{j=0}^{k} e_j$ with $\mathrm{mes}(e) \le \delta/2$,

$$\begin{aligned}
&|f^{(r)}(x+h) - f^{(r)}(x)| \\
&\quad \le 2I_{2^k}(f^{(r)}, A^*) + C(r)h\delta^{-r-1}\left\{(\varepsilon_0)^{1/(r+2)} + \sum_{i=1}^{k}(\varepsilon_{2^i}2^{i(1-\alpha)})^{1/(r+2)}\right\}^{r+2} \\
&\quad \le C(r)\delta^{-r}\varepsilon_{2^k}2^{-\alpha k} + C(r)\delta^{-r-1}h \\
&\qquad \times\left(\sum_{0\le j\le k/2}\varepsilon_{2^j}^{1/(r+2)}2^{j(1-\alpha)/(r+2)} + \max_{j\ge k/2}\{\varepsilon_{2^j}^{1/(r+1)}\}\sum_{k/2<j\le k}2^{j(1-\alpha)/(r+2)}\right)^{r+2} \\
&\quad \le C(r)\delta^{-r-1}\left(h^{(1+\alpha)/2} + \max_{j\ge k/2}\{\varepsilon_{2^j}\}h^{\alpha}\right) \\
&\quad := \sigma_h\delta^{-r-1}h^{\alpha},
\end{aligned}$$

where

$$\lim_{x+h\in A^*, h\to 0+} \sigma_h = 0$$

for each $x \in A^*$.

From the above discussion we obtain

$$\mathrm{mes}\{\{x\} \cup \{x+h\}: h^{-\alpha}|f^{(r)}(x+h) - f^{(r)}(x)| > \sigma_h\delta^{-r-1}\} := \mathrm{mes}(E_\delta^h) \le 2\delta.$$

Thus, for any given $\varepsilon > 0$, we have

$$\begin{aligned}
&h^{-\alpha}\int_{-1}^{1-h}|f^{(r)}(x+h) - f^{(r)}(x)|\,dx \\
&\quad \le h^{-\alpha}\left(\int_{E_\varepsilon^h} + \int_{[-1,1-h]\backslash E_\varepsilon^h}\right)|f^{(r)}(x+h) - f^{(r)}(x)|\,dx \\
&\quad \le 2\varepsilon + 2\sigma_h\varepsilon^{-r-1}
\end{aligned}$$

since $f^{(r)} \in \text{Lip}\ \alpha$, so, for small enough h,

$$h^{-\alpha} \int_{-1}^{1-h} |f^{(r)}(x+h) - f^{(r)}(x)|\, dx \leq 3\varepsilon.$$

So we have proved Theorem 1. ∎

4. Proof of Theorem 3

Fix an ε, $0 < \varepsilon < \frac{1}{2}$, and assume $r = 0$. There is an $h_\varepsilon > 0$ such that, for every $0 < h \leq h_\varepsilon$,

$$\text{mes}(S_h) := \text{mes}\{x: F_h^0(x) > \varepsilon\} < \varepsilon,$$

we then have, for all $0 < h_1, h_2 < h$, and $x \in [-1, 1] \backslash S_h$,

$$\left| \frac{f(x+h_1) - f(x)}{h_1} - \frac{f(x) - f(x-h_2)}{h_2} \right| \leq \varepsilon. \tag{21}$$

Given n such that $N = [\varepsilon^{4/5} n/84] \geq h_\varepsilon^{-1}$, set $S := S_{N^{-1}}$,

$$I_j = \left[-1 + \frac{j}{N}, -1 + \frac{j+1}{N}\right], \qquad j = 0, 1, \ldots, 2N-1,$$

$$A = \left\{I_j: \text{mes}(I_j \cap S) \leq \frac{1}{4N}, j = 0, 1, \ldots, 2N-1\right\},$$

and break up each interval in

$$\left\{I_j: \text{mes}(I_j \cap S) > \frac{1}{4N}, j = 0, 1, \ldots, 2N-1\right\}$$

into $K = [1/\varepsilon]$ equal subintervals. Denote the collection of all (at most $4N\varepsilon K \leq 4N$) such small subintervals by B.

Lemma 11. *Let $I = [a, b] \in B$,*

$$l(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a),$$

then, for all $x \in I$,

$$|f(x) - l(x)| \leq \varepsilon N^{-1}.$$

Proof. Let

$$g(x) = f(x) - l(x)$$

and

$$|g(y)| = \max_{x \in [a,b]} |g(x)|.$$

Without loss we may suppose $y \le (b-a)/2$. From $g(a) = 0$ and $f \in Z$,

$$\begin{aligned}|g(x)| &\le 2|g(y)| - |g(2y-a)| \le |g(a) + g(2y-a) - 2g(y)| \\ &= |f(a) + f(2y-a) - 2f(y)| \le |y-a| \le \varepsilon N^{-1},\end{aligned}$$

which is the required result. ■

Lemma 12. *Let $I = [a, b] \in A$,*

$$l(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a),$$

then, for all $x \in I$,

$$|f(x) - l(x)| \le 16\varepsilon N^{-1}.$$

Proof. Write

$$g(x) = f(x) - l(x)$$

and

$$|g(y)| = \max_{x \in [a,b] \setminus S} |g(x)|,$$

then, from (21),

$$\left|\frac{g(y)(b-a)}{(y-a)(y-b)}\right| = \left|\frac{f(b) - f(y)}{b - y} - \frac{f(y) - f(a)}{y - a}\right| \le \varepsilon,$$

that is,

(22) $$|g(y)| \le \varepsilon N^{-1}.$$

Now we show that

(23) $$\sup_{x \in [a,b] \cap S} |g(x)| \le 16\varepsilon N^{-1}.$$

Suppose (23) does not hold, which is equivalent to the existence of a $z \in S$ such that

(24) $$|g(z)| > 16\varepsilon N^{-1}.$$

Assume by symmetry that

$$z \ge \frac{b-a}{2}.$$

Since

$$\text{mes}([a, b] \cap S) \le \frac{1}{4N},$$

there is a point $u \in [a, (b-a)/2] \backslash S$ such that

$$u - a \geq \frac{1}{8N}, \tag{25}$$

consequently,

$$\begin{aligned}\left|\frac{g(u)(z-a)}{(u-a)(u-z)} - \frac{g(z)}{u-z}\right| &= \left|\frac{g(z)-g(u)}{z-u} - \frac{g(u)-g(a)}{u-a}\right| \\ &= \left|\frac{f(z)-f(u)}{z-u} - \frac{f(u)-f(a)}{u-a}\right| \\ &\leq \varepsilon,\end{aligned}$$

which follows from (21) together with $u \in [a, (b-a)/2] \backslash S$. At the same time, by (22) and (25),

$$\left|\frac{g(u)(z-a)}{(u-a)(u-z)}\right| \leq 8\varepsilon N^{-1}|u-z|^{-1},$$

and, with assumption (24), we get

$$\left|\frac{g(u)(z-a)}{(u-a)(u-z)} - \frac{g(z)}{u-z}\right| > \frac{|g(z)|}{2|u-z|}.$$

That is,

$$|g(z)| < 2\varepsilon|u-z| < 2\varepsilon N^{-1},$$

which leads to a contradiction to (24). Therefore Lemma 12 follows from combining (22) and (23). ■

Define $L(f, x)$ as follows: For each $I \in A \cup B$, let $L(f, x)$ be equal to $f(x)$ at the endpoints of the interval I, and be linear in between. Combining Lemmas 11 and 12, we get, for all $x \in [-1, 1]$,

$$|f(x) - L(f, x)| \leq 16\varepsilon N^{-1}. \tag{26}$$

Applying Lemma 3, we can find an Mth-degree polynomial $p_M(f, x)$ such that

$$\begin{aligned}\|f - p_M(f)\| &\leq C(r)M^{-1}, \\ \|p_M''(f)\| &\leq C(r)M.\end{aligned} \tag{27}$$

Hence, for $I = [a, b] \in A \cup B$ and $x \in I$, on considering

$$p_M(f, a) = L(p_M, a), \qquad p_M(f, b) = L(p_M, b),$$

we have

$$\begin{aligned}\left|\frac{(p_M(f,x) - L(p_M, x))(b-a)}{(x-a)(x-b)}\right| &= |p_M'(f, \beta) - p_M'(f, \gamma)| \\ &\leq \|p_M''(f)\|\ |\beta - \gamma| \\ &\leq CN^{-1}M\end{aligned}$$

from (27), where $\beta, \gamma \in (a, b)$, that is,

$$|p_M(f, x) - L(p_M, x)| \le CN^{-2}M \tag{28}$$

for $x \in I$.

At last, as in the proof of Theorem 2, using Lemmas 4 and 5, we can deduce that there is a rational $R(x) \in R_{42Nm,42Nm}$ such that

$$\|L(f) - L(p_M) - R\| \le C\varepsilon^{-2}M^{-1}m^{-5}. \tag{29}$$

Altogether from (26), (28), and (29),

$$\|f(x) - p_M(f, x) - R(x)\| \le C\{\varepsilon N^{-1} + N^{-2}M + \varepsilon^{-2}M^{-1}m^{-5}\}.$$

Now choose

$$N = \left[\frac{n}{84}\varepsilon^{4/5}\right],$$

$$m = [\varepsilon^{-4/5}],$$

and

$$M = [n\varepsilon^{9/5}].$$

We have

$$\|f - p_M(f) - R\| \le C\varepsilon^{1/5}n^{-1}$$

and

$$\deg(p_M(f) + R) = 42Nm + M \le n.$$

For general $r \ge 1$, the result (9) follows by applying the above result and Lemma 8. The proof is complete. ■

Acknowledgment. P. B. Borwein was supported in part by NSERC of Canada.

References

1. N. K. Bari, S. B. Stechkin (1956): *Best approximation and differential properties of two conjugate functions.* Trudy Moskov. Mat. Obshch., **5**:483–522 (MR 18:303).
2. D. Leviatan (1982): *The behavior of the derivatives of the algebraic polynomials of best approximation.* J. Approx. Theory, **35**:169–176.
3. G. G. Lorentz (1966): Approximation of Functions. New York: Holt, Rinehart, and Winston.
4. D. J. Newman (1978): *Approximation with Rational Functions.* Providence, RI: American Mathematical Society.
5. D. J. Newman (1981): *Polynomials and rational functions.* In: Approximation Theory and Applications (Proc. Workshop, Technion–Israel Inst. Tech., Haifa, 1980), New York: Academic Press, pp. 265–282.
6. P. P. Petrushev, V. A. Popov (1987): Rational Approximation of Real Functions. Cambridge: Cambridge University Press.

7. V. A. POPOV (1977): *Uniform rational approximation of the class V_r and its applications.* Acta Math. Acad. Sci. Hungar., **29**:119–129.
8. H. L. ROYDEN (1968): Real Analysis. New York: Macmillan.
9. J. SZABADOS (1967): *Negative results in the theory of rational approximation.* Studia Sci. Math. Hungar., **2**:385–390.

P. B. Borwein
Department of Mathematics, Statistics,
and Computing Science
Dalhousie University
Halifax
Nova Scotia
Canada B3H 3J5

S. P. Zhou
Department of Mathematics, Statistics,
and Computing Science
Dalhousie University
Halifax
Nova Scotia
Canada B3H 3J5