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ON APPROXIMATION BY TRIGONOMETRIC LAGRANGE INTERPOLATING POLYNOMIALS II

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We show that trigonometric Lagrange interpolating approximation with arbitrary real distinct nodes in L^p space for $1 \leqslant p < \infty$, as that with equally spaced nodes in L^p space for $1 < p < \infty$ in an earlier paper by T.F. Xie and S.P. Zhou, may also be arbitrarily "bad". This paper is a continuation of this earlier work by Xie and Zhou, but uses a different method.

Let $L^p_{2\pi}$, $1 \leqslant p \leqslant \infty$ be the class of real integrable functions of power p and of period 2π and let $L^\infty_{2\pi} = C_{2\pi}$ the class of all real continuous functions of period 2π.

For $f \in L^1_{2\pi}$, $S_n(f,x)$ is the nth partial sum of the Fourier series of $f(x)$; for $f \in L^p_{2\pi}$, $E_n(f)_p$ is the nth best approximation of $f(x)$ in L^p; for $f \in C_{2\pi}$, $L^X_n(f,x)$ is the nth trigonometric Lagrange interpolating polynomial of $f(x)$ with distinct nodes $X_n = \{x_{n,j}\}_{j=0}^{2n}$ (by $a \neq b$ we mean that $a \not\equiv b (\bmod 2\pi)$). In particular,

$$L_n(f,x) = \sum_{k=0}^{2n} f(x_k) l_k(x)$$

is the nth trigonometric Lagrange interpolating polynomial of $f(x)$ with equally spaced nodes, where

$$l_k(x) = \frac{1}{2n+1} \frac{\sin(n+1/2)(x-x_k)}{\sin(x-x_k)/2},$$
$$x_k = \frac{2k\pi}{2n+1}, \quad k = 0,1,\cdots,2n.$$

The norm of $f \in L^p_{2\pi}$ is defined as follows.

$$\|f\|_{L^p} = \left(\int_0^{2\pi} |f(x)|^p dx\right)^{1/p}, \quad 1 \leqslant p < \infty,$$
$$\|f\| = \|f\|_{L^\infty} = \max_{0 \leqslant x \leqslant 2\pi} |f(x)|.$$

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Although

$$\|L_n\| = \sup\{\|L_n f\| : \|f\| = 1\} \sim \|S_n\| \sim \log(n+1),$$

(whereby $A_n \sim B_n$ we indicate that there exists a positive constant M independent of n such that $M^{-1} \leqslant A_n/B_n \leqslant M$) the story for the behaviour of these two linear operators in L^p space is different. Throughout the paper, $C(x)$ always indicates a positive constant depending upon x and C indicates a positive absolute constant, which may have different values at different places. For Fourier partial sums, by applying the well-known Riesz theorem (see, for example, Zygmund [4]) one has

$$\|f - S_n(f)\|_{L^p} \leqslant C(p)E_n(f)_p, \quad 1 < p < \infty,$$

while for Lagrange interpolation with equally spaced nodes, the work [3] proved that there exists an infinitely differentiable function $f \in C_{2\pi}$ such that

$$\limsup_{n\to\infty} \frac{\|f - L_n(f)\|_{L^p}}{\lambda_n^{-1} E_n(f)_p} > 0, \quad 1 < p < \infty,$$

where $\{\lambda_n\}$ is any given positive decreasing sequence with

$$n^s \lambda_n \to 0$$

for any $s > 0$.

One might ask what happens in L^1 space? Though in many cases L^1 possesses similar properties to L^∞ by duality, it appears not to happen in this case. Furthermore, what happens for Lagrange interpolation with arbitrary real distinct nodes in L^p space for $1 \leqslant p < \infty$? Since the constructive method used in [3] is no longer valid in these cases, the present paper will use a different idea to construct the required counterexample.

THEOREM. *Let $1 \leqslant p < \infty$. Suppose that $\{X_n\}$ is a given sequence of real distinct nodes and $\{\lambda_n\}$ is any given positive decreasing sequence. Then there exists an infinitely differentiable function $f \in C_{2\pi}$ such that*

$$\limsup_{n\to\infty} \frac{\|f - L_n^X(f)\|_{L^p}}{\lambda_n^{-1} \|f - S_n(f)\|_{L^p}} > 0.$$

COROLLARY. *Let $1 \leqslant p < \infty$. Suppose that $\{X_n\}$ is a given sequence of real distinct nodes and $\{\lambda_n\}$ is any given positive decreasing sequence. Then there exists an infinitely differentiable function $f \in C_{2\pi}$ such that*

$$\limsup_{n\to\infty} \frac{\|f - L_n(f)\|_{L^p}}{\lambda_n^{-1} E_n(f)_p} > 0.$$

Lemma 1. *Let $1 \leqslant p < \infty$. Suppose that $X_n = \{x_{n,j}\}_{j=0}^{2n}$ is a sequence of real distinct nodes and N_n is a natural number. Then there exists a function $h_n \in C_{2\pi}$ such that*

$$h_n(x_{n,0}) = 0,$$

(1) $$1 \leqslant h_n(x_{n,j}) \leqslant \|h_n\| \leqslant 2n, \quad j = 1, 2, \cdots, 2n,$$

and

(2) $$\|h_n\|_{L^p} \leqslant CnN_n^{-2/p}.$$

Proof: Because of the period 2π, without loss of generality we can assume that

$$0 = x_{n,0} < x_{n,1} < x_{n,2} < \cdots < x_{n,2n} < 2\pi.$$

Let

$$N^*_{n_j} = \frac{2\pi - x_{n,j}}{x_{n,j}} N_n$$

for $1 \leqslant j \leqslant 2n$. Then it is clear that $x^{N_n}(2\pi - x)^{N^*_{n_j}}$ has a maximum point $x_{n,j}$. Write

$$\rho_{n,j} := x_{n,j}^{N_n}(2\pi - x_{n,j})^{N^*_{n_j}},$$

set

$$h_n(x) = \sum_{k=1}^{2n} \rho_{n,k}^{-1} x^{N_n}(2\pi - x)^{N^*_{n_k}}$$

for $x \in [0, 2\pi)$, and extend it to the whole line with period 2π. Evidently, $h_n \in C_{2\pi}$ and

$$h_n(0) = h_n(2\pi) = 0.$$

We clearly have

$$h_n(x_{n,j}) \geqslant \rho_{n,j}^{-1} x_{n,j}^{N_n}(2\pi - x_{n,j})^{N^*_{n_j}} = 1$$

for $1 \leqslant j \leqslant 2n$. At the same time,

$$h_n(x_{n,j}) \leqslant \|h_n\| \leqslant \sum_{k=1}^{2n} \rho_{n,k}^{-1} x_{n,j}^{N_n}(2\pi - x_{n,j})^{N^*_{n_k}} = 2n.$$

On the other hand, a calculation yields

$$\left\| x^{N_n}(2\pi - x)^{N^*_{n_j}} \right\|_{L^p} = (2\pi)^{N_n + N^*_{n_j} + 1/p} \left(\frac{\Gamma(N_n p + 1)\Gamma\left(N^*_{n_j} p + 1\right)}{\Gamma\left(N_n p + N^*_{n_j} p + 2\right)} \right)^{1/p}$$

$$\leqslant C\rho_{n,j} N_n^{-p/2},$$

so

$$\|h_n\|_{L^p} \leqslant CnN_n^{-p/2}.$$

The proof of Lemma 1 is thus completed. ∎

Lemma 2. *Let $1 \leqslant p < \infty$. Suppose that $X_n = \{x_{n,j}\}_{j=0}^{2n}$ is a sequence of real distinct nodes and that $\{\lambda_n\}$ is a given positive decreasing sequence. Then there exists a trigonometric polynomial $g_n(x)$ of degree M_n such that for large enough n,*

$$\|g_n\| = O(n\delta_n^{-1}),$$
$$\|g_n - S_n(g_n)\|_{L^p} = O(\lambda_n),$$

and

$$\left\|g_n - L_n^X(g_n)\right\|_{L^p} \geqslant C,$$

where
$$\delta_n = 2^{-2n/p} \prod_{0\leqslant i\neq j\leqslant 2n} \left\|\sin\frac{x_{n,i}-x_{n,j}}{2}\right\|^{1/p}.$$

Proof: Let $h_n(x)$ be the function defined in Lemma 1. We first establish

(3)
$$\|h_n - S_n(h_n)\|_{L^p} = O\left(n\log(n+1)N_n^{-2/p}\right),$$

and

(4)
$$\left\|h_n - L_n^X(h_n)\right\|_{L^p} \geqslant C2^{-2n/p}\eta_n^{1/p} - CnN_n^{-2/p},$$

where
$$\eta_n = \prod_{0\leqslant i\neq j\leqslant 2n} \left\|\sin\frac{x_{n,i}-x_{n,j}}{2}\right\|.$$

Inequality (3) is straightforward: we just need to apply (2) and the estimation of the Lebesgue constant. Now write

$$L_n^X(h_n,x) = \sum_{j=0}^{2n} h_n(x_{n,j})l_j^X(x),$$

where
$$l_j^X(x) = \frac{\prod_{k\neq j}\sin\frac{x-x_{n,k}}{2}}{\prod_{k\neq j}\sin\frac{x_{n,j}-x_{n,k}}{2}}.$$

Since

$$\sin\frac{x-x_{n,k}}{2} = \sin\frac{x_{n,j}-x_{n,k}}{2}\cos\frac{x-x_{n,j}}{2} + \cos\frac{x_{n,j}-x_{n,k}}{2}\sin\frac{x-x_{n,j}}{2},$$

for $x \in [x_{n,j} - n^{-1}2^{-2n}\eta_n, x_{n,j} + n^{-1}2^{-2n}\eta_n]$, we have

(5)
$$l_j^X(x) = 1 + O(n^{-1}).$$

Meanwhile, for $x \in [x_{n,j} - n^{-1}2^{-2n}\eta_n, x_{n,j} + n^{-1}2^{-2n}\eta_n]$ and $i \neq j$,

$$(6) \qquad |l_i^X(x)| = \frac{|\prod_{k\neq i} \sin\frac{x-x_{n,k}}{2}|}{\prod_{k\neq i} |\sin\frac{x_{n,i}-x_{n,k}}{2}|} \leqslant \frac{|x - x_{n,j}| \left\| \frac{d}{dx}\left(\prod_{k\neq i} \sin\frac{x-x_{n,k}}{2}\right)\right\|}{\eta_n} \leqslant 2^{-2n}.$$

Combining (5), (6) and (1), for sufficiently large n we get

$$\begin{aligned} \|h_n - L_n^X(h_n)\|_{L^p} &\geqslant \left(\sum_{j=1}^{2n} \int_{x_{n,j}-n^{-1}2^{-2n}\eta_n}^{x_{n,j}+n^{-1}2^{-2n}\eta_n} \left\|\sum_{k=0}^{2n} h_n(x_{n,j}) l_k^X(x)\right\|^p dx\right)^{1/p} - \|h_n\|_{L^p} \\ &\geqslant \left(\sum_{j=1}^{2n} C^p n^{-1} 2^{-2n}\eta_n\right)^{1/p} - CnN_n^{-1/(2p)} \\ &\geqslant C2^{-2n/p}\eta_n^{1/p} - CnN_n^{-1/(2p)}, \end{aligned}$$

that is, (4).

Without loss of generality suppose that $\lambda_n \leqslant 1$. Now choose

$$N_n = [n^{2p} \log^{2p}(n+1) 2^{4n} \eta_n^{-2} \lambda_n^{-2p} + 1];$$

then (3), (4) become

$$(7) \qquad \|h_n - S_n(h_n)\|_{L^p} = O(\delta_n \lambda_n),$$

and

$$(8) \qquad \|h_n - L_n^X(h_n)\|_{L^p} \geqslant C\delta_n,$$

where

$$\delta_n = 2^{-2n/p}\eta_n^{1/p}.$$

Because $h_n \in C_{2\pi}$, we may select a trigonometric polynomial g_n^* with sufficiently large degree $M_n \geqslant n$ such that

$$(9) \qquad \|h_n - g_n^*\| \leqslant \delta_n^2 \lambda_n \min\{\log^{-1}(n+1), (\|L_n^X\| + 1)^{-1}\}.$$

Hence by (7) and (9),

$$\begin{aligned} \|g_n^* - S_n(g_n^*)\|_{L^p} &\leqslant \|g_n^* - h_n\| + \|S_n(h_n) - S_n(g_n^*)\| + \|h_n - S_n(h_n)\|_{L^p} \\ &\leqslant \delta_n^2 \lambda_n \log^{-1}(n+1)(1 + \|S_n\|) + C\delta_n\lambda_n \\ &\leqslant C\delta_n\lambda_n. \end{aligned}$$

Similarly, from (8) and (9),

$$\begin{aligned}\|g_n^* - L_n^X(g_n^*)\|_{L^p} &\geqslant \|h_n - L_n^X(h_n)\|_{L^p} - \|g_n^* - h_n\| - \|L_n^X(h_n) - L_n^X(g_n^*)\| \\ &\geqslant C\delta_n - \delta_n^2\lambda_n(\|L_n^X\| + 1)^{-1}(1 + \|L_n^X\|) \\ &\geqslant C\delta_n\end{aligned}$$

for large enough n. Set

$$g_n(x) = \delta_n^{-1}g_n^*(x);$$

then from the above discussion we get the required inequality. □

PROOF OF THE THEOREM: Select a sequence $\{n_j\}$ inductively: Let $n_1 = 1$. After n_j, choose

$$n_{j+1} = \left[m_{n_j}^2 \lambda_{n_j}^{-1/n_j}\left(\left\|L_{n_j}^X\right\| + \log n_j\right) + 1\right], \tag{10}$$

where

$$m_n = M_n\left(n^2\delta_n^{-2/n} + 1\right).$$

Define

$$f(x) = \sum_{j=1}^{\infty} m_{n_j}^{-n_j} g_{n_j}(x).$$

Clearly $f(x) \in C_{2\pi}$ is infinitely differentiable since $g_{n_j}(x)$ is a trigonometric polynomial of degree m_{n_j} and $\|g_n\| = O(n\delta_n^{-1})$. Together with (10), Lemma 2 implies that

$$\begin{aligned}\left\|f - L_{n_j}^X(f)\right\|_{L^p} &\geqslant m_{n_j}^{-n_j}\left\|g_{n_j} - L_{n_j}^X\left(g_{n_j}\right)\right\|_{L^p} - C\left(\left\|L_{n_j}^X\right\| + 1\right)\sum_{k=j+1}^{\infty} m_{n_k}^{-n_k}\left\|g_{n_k}\right\| \\ &\geqslant Cm_{n_j}^{-n_j} - Cm_{n_{j+1}}^{-n_{j+1}/2}\lambda_{n_j} \geqslant Cm_{n_j}^{-n_j}.\end{aligned}$$

At same time, by (10) and Lemma 2 again,

$$\begin{aligned}\left\|f - S_{n_j}(f)\right\|_{L^p} &= O\left(m_{n_j}^{-n_j}\left\|g_{n_j} - S_{n_j}\left(g_{n_j}\right)\right\|_{L^p} + \left(\left\|S_{n_j}\right\| + 1\right)\sum_{k=j+1}^{\infty} m_{n_k}^{-n_k}\left\|g_{n_k}\right\|\right) \\ &= O\left(m_{n_j}^{-n_j}\lambda_{n_j} + m_{n_{j+1}}^{-n_{j+1}/2}\right) = O\left(m_{n_j}^{-n_j}\lambda_{n_j}\right).\end{aligned}$$

Altogether,

$$\frac{\left\|f - L_{n_j}^X(f)\right\|_{L^p}}{\lambda_{n_j}^{-1}\left\|f - S_{n_j}(f)\right\|_{L^p}} \geqslant C > 0,$$

which is the required result. □

REMARK. In spite of the counterexample in the present paper, there are several positive results in this direction. For example, [1, 2] discuss the rate of convergence of $L_n(f, x)$ to $f(x)$ in L^p, in terms of the sequence of best approximation of the function in L^p.

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