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MARKOV'S AND BERNSTEIN'S INEQUALITIES ON DISJOINT INTERVALS

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1. Introduction. In 1889, A. A. Markov proved the following inequality:

INEQUALITY 1. (Markov [**4**]). *If p_n is any algebraic polynomial of degree at most n then*

$$\|p_n'\|_{[a,b]} \leqq \frac{2n^2}{b-a}\|p_n\|_{[a,b]}$$

where $\| \ \|_A$ *denotes the supremum norm on A.*

In 1912, S. N. Bernstein established

INEQUALITY 2. (Bernstein [**2**]). *If p_n is any algebraic polynomial of degree at most n then*

$$|p_n'(x)| \leqq \frac{n}{((x-a)(b-x))^{1/2}}\|p_n\|_{[a,b]}$$

for $x \in (a, b)$.

In this paper we extend these inequalities to sets of the form $[a, b] \cup [c, d]$. Let Π_n denote the set of algebraic polynomials with real coefficients of degree at most n.

THEOREM 1. *Let $a < b \leqq c < d$ and let $p_n \in \Pi_n$. Then*

$$|p_n'(x)| \leqq \left(\frac{c-x}{d-x}\right)^{1/2}\frac{n}{((b-x)(x-a))^{1/2}}\|p_n\|_{[a,b]\cup[c,d]}$$

for $x \in (a, b)$.

We note that Inequality 2 is a special case ($b = c = d$) of the above theorem.

COROLLARY 1. *Let $a < b \leqq c < d$ and let $p_n \in \Pi_n$. Then*

$$|p'(x)| \leqq \left(\frac{x-b}{x-a}\right)^{1/2}\frac{n}{((x-c)(d-x))^{1/2}}\|p_n\|_{[a,b]\cup[c,d]},$$

for $x \in (c, d)$.

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COROLLARY 2. *Let $a < b \leqq c < d$ and let $p_n \in \Pi_n$. Then,*

$$||p_n'||_{[c,d]} \leqq \left(\frac{d-b}{d-a}\right)^{1/2} \frac{2n^2}{d-c} ||p_n||_{[a,b] \cup [c,d]}.$$

Thus, we obtain sharper bounds than those we achieve by applying Inequality 1 or Inequality 2 directly to $[c, d]$.

On sets of the form $[-b, -a] \cup [a, b]$ we can derive an asymptotically "best possible" form of Markov's inequality.

THEOREM 2. a) *If $0 < a < b$, n is even and $p_n \in \Pi_n$, then*

$$||p_n'||_{[-b,-a] \cup [a,b]} \leqq \left(1 + \frac{9}{n^2}\right) \frac{n^2 b}{b^2 - a^2} ||p_n||_{[-b,-a] \cup [a,b]}$$

provided that n is large enough to satisfy

$$\frac{b^2 - a^2}{3abn} + \frac{(b+a)}{2b}\left(1 + \frac{6}{n}\right)^2 e^{6(b^2-a^2)5abn} \leqq 1.$$

b) *For each even n there exists $p_n \in \Pi_n$ so that*

$$||p_n'||_{[-b,-a] \cup [a,b]} = \frac{n^2 b}{b^2 - a^2} ||p_n||_{[-b,-a] \cup [a,b]}.$$

COROLLARY 3. *Suppose n is even and $n \geqq 50$. If $p_n \in \Pi_n$ then*

$$||p_n'||_{[-2,-1] \cup [1,2]} \leqq \left(1 + \frac{9}{n^2}\right) \frac{2n^2}{3} ||p_n||_{[-2,-1] \cup [1,2]}.$$

2. Characterizing polynomials that maximize Markov's or Bernstein's inequalities. In this section we show that polynomials that maximize $|p_n'(t)|$, subject to $||p_n||_I \leqq 1$ where I is compact, must be of the form

$$\alpha x^n + \beta x^{n-1} - q_{n-2}(x)$$

where $q_{n-2} \in \Pi_{n-2}$ is the best approximation to $\alpha x^n + \beta x^{n-1}$ on I. In particular, we show, as Bernstein did for the interval $[0, 1]$ (see [**2**]), that the polynomial that satisfies $||p_n||_I \leqq 1$ and has maximum derivative at $\max I$ is of the form

$$p_n(x) = ax^n - q_{n-1}(x)$$

where $q_{n-1} \in \Pi_{n-1}$ and q_{n-1} is the best approximation to ax^n on I.

THEOREM 3. *Let I be any infinite compact set of real numbers and let $\zeta \in R$. Suppose $p_n \in \Pi_n$ satisfies*

$$(1) \quad \frac{|p_n'(\zeta)|}{||p_n||_I} = \max_{\substack{q_n \in \Pi_n \\ q_n \neq 0}} \frac{|q_n'(\zeta)|}{||q_n||_I}.$$

Then, there exist α *and* β *so that* $p_n(x) = \alpha x^n + \beta x^{n-1} - s_{n-2}(x)$ *where* $s_{n-2} \in \Pi_{n-2}$ *is the best Chebyshev approximation to* $\alpha x^n + \beta x^{n-1}$ *on* I. *(The best Chebyshev approximation is the one that minimizes the supremum norm.)*

We need the following lemma for the proof of this theorem:

LEMMA 1. *Let* $p_n \in \Pi_n$ *and let* ζ *be any point that is not a root of* p_n. *Suppose that there exist at most* $k \leqq n-2$ *points* $x_1 < x_2 < \ldots < x_k$ *where* p_n *changes sign. Then there exists* $q_n \in \Pi_n$ *so that*

a) $\operatorname{sgn} q_n'(\zeta) = \operatorname{sgn} p_n'(\zeta)$,

b) $\operatorname{sgn} q_n(x) = -\operatorname{sgn} p_n(x)$, *except possibly at the roots of* q_n.

Proof. Let

$$s(x) = -(\operatorname{sgn} p_n(-\infty))(-1)^k \prod_{i=1}^{k} (x - x_i)$$

and consider $q_n^y(x) = s(x)(x-y)^2$. Then, if $s(\zeta) \neq 0$,

$$\left.\frac{dq_n^y(x)}{dx}\right|_{\zeta} = (\zeta - y)(2s(\zeta) + (\zeta - y)s'(\zeta))$$

which as a function of y changes sign at ζ. Thus, for an appropriate y close to ζ, q_n^y satisfies a) and b).

Proof of Theorem 3. Let p_n satisfy the assumptions of the theorem (that such a p_n exists is a simple consequence of Π_n being finite dimensional).

Suppose p_n has at most $n-2$ changes of sign and suppose $p_n(\zeta) \neq 0$. If q_n satisfies the conclusion of Lemma 1, then for sufficiently small $\epsilon > 0$,

$$\|p_n + \epsilon q_n\|_I \leqq \|p_n\|_I \quad \text{and} \quad |p_n'(\zeta) + \epsilon q_n'(\zeta)| > |p_n'(\zeta)|$$

which contradicts the assumption that p_n satisfies (1). Now suppose $p_n(\zeta) = 0$ and p_n changes sign at $x_1 < \ldots < x_k$. If

$$q_n(x) = -(\operatorname{sgn} p_n(-\infty))(-1)^k\left(\prod_{i=1}^{k} (x - x_i)\right)(x - \zeta)^2$$

then, for sufficiently small $\epsilon > 0$,

$$\|p_n + \epsilon q_n\|_I < \|p_n\|_I \quad \text{and} \quad |p_n'(\zeta) + \epsilon q_n'(\zeta)| = |p_n'(\zeta)|$$

which also contradicts the assumption that p_n satisfies (1). Thus, p_n has at least $n-1$ sign changes.

We now suppose that the coefficient of x^n is non-zero for p_n. It follows that p_n has n real roots $x_1 < x_2 < \ldots < x_n$. We claim that in each interval (x_j, x_{j+1}) there exists a point $y_i \in I$ so that

(2) $|p_n(y_i)| = \|p_n\|_I$.

If (2) is false then as in the proof of the lemma, we can, for a suitably chosen y, construct

$$q_n(x) = -(\operatorname{sgn} p_n(-\infty))(-1)^n\left(\prod_{i=1}^{j-1}(x-x_i)\right)\left(\prod_{i=j+2}^{n}(x-x_i)\right) \times (x-y)^2$$

where

a) $\operatorname{sgn} q_n'(\zeta) = \operatorname{sgn} p_n'(\zeta)$

and

b) $\operatorname{sgn} q_n(x) = -\operatorname{sgn} p_n(x)$,

except possibly for $x \in \{x_1, \ldots, x_n, y\} \cup [x_j, x_{j+1}]$. We note that since the y of Lemma 1 can be chosen from an interval, we may assume that $|p_n(y)| \neq \|p_n\|_I$. It follows from a), b) and the assumption

$$\|p_n\|_{[x_j, x_{j+1}]} < \|p_n\|_I$$

that for sufficiently small $\epsilon > 0$,

$$\|p_n + \epsilon q_n\|_I < \|p_n\|_I$$

and

$$|p_n'(\zeta) + \epsilon q_n'(\zeta)| \geqq |p_n'(\zeta)|.$$

This contradiction establishes (2).

We may by a similar argument show that there exists y_n so that

$$y_n \in I \cap (-\infty, x_1) \quad \text{or} \quad y_n \in I \cap (x_n, \infty)$$

and

$$|p_n(y_n)| = \|p_n\|_I.$$

Thus, if $p_n(x) = \alpha x^n + \beta x^{n-1} - s_{n-2}(x)$ where $\alpha \neq 0$, then p_n achieves its maximum norm, with alternate sign, at n points $y_1 < y_2 < \ldots < y_n$ in I. This suffices to establish the theorem.

If p_n is actually of degree $n - 1$, then $p_n(x) = \beta x^{n-1} - q_{n-2}(x)$. A similar argument shows that $q_{n-2}(x)$ is the best approximation to βx^{n-1} on I.

THEOREM 4. *Let I be any infinite compact set and let $\zeta \geqq \delta = \max I$. Suppose $p_n \in \Pi_n$ satisfies*

$$(1) \quad \frac{|p_n'(\zeta)|}{\|p_n\|_I} = \max_{\substack{q_n \in \Pi_n \\ q_n \neq 0}} \frac{|q_n'(\zeta)|}{\|q_n\|_I}.$$

Then $p_n(x) = \alpha x^n - q_{n-1}(x)$ where $q_{n-1} \in \Pi_{n-1}$ and q_{n-1} is the best Chebyshev approximation to αx^n on I.

Proof. Let $\gamma = \min I$. The preceding theorem guarantees the existence of $n-1$ points $\gamma < x_1 < \dots < x_{n-1} < \delta$ where p_n changes sign. We first show that p_n has n distinct roots in $[\gamma, \delta]$. Suppose p_n does not change sign at any point in $[\gamma, \delta]$ other than $x_1, \dots, x_{n-1}$. Consider

$$q_n^y(x) = -\operatorname{sgn}(p_n(\delta))\left(\prod_{k=1}^{n-1}(x - x_k)\right)(y - x)$$

$$= s_n(x)(y - x)$$

then

$$\left.\frac{dq_n^y(x)}{dx}\right|_\zeta = s_n'(\zeta)(y - \zeta) - s_n(\zeta).$$

Since $\operatorname{sgn} s_n'(\zeta) = \operatorname{sgn} s_n(\zeta) \neq 0$ we may, for a suitable choice of $y > \zeta$, set $t_n = q_n^y$ where

a) $\operatorname{sgn} t_n'(\zeta) = \operatorname{sgn} p_n'(\zeta)$
b) $\operatorname{sgn} t_n = -\operatorname{sgn} p_n$ on I.

Thus, for sufficiently small $\epsilon > 0$,

$$\|p_n + \epsilon t_n\|_I < \|p_n\|_I \quad \text{and} \quad |p_n'(\zeta) + \epsilon t_n'(\zeta)| > |p_n'(\zeta)|$$

which is a contradiction. Thus, p_n has n distinct roots $\gamma \leqq x_1 < x_2 < \dots < x_n \leqq \delta$. We now show that

$$|p_n(\delta)| = |p_n(\gamma)| = \|p_n\|_I.$$

This, coupled with (2) of the proof of Theorem 3, suffices to complete the result. We will only show that $|p_n(\delta)| = \|p_n\|_I$ since the proof that $|p_n(\gamma)| = \|p_n\|_I$ is similar. Suppose $|p_n(\delta)| < \|p_n\|_I$. Let

$$q_n(x) = -(\operatorname{sgn} p_n(-\infty))(-1)^{n-1}\left(\prod_{i=1}^{n-1}(x - x_i)\right)(y - x)$$

where, as before, $y > \zeta$ is chosen so that

$$\operatorname{sgn} q_n'(\zeta) = \operatorname{sgn} p_n'(\zeta).$$

Then, for sufficiently small $\epsilon > 0$, $p_n + \epsilon q_n$ contradicts the assumption that p_n satisfies (1).

3. Bernstein's inequality on $[a, b] \cup [c, d]$.

Proof of Theorem 1. Let $A = [a, b] \cup [c, d]$ and let $\tau \in A$. Let $p_n \in \Pi_n$ satisfy

$$\frac{|p_n'(\tau)|}{\|p_n\|_A} = \max_{q_n \in \Pi_n} \frac{|q_n'(\tau)|}{\|q_n\|_A}$$

and

$$\|p_n\|_A = 1.$$

We may, by the proof of Theorem 3, assume that p_n has all its roots in A with the possible exceptions of a root $\lambda_1 \in (b, c)$ and a root $\lambda_2 > d$ or $\lambda_2 < a$. We treat the case where $\lambda_1 \in (b, c)$ and $\lambda_2 > d$. The other cases proceed analogously. We observe that if we increase c or a and if we decrease b or d we strengthen the inequality in the statement of the theorem. Thus, we may also assume that for $y \in \{a, b, c, d\}$,

$$|p_n(y)| = 1 \quad \text{and} \quad |p_n'(y)| \neq 0.$$

(If there is no point $z \in (b, c)$ where $|p_n(z)| \geqq 1$ then we can deduce the result from Inequality 2.) We have guaranteed the existence of points

$$b < \epsilon_1 < \delta_1 < \lambda_1 < \delta_2 < \epsilon_2 < c$$

and

$$d < \epsilon_3 < \delta_3 < \lambda_2 < \delta_4$$

so that

$$|p_n'(\epsilon_i)| = 0 \qquad i = 1, 2, 3$$

and

$$|p_n(\delta_i)| = 1 \qquad i = 1, 2, 3, 4.$$

We deduce from Theorem 3 and a comparison of roots and leading terms that

$$(p_n'(x))^2(x-a)(x-b)(x-c)(x-d)(x-\delta_1)(x-\delta_2)(x-\delta_3) \times (x-\delta_4)$$
$$= n^2((p_n(x))^2 - 1)(x-\epsilon_1)^2(x-\epsilon_2)^2(x-\epsilon_3)^2.$$

Thus, if $\tau \in (a, b)$,

$$(p_n'(\tau))^2 \leqq \frac{n^2(\tau-\epsilon_2)^2}{|(\tau-a)(\tau-b)(\tau-c)(\tau-d)|} \cdot \frac{(\tau-\epsilon_1)^2}{(\tau-\delta_1)(\tau-\delta_2)} \cdot \frac{(\tau-\epsilon_3)^2}{(\tau-\delta_3)(\tau-\delta_4)} \leqq \frac{n^2(\tau-c)^2}{|(\tau-a)(\tau-b)(\tau-c)(\tau-d)|}$$

and the result now follows.

Corollary 1 follows immediately from Theorem 1. Corollary 2 is a consequence of Corollary 1 and the next inequality.

INEQUALITY 3. (Schur [3] p. 41). *If* $p_{n-1} \in \Pi_{n-1}$ *and*

$$|p_{n-1}(x)| \leqq \frac{L}{((x-a)(b-x))^{1/2}} \quad \text{for } a < x < b,$$

then

$$||p_{n-1}(x)||_{[a,b]} \leqq \frac{2Ln}{b-a}.$$

4. Markov's inequality on $[-b, -a] \cup [a, b]$. We require the following results for the proof of Theorem 2.

THEOREM 5. (Achieser [**1**], p. 287). *Let n be an even integer. The polynomial $p_n \in \Pi_n$ with leading coefficient 1 that deviates least from zero on $[-b, -a] \cup [a, b]$ is*

$$S_n(x) = \frac{(b^2 - a^2)^{n/2}}{2^{n-1}} T_{n/2}\left(\frac{2x^2 - b^2 - a^2}{b^2 - a^2}\right)$$

where T_n is the n^{th} Chebyshev polynomial ($T_n = \cos n \cos^{-1}x$).

LEMMA 2. *Let n be even and let S_n be defined as in Theorem* 5. *Then,*

$$\frac{||S'||_{[-b,-a] \cup [a,b]}}{||S||_{[-b,-a] \cup [a,b]}} = \frac{|S'(b)|}{||S||_{[-b,-a] \cup [a,b]}} = \frac{n^2 b}{b^2 - a^2}.$$

The proof of Lemma 2 is straightforward and is omitted.

LEMMA 3. *Suppose n is even. Then*

$$\max_{\substack{p_n \in \Pi_n \\ p_n \neq 0}} \frac{|p_n'(b)|}{||p_n||_{[-b,-a] \cup [a,b]}} = \frac{n^2 b}{b^2 - a^2}.$$

Proof. This is a direct consequence of Theorem 4, Theorem 5 and Lemma 2.

LEMMA 4. (Soble [**5**]). *If $p_n \in \Pi_n$ has non-negative coefficients then, for* $x > 0$

$$|p_n'(x)| \leqq \frac{n}{x} |p_n(x)|.$$

Proof of Theorem 2. Suppose $p_n \in \Pi_n$ satisfies

$$\frac{||p_n'||_{[-b,-a] \cup [a,b]}}{||p_n||_{[-b,-a] \cup [a,b]}} = \max_{q_n \in \Pi_n} \frac{||q_n'||_{[-b,-a] \cup [a,b]}}{||q_n||_{[-b,-a] \cup [a,b]}}.$$

Suppose $\zeta \in [a, b]$ is a point where

$$|p'(\zeta)| = ||p'||_{[-b,-a] \cup [a,b]}$$

and

$$(1) \quad |p_n'(\zeta)| > \frac{n^2 b}{b^2 - a^2} ||p_n||_{[-b,-a] \cup [a,b]}.$$

Then, by Inequality 2 applied to $[a, b]$

$$\frac{n^2 b}{b^2 - a^2} \leqq \frac{n}{((b - \zeta)(\zeta - a))^{1/2}}$$

and

$$(b - \zeta)(\zeta - a) \leqq \frac{(b^2 - a^2)^2}{n^2 b^2}.$$

Since either $(b - \zeta) \geqq \frac{1}{2}(b - a)$ or $(\zeta - a) \geqq \frac{1}{2}(b - a)$, either

$$(b - \zeta) \leqq \frac{2(b + a)(b^2 - a^2)}{b^2 n^2} \quad \text{or} \quad (\zeta - a) \leqq \frac{2(b + a)(b^2 - a^2)}{b^2 n^2}.$$

Suppose

$$(2) \quad (b - \zeta) \leqq \frac{2(b + a)}{b} \cdot \frac{(b^2 - a^2)}{bn^2} \leqq \frac{4(b^2 - a^1)}{bn^2}.$$

Then, by Lemma 3 and (2), for $n \geqq 10$,

$$\max_{\substack{p_n \in \Pi_n \\ p_n \neq 0}} \frac{\|p_n'\|_{[-b,-a] \cup [a,b]}}{\|p_n\|_{[-b,-a] \cup [a,b]}} \leqq \max_{\substack{p_n \in \Pi_n \\ p_n \neq 0}} \frac{|p_n'(\zeta)|}{\|p_n\|_{[-\zeta,-a] \cup [a, \zeta]}} \leqq \frac{n^2 \zeta}{\zeta^2 - a^2}$$

$$\leqq \frac{n^2 b}{\left(b - \dfrac{4(b^2 - a^2)}{bn^2}\right)^2 - a^2} \leqq \left(1 + \frac{9}{n^2}\right) \frac{n^2 b}{b^2 - a^2}.$$

Suppose now that $(\zeta - a) \leqq 4(b^2 - a^2)/bn^2$. Write $p_n(x) = q_m(x) r_h(x)$ where $q_m(x)$ has all its roots in $[-b, -a]$ and $r_h(x)$ has no roots in $[-b, -a]$. By Theorem 3, p_n oscillates between its maximum and minimum at least n times on $[-b, -a] \cup [a, b]$. Hence, p_n has at least $n - 2$ distinct roots in $[-b, -a] \cup [a, b]$. By the proof of Theorem 3, between any two roots of p_n there is a point of $[-b, -a] \cup [a, b]$ where p_n attains its norm. Suppose now that $m \geqq 2 + n/2$. Then, $p_n(x) - p_n(-x) \in \Pi_{n-1}$ has at least $n/2$ roots in $[-b, -a]$ and at least $n/2$ roots in $[a, b]$ and hence, $p_n(x) = -p_n(-x)$. However, if p_n is even, then it follows from Theorem 3, Theorem 5 and Lemma 3 that $p_n = S_n$ and we are done. Thus, we may assume $m \leqq n/2 + 1$. Similarly, since r_n has at least $h - 2$ roots in $[a, b]$, we may assume that $h \leqq n/2 + 3$. We may also assume that $n \geqq 10$.

$$(3) \quad |q_m'(\zeta)| \leqq \frac{m}{a + \zeta} |q_m(\zeta)| \leqq \frac{n + 2}{4a} |q_m(\zeta)|.$$

Also, since $q_m(x) = \alpha \Pi (x + x_i)$ with $x_i \geqq a$,

$$(4) \quad \frac{|q_m(\zeta)|}{|q_m(a)|} = \Pi\left(\frac{\zeta + x_i}{a + x_i}\right) \leqq \Pi\left(1 + \frac{\zeta - a}{a + x_i}\right) \leqq \left(1 + \frac{2(b^2 - a^2)}{abn^2}\right)^{(n+2)/2}$$

$$\leqq e^{6(b^2 - a^2)/5abn}.$$

By Inequality 1,

$$(5)\quad |r_h'(\zeta)| \leqq \frac{2h^2}{b-a}\,\|r_h\|_{[a,b]} \leqq \frac{2\left(\frac{n}{2}+3\right)^2}{b-a}\,\|r_h\|_{[a,b]}.$$

Thus, by (3), (4) and (5),

$$\begin{aligned}
|p_n'(\zeta)| &\leqq |q_m'(\zeta)|\,|r_h(\zeta)| + |r_h'(\zeta)|\,|q_m(\zeta)| \\
&\leqq \frac{n+2}{4a}\,\|p_n\|_{[a,b]} + \frac{2\left(\frac{n}{2}+3\right)^2}{b-a}\,\|r_h\|_{[a,b]}\,|q_m(\zeta)| \\
&\leqq \frac{n+2}{4a}\,\|p_n\|_{[a,b]} + \frac{2\left(\frac{n}{2}+3\right)^2}{b-a}\,\|p_n\|_{[a,b]}\,\frac{|q_m(\zeta)|}{|q_m(a)|} \\
&\leqq \left(\frac{b^2-a^2}{3abn} + \frac{(b+a)}{2b}\left(1+\frac{6}{n}\right)^2 e^{6(b^2-a^2)/5abn}\right)\frac{n^2 b}{b^2-a^2} \\
&\qquad \times \|p_n\|_{[a,b]}.
\end{aligned}$$

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