Reprinted from JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS Vol. 154, No. 2, January 15, 1991
Printed in Belgium

Quadratically Converging Rational Mean Iterations

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Submitted by Bruce C. Berndt

Received April 7, 1989

Various quadratic mean iterations like

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \frac{(a_n + b_n)^3}{8a_n b_n}$$

are explicitly analysed. The limits of these iterations and the uniformizing functions are usually expressible in terms of infinite products of the form

$$\prod_{n=0}^{\infty} (1 + aq^{2^n}).$$

Thus we provide a storehouse of examples of mean iterations which are neither elementary nor intimately related to the arithmetic-geometric mean iteration.

1. INTRODUCTION

Various mean iterations, for example,

$$a_{n+1} := \frac{a_n + b_n}{2}$$

and

$$b_{n+1} := \frac{2}{1/a_n + 1/b_n}$$

have algebraic limits. In the above case, if we commence with $a_0, b_0 > 0$ then the common limit of $\{a_n\}$ and $\{b_n\}$ is $\sqrt{a_0 b_0}$. While another small

* Research supported in part by NSERC of Canada.

0022-247X/91 $3.00

class of examples are related to the arithmetic–geometric mean iteration of Gauss, Lagrange, and Legendre, that is

$$a_{n+1} := \frac{a_n + b_n}{2}$$

and

$$b_{n+1} := \sqrt{a_n b_n}.$$

In this case the common limit commencing with $a_0 := 1$ and $b_0 := x$ is,

$$\frac{\pi/2}{\int_0^{\pi/2} \frac{d\Theta}{\sqrt{1-(1-x^2)\sin^2\Theta}}}$$

which is (essentially) an elliptic integral of the second kind and is the solution of a linear differential equation of the second order with algebraic coefficients. For much more on these matters see [3].

The kind of examples we wish to explore in this paper are iterations like

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \frac{(a_n + b_n)^3}{8 a_n b_n}$$

and

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \frac{(a_n + b_n)\, a_n b_n}{a_n^2 + b_n^2}.$$

In both cases the limits are expressible in terms of non-elementary (in fact hypertranscendental) infinite products, as are the uniformizing parameters. Definitions of these terms are provided in the next section.

2. The Main Construction

For the purposes of this paper a *mean* is any function of two variables

$$M: R \times R \to R$$

defined in some neighbourhood U_M of $(1, 1) \cap \{x > 0, y > 0\}$ that satisfies

$$M(1, 1) = 1 \tag{2.1}$$

$$M(\lambda x, \lambda y) = \lambda M(x, y), \qquad (x, y), (\lambda x, \lambda y) \in U_M \tag{2.2}$$

$$\min(x, y) \leqslant M(x, y) \leqslant \max(x, y), \qquad (x, y) \in U_M. \tag{2.3}$$

A *mean iteration* is a two term iteration given by

$$a_{n+1}=M_1(a_n, b_n); \qquad b_{n+1}=M_2(a_n, b_n) \tag{2.4}$$

with starting values (a_0, b_0) in some neighbourhood of $(1, 1)$. In all the cases we will look at, the means are analytic functions in each variable in some complex neighbourhood of $(1, 1)$. Such means we call *analytic* means. In this case, there exists a complex neighbourhood U of $(1, 1)$ so that if

$$(a_0, b_0)\in U$$

then $\{a_n\}$ and $\{b_n\}$ converge to a common limit which we denote by

$$\mathbf{M}_1\otimes\mathbf{M}_2(a_0, b_0). \tag{2.5}$$

Furthermore, the convergence is quadratic, that is

$$|a_{n+1}-b_{n+1}|=O(|a_n-b_n|^2). \tag{2.6}$$

(This is all dealt with in [3, Chap. 8].)

F and G are called uniformizing parameters of the iteration if, F and G are functions that satisfy

$$F(q^2)=M_1(F(q), G(q)) \tag{2.7}$$

and

$$G(q^2)=M_2(F(q), G(q)), \tag{2.8}$$

where F and G are defined in a neighbourhood of the origin and $F(0)=G(0)=1$.

For the arithmetic-geometric mean iteration the uniformizing parameters are squares of theta functions, namely,

$$F(q)=\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2=:(\Theta_3(q))^2 \tag{2.9}$$

and

$$G(q)=\left(\sum_{n=-\infty}^{\infty} (-q)^{n^2}\right)^2=:(\Theta_4(q))^2 \tag{2.10}$$

note that if F and G uniformize $\mathbf{M}_1\otimes\mathbf{M}_2$, then, with (2.7) and (2.8)

$$\begin{aligned}\mathbf{M}_1\otimes\mathbf{M}_2(F(q), G(q)) &= \cdots = \mathbf{M}_1\otimes\mathbf{M}_2(F(q^{2^n}), G(q^{2^n}))\\ &=\mathbf{M}_1\otimes\mathbf{M}_2(1, 1)=1\end{aligned} \tag{2.11}$$

so with (2.2)

$$\mathbf{M}_1 \otimes \mathbf{M}_2 \left(\frac{F(q)}{G(q)}, 1 \right) = \frac{1}{G(q)}. \tag{2.12}$$

In particular identifying the uniformizing parameter goes a long way towards analyzing the iteration. (See, in particular, [1, 3, 5].)

We will call $T(\cdot, \cdot)$ an (analytic) *multiplier* if T maps some neighbourhood U of $(1, 1)$ to R and satisfies

$$T(1, 1) = 1 \tag{2.13}$$

and

$$T(x, y) = T\left(\frac{x}{y}, 1 \right). \tag{2.14}$$

We use the notation $T(x) := T(x, 1)$. Note the ratio of two means forms a multiplier. The main construction is the content of the next theorem. In all the cases we consider $T(x)$ is analytic in a (complex) neighbourhood of 1.

THEOREM 1. *Let $M(\cdot, \cdot)$ be an analytic mean and $T(\cdot, \cdot)$ be an analytic multiplier. Let R be a solution of the functional equation*

$$R(q^2) = T(R(q)) \tag{2.15}$$

analytic is a neighbourhood of 0, *and satisfying* $R(0) = 1$. *Then the iteration*

$$a_{n+1} = M(a_n, b_n) \cdot T(a_n, b_n) := N(a_n, b_n) \tag{2.16}$$

and

$$b_{n+1} = M(a_n, b_n) \tag{2.17}$$

satisfies

$$\mathbf{N} \otimes \mathbf{M}(R(q), 1) = \prod_{n=0}^{\infty} M(R(q^{2^n}), 1). \tag{2.18}$$

The uniformizing parameters are given by

$$F(q) = \frac{R(q)}{\prod_{n=0}^{\infty} (M(R(q^{2^n}), 1))} \tag{2.19}$$

and

$$G(q) = \frac{1}{\prod_{n=0}^{\infty} M(R(q^{2^n}), 1)} \tag{2.20}$$

so

$$\mathbf{N} \otimes \mathbf{M}(F(q), G(q)) = 1 \tag{2.21}$$

for q in some complex neighbourhood of zero.

Proof. It is standard that (2.15) has an analytic solution of the requisite form (see, for example, [5]) though in the applications of the Theorem we exhibit the solution explicitly.

Note also, since M and R are "analytic" and $M(R(0), 1) = 1$, that F and G and the product (2.18) are well-defined analytic functions in some neighbourhood of the origin. Furthermore, by (2.19) and (2.20)

$$\frac{F(q)}{G(q)} = R(q) \tag{2.22}$$

so, by (2.15)

$$\frac{F(q^2)}{G(q^2)} = T\left(\frac{F(q)}{G(q)}\right). \tag{2.23}$$

In order to establish the limit formula (2.18) it suffices to observe that

$$\mathbf{N} \otimes \mathbf{M}(F(q), G(q)) = \mathbf{N} \otimes \mathbf{M}(N(F(q), G(q)), M(F(q), G(q)))$$

since the limit must be invariant on replacing (a_n, b_n) by (a_{n+1}, b_{n+1}) thus

$$\begin{aligned} \mathbf{N} \otimes \mathbf{M}(F(q), G(q)) &= \mathbf{N} \otimes \mathbf{M}(N(F(q), G(q)), M(F(q), G(q))) \\ &= M(F(q), G(q))\, \mathbf{N} \otimes \mathbf{M}(T(F(q), G(F(q)), 1) \\ &= M(F(q), G(q))\, \mathbf{N} \otimes \mathbf{M}\left(T\left(\frac{F(q)}{G(q)}\right), 1\right) \\ &= M(F(q), G(q))\, \mathbf{N} \otimes \mathbf{M}(R(q^2), 1), \end{aligned}$$

where we have used homogeneity and (2.23) to deduce the equalities. If we divide through by $G(q)$ we derive

$$\begin{aligned} \mathbf{N} \otimes \mathbf{M}(R(q), 1) &= M(r(q), 1)\, \mathbf{N} \otimes \mathbf{M}(R(q^2), 1) \\ &= \prod_{n=0}^{\infty} M(R(q^{2^n}), 1). \end{aligned}$$

While we assumed the existence of the limit of the iteration to derive the result, and the existence is apparent in the examples we treat, the existence is, in fact, guaranteed by the convergence of the above product (see [3]). Since

$$G(q)=\frac{1}{\prod_{n=0}^{\infty} M(R(q^{2^n}),1)}$$

it follows that

$$\mathbf{N}\otimes\mathbf{M}(F(q),G(q))=G(q)\,\mathbf{N}\otimes\mathbf{M}(R(q),1)=1$$

and that

$$\mathbf{N}\otimes\mathbf{M}(F(q^2),G(q^2))=G(q^2)\,\mathbf{N}\otimes\mathbf{M}(R(q^2),1)=1.$$

However, we deduced above that

$$\mathbf{N}\otimes\mathbf{M}(F(q),G(q))=M(F(q),G(q))\,\mathbf{N}\otimes\mathbf{M}(R(q^2),1)$$

so

$$G(q^2)=M(F(q),G(q)).$$

Similarly

$$F(q^2)=N(F(q),G(q))$$

and we see that F and G are the uniformizing parameters. ∎

We should note that, the iteration of the above theorem need not be a mean iteration because $N(\cdot,\cdot)$ need not satisfy property (2.3) of being a mean.

3. Examples with T Rational

Particularly interesting examples are generated from Theorem 1 in the case where T and R are both rational functions. This can arise in the following way

LEMMA 1. (A) *If*

$$R(q) := \frac{1+aq}{1+bq}$$

and

$$T(x, y) := \frac{(bx-ay)^2 + a(x-y)^2}{(bx-ay)^2 + b(x-y)^2}$$

then

$$R(q^2) = T(R(q), 1).$$

(B) *If*

$$R(q) = \frac{1+aq+q^2}{1-aq+q^2}$$

and

$$T(x, y) = \frac{(a^2+a-2)(x^2+y^2) + 2(a^2-a+2)\,xy}{(a^2-a-2)(x^2+y^2) + 2(a^2+a+2)\,xy}$$

then

$$R(q^2) = T(R(q), 1).$$

Proof. In both cases the proof is an entirely rational calculation, and as such is most amenable to being checked symbolically (in Maple, or Macsyma or one of the other symbol manipulation packages). ∎

It is in fact, rather hard to find non-trivial rational R and T, that satisfy

$$R(q^2) = T(R(q), 1)$$

(with $R(0) = 1$). For such a solution to exist $T(x, 1)$ must be of the form $(ax^2 + bx + c)/(dx^2 + ex + f)$.

We list, in tabular form a number of specializations corresponding to different a and b in the above lemma, coupled with a variety of different means in Theorem 1. The notation is that of Theorem 1. All the following examples are derived in this fashion, the mean and the choices of a and b are specified in each case. In every case the underlying convergence is quadratic. In a number of places we have simplified using the identity of Euler

$$\prod_{n=0}^{\infty} (1+q^{2^n}) = \frac{1}{1-q}.$$

It is the judicious choice of M, a, and b, that determines the simplicity of the examples. See Table I.

TABLE I

	$N(a_n, b_n)$	$M(a_n, b_n)$	Limit
1.	$\frac{(a_n+b_n)(a_n^2+b_n^2)}{4a_nb_n}$	$\frac{a_n+b_n}{2}$	$\mathbf{N}\otimes\mathbf{M}\left(\frac{1+q}{1-q}, 1\right) = \prod_{n=0}^{\infty} \frac{1}{(1-q^{2^n})}$
2.	$\frac{(a_n+b_n)(a_n^2+6a_nb_n+b_n^2)}{3a_n^2+2a_nb_n+3b_n^2}$	$\frac{a_n+b_n}{2}$	$\mathbf{N}\otimes\mathbf{M}\left(\frac{1-2q}{1+2q}\right) = \prod_{n=0}^{\infty} \frac{1}{(1+2q^{2^n})}$
3.	$\frac{a_n^2+b_n^2}{2\sqrt{a_nb_n}}$	$\sqrt{a_nb_n}$	$\mathbf{N}\otimes\mathbf{M}\left(\frac{1+q}{1-q}, 1\right) = \left(\prod_{n=0}^{\infty} \frac{1+q^{2^n}}{1-q^{2^n}}\right)^{1/2}$
4.	$\frac{(a_n^2+b_n^2)^2}{2(a_n+b_n)\,a_nb_n}$	$\frac{a_n^2+b_n^2}{a_n+b_n}$	$\mathbf{N}\otimes\mathbf{M}\left(\frac{1+q}{1-q}, 1\right) = \frac{1}{(1-q^2)} \prod_{n=0}^{\infty} \frac{1}{(1-q^{2^n})}$
5.	$\frac{a_n^3+b_n^3}{2a_nb_n}$	$\frac{a_n^3+b_n^3}{a_n^2+b_n^2}$	$\mathbf{N}\otimes\mathbf{M}\left(\frac{1+q}{1-q}, 1\right) = (1-q^2) \prod_{n=0}^{\infty} \left(\frac{1+3q^{2^{n+1}}}{1-q^{2^n}}\right)$
6.	$\frac{4a_n^2b_n^2}{(a_n+b_n)(a_n^2+b_n^2)}$	$\frac{2}{1/a_n+1/b_n}$	$\mathbf{N}\otimes\mathbf{M}\left(\frac{1-q}{1+q}, 1\right) = \prod_{n=0}^{\infty} (1-q^{2^n})$
7.	$\frac{a_n^2+\alpha b_n^2}{(1+\alpha)\,b_n}$	$\frac{2a_n+(\alpha-1)\,b_n}{\alpha+1}$	$\mathbf{N}\otimes\mathbf{M}\left(\frac{1+\alpha q}{1-q}, 1\right) = \frac{1}{(1-q)} \prod_{n=0}^{\infty} \frac{1}{(1-q^{2^n})}$
8.	$\frac{\alpha a_n^2}{(\alpha-1)\,a_n+b_n}$	$\frac{(\alpha+1)\,a_n-b_n}{\alpha}$	$\mathbf{N}\otimes\mathbf{M}\left(\frac{1}{1-\alpha^2 q}, 1\right) = \frac{1}{(1-\alpha q)} \prod_{n=0}^{\infty} \frac{1}{(1-\alpha q^{2^n})}$
9.	$\frac{2a_n^2}{3a_n-b_n}$	$\frac{a_n+b_n}{2}$	$\mathbf{N}\otimes\mathbf{M}\left(\frac{1}{1-4q}, 1\right) = \frac{1}{(1-4q)} \prod_{n=0}^{\infty} \frac{1}{(1-2q^{2^n})}$
10.	$\frac{(a_n+b_n)^3}{8a_nb_n}$	$\frac{a_n+b_n}{2}$	$\mathbf{N}\otimes\mathbf{M}\left(\left(\frac{1+q}{1-q}\right)^2, 1\right) = \frac{1}{(1-q^2)} \prod_{n=0}^{\infty} \frac{1}{(1-q^{2^n})^2}$
11.	$\frac{(a_n+b_n)^2}{4\sqrt{a_nb_n}}$	$\sqrt{a_nb_n}$	$\mathbf{N}\otimes\mathbf{M}\left(\left(\frac{1+q}{1-q}\right)^2, 1\right) = \frac{1}{(1-q)} \prod_{n=0}^{\infty} \frac{1}{(1-q^{2^n})}$

Table continued

TABLE I—*Continued*

	Uniformizing Parameters	Derivation
1.	$\prod_{n=0}^{\infty}(1-(-q)^{2^n})$ $\prod_{n=0}^{\infty}(1-q^{2^n})$	$a=1, b=-1$ in Lemma 1, (a)
2.	$\prod_{n=0}^{\infty}(1+2(-q)^{2^n})$ $\prod_{n=0}^{\infty}(1+2q^{2^n})$	$a=-2, b=2$, in Lemma 1, (a)
3.	$\left(\prod_{n=0}^{\infty}\frac{1-(-q)^{2^n}}{1+(-q)^{2^n}}\right)^{1/2}$ $\left(\prod_{n=0}^{\infty}\frac{1-q^{2^n}}{1+q^{2^n}}\right)^{1/2}$	$a=1, b=-1$, in Lemma 1, (a)
4.	$(1-q^2)\prod_{n=0}^{\infty}(1-(-q)^{2^n})$ $(1-q^2)\prod_{n=0}^{\infty}(1-q^{2^n})$	$a=1, b=-1$, in Lemma 1, (a)
5.	$\frac{1}{1-q^2}\prod_{n=0}^{\infty}\frac{1-(-q)^{2^n}}{1+3q^{2^{n+1}}}$ $\frac{1}{1-q^2}\prod_{n=0}^{\infty}\frac{1-q^{2^n}}{1+3q^{2^{n+1}}}$	$a=1, b=-1$, in Lemma 1, (a)
6.	$\prod_{n=0}^{\infty}(1-(-q^{2^n}))^{-1}$ $\prod_{n=0}^{\infty}(1-q^{2^n})^{-1}$	$a=-1, b=1$, in Lemma 1, (a)
7.	$(1+\alpha q)\prod_{n=0}^{\infty}(1-q^{2^n})$ $(1-q)\prod_{n=0}^{\infty}(1-q^{2^n})$	$a=\alpha, b=-1$, in Lemma 1, (a)

Table continued

TABLE I—*Continued*

	Uniformizing Parameters	Derivation
8.	$\left(\frac{1-\alpha q}{1-\alpha^2 q}\right)\prod_{n=0}^{\infty}(1-\alpha q^{2^n})$	$a=0$, $b=-\alpha^2$, in Lemma 1, (a)
	$(1-\alpha q)\prod_{n=0}^{\infty}(1-\alpha q^{2^n})$	
9.	$\prod_{n=0}^{\infty}(1-2q^{2^n})$	$a=0$, $b=-4$, in Lemma 1, (a)
	$(1-4q)\prod_{n=0}^{\infty}(1-2q^{2^n})$	
10.	$(1-q^2)\prod_{n=0}^{\infty}(1-(-q)^{2^n})^2$	$a=2$, in Lemma 1, (b)
	$(1-q^2)\prod_{n=0}^{\infty}(1-q^{2^n})^2$	
11.	$(1+q)\prod_{n=0}^{\infty}(1-(-q)^{2^n})$	$a=2$, in Lemma 1, (b)
	$(1-q)\prod_{n=0}^{\infty}(1-q^{2^n})$	

4. An Example Related to the AGM

Let

$$AG(1, x)$$

denote the common limit of the arithmetic geometric mean iteration

$$a_{n+1} = \frac{a_n + b_n}{2}$$

and

$$b_{n+1} = \sqrt{a_n b_n}$$

commencing with $a_0 := 1$ and $b_0 := x$. Then

$$AG(1, x) = \frac{\pi/2}{\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-(1-x^2)\sin^2\theta}}}.$$

We now state the following theorem.

THEOREM 2. *Let m be an integer. The iteration*

$$a_{n+1} = \frac{(a_n + b_n)^{2m}}{4^m(\sqrt{a_n b_n})^{2m-1}} := M(a_n, b_n)$$

$$b_{n+1} = \frac{(a_n + b_n)^{2m-1}}{2^{2m-1}(a_n b_n)^{m-1}} := N(a_n, b_n)$$

commencing with $a_0 := 1$ and $b_0 := x$ has limit

$$\mathbf{M} \otimes \mathbf{N}(1, x) = \frac{(AG(1, x))^{4n-1}}{x^{2n-1}}.$$

Proof. We use the following identities that may be found in [3]. If

$$h(q) := \left(\sum_{n=-\infty}^{\infty} (-q)^{n^2} \Big/ \sum_{n=-\infty}^{\infty} q^{n^2} \right)^2 \tag{4.1}$$

and

$$T(x) := \frac{2\sqrt{x}}{1+x} \tag{4.2}$$

then

$$h(q^2) = T(h(q)). \tag{4.3}$$

Also

$$AG(h(q), 1) = \prod_{n=0}^{\infty} \left(\frac{1 + h(q^{2^n})}{2} \right) \tag{4.4}$$

and

$$AG\left(\frac{1}{h(q)}, 1\right) = \prod_{n=0}^{\infty} \sqrt{\frac{1}{h(q^{2^n})}}. \tag{4.5}$$

We now apply Theorem 1 with $R := h$; T as above, and with

$$M(x, y) := \frac{(x+y)^{2m}}{4^m(\sqrt{xy})^{2m-1}}$$

to deduce that

$$\mathbf{M} \otimes \mathbf{N}(h(q), 1) = \prod_{n=0}^{\infty} M(h(q^{2^n}), 1)$$

which simplifies with (4.4) and (4.5) to the desired result. ▮

5. Comments

The function

$$P_c(x) := \prod_{n=0}^{\infty} (1 + cx^{2^n}) \tag{5.1}$$

that arises in many of examples, except for $c \neq 0, 1$ where it is rational, is a *hypertranscendental function*. That is, it satisfies no algebraic differential equation of any order. See [9, 10, 11]. The theta function parametrizing the AGM as well as the limit are not hypertranscendental. Thus, these examples are analytically quite distinct. The results of [10, 11] also show that $P_c(x)$ is transcendental for x and c algebraic ($\neq 0, 1$). Some alternative representations of P_c include

$$P_c(x) := \sum_{n=0}^{\infty} (c)^{\delta(n)} x^n, \tag{5.2}$$

where $\delta(n)$ is the number of ones in the binary expansion of n. Also, for $|x| < 1$,

$$P_c(x) := 1 + cx + cx \sum_{n=0}^{\infty} \prod_{k=0}^{n} x^{2^k}(1 + cx^{2^k}) \tag{5.3}$$

and

$$\frac{1}{P_c(x)} = 1 + cx - (1 + cx)x \sum_{n=0}^{\infty} \prod_{k=0}^{n} \frac{x^{2^k}}{1 + cx^{2^{k+1}}}. \tag{5.4}$$

In most of the examples we considered the ratio $M(x, y)/N(x, y)$ is a quadratic rational function. Another example with quadratic ratio is

$$a_{n+1} = \frac{a_n + b_n}{2}$$

$$b_{n+1} = \frac{a_n^2 + b_n^2}{a_n + b_n}$$

which is studied in [8]. This does not appear to be intimately related to any of the examples we have raised here. (See [6], see also [2, 3, 7] for additional material on quadratic mean iterations.) Reference [12] has some related mean iterations, particularly ones with algebraic limits.

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Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium