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Approximations by Rational Functions with Positive Coefficients

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We consider the problem of approximating analytic functions with positive coefficients by rational functions with positive coefficients. In the spirit of investigations by Reddy, Newman, and Erdös [3, 6, 7], we show that for e^x and various classes of analytic functions on [0, 1], best uniform rational approximations with either positive coefficients or positive coefficiented denominators reduce to polynomial approximations.

Let Π_n denote the algebraic polynomials with real coefficients of degree at most n and let $\Pi_n{}^+$ denote those elements of Π_n that have nonnegative coefficients. Let

$$R_{n,m} = \{p_n/q_m \mid p_n \in \Pi_n, q_m \in \Pi_m\},$$

$$R_{n,m}^{+} = \{p_n/q_m \mid p_n \in \Pi_n, q_m \in \Pi_m{}^+\},$$

$$R_{n,m}^{++} = \{p_n/q_m \mid p_n \in \Pi_n{}^+, q_m \in \Pi_m{}^+\}.$$

Let $\|\cdot\|_{[a,b]}$ denote the uniform on $[a, b]$ and define

$$\Pi_n(f\colon [a, b]) = \inf_{p_n \in \Pi_n} \|f - p_n\|_{[a,b]} .$$

Similarly, define $\Pi_n{}^+(f\colon [a, b])$, $R_{n,m}(f\colon [a, b])$, $R_{n,m}^{+}(f\colon [a, b])$, and $R_{n,m}^{++}(f\colon [a, b])$, respectively, as the distances from f to $\Pi_n{}^+$, $R_{n,m}$, $R_{n,m}^{+}$, and $R_{n,m}^{++}$. When $n = m$ we contract $R_{n,m}$ to R_n, $R_{n,m}(f\colon [a, b])$ to $R_n(f\colon [a, b])$, etc.

THEOREM 1 (Reddy [7]).

$$R_n{}^+(x^{n+1}\colon [0, 1]) = \Pi_n(x^{n+1}\colon [0, 1]).$$

THEOREM 2 (Newman and Reddy [6]). *If* $1 \leq k \leq n$,

$$R_k^{++}(x^{n+1}\colon [0, 1]) = \Pi_k{}^+(x^{n+1}\colon [0, 1]).$$

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Thus, allowing positive coefficiented polynomials in the denominator does not improve approximations to x^n. We show that this interesting phenomenon extends to the function e^x. For example:

THEOREM 3. *Suppose for each* $n \geq 0$ *that* $f^{(n)}(x) \geq f^{(n+1)}(x) \geq 0$ *for* $x \in [0, 1]$. *Then*

$$R_n^+(f\colon [0, 1]) = \Pi_n(f\colon [0, 1]) = \Pi_n^+(f\colon [0, 1]) = R_n^{++}(f\colon [0, 1]).$$

RATIONAL APPROXIMATIONS WITH RESTRICTED COEFFICIENTS

An examination of the proof of Cheney's characterization theorem for best rational approximation allows us to reformulate this theorem in the following two ways [1, p. 159]:

THEOREM 4. *Suppose that* $A, B \subset \Pi_n$ *and suppose that* $f \in C[a, b]$ *and* $f \notin R_n$. *Suppose* p/q, $q > 0$ *on* $[a, b]$ *is a best uniform approximation to f from the rational functions whose numerators are in A and denominators are in B. Then there exist no* $p_n \in \Pi_n$, $q_n \in \Pi_n$ *so that*

(a) $p_n - (p/q)q_n$ *has the same sign as* $f - p/q$ *on the set of points*

$$Y = \{y\colon |f(y) - p(y)/q(y)| = \|f - p/q\|_{[a,b]}\}$$

and

(b) *for some* $\epsilon > 0$ *and all* $0 \leq \lambda \leq \epsilon$,

$$p + \lambda p_n \in A \qquad \text{and} \qquad q + \lambda q_n \in B$$

and as a characterization theorem for best approximations

THEOREM 5. *An element* $p/q \in R_{n,m}^+$ *(respectively* $R_{n,m}^{++}$*) is a best approximation to* $f \in C[a, b]$, $f \notin R_{n,m}^+$ *(resp.* $R_{n,m}^{++}$*), if and only if no* $\phi \in \Pi_n - p/q \cdot \Pi_m^+$ *(resp.* $\Pi_n^+ - p/q \cdot \Pi_m^+$*) has the same sign as* $f - p/q$ *on the set*

$$Y = \{y\colon |f(y) - p(y)/q(y)| = \|f - p/q\|_{[a,b]}\}.$$

THEOREM 6. *Suppose* $f(x) = \sum_{i=0}^{\infty} a_i x^i$, *where* $a_i \geq a_{i+1} \geq 0$ *for all i. Then for fixed* $k \leq n$ *and* $0 \leq a \leq b \leq 1$ *the rational function that satisfies*

$$\min_{p_n \in \Pi_n, \lambda \text{real}} \left\| \frac{p_n(x)}{1 - \lambda x^k} - f(x) \right\|_{[a,b]}$$

reduces to a polynomial (i.e., has $\lambda = 0$), if and only if f is a polynomial of degree at most n.

Proof. Assume f is not a polynomial of degree n. Let $x_1, x_2, \ldots, x_{n+2}$ be any $n+2$ distinct points in $[a, b]$. Since $f^{(n+1)} > 0$ and $(x^k f)^{(n+1)} > 0$ on $(a, b]$, we have

$$\det \begin{vmatrix} 1, x_1, & \ldots, x_1^n, & f(x_1) \\ 1, x_2, & \ldots, x_2^n, & f(x_2) \\ \vdots & \vdots & \vdots \\ 1, x_{n+2}, & \ldots, x_{n+2}^n, & f(x_{n+2}) \end{vmatrix} = \beta \neq 0$$

and

$$\det \begin{vmatrix} 1, x_1, & \ldots, x_1^n, & f(x_1)\, x_1^k \\ 1, x_2, & \ldots, x_2^n, & f(x_2)\, x_2^k \\ \vdots & \vdots & \vdots \\ 1, x_{n+2}, & \ldots, x_{n+2}, & f(x_{n+2})\, x_{n+2}^k \end{vmatrix} = \alpha \neq 0.$$

Thus, there exists a polynomial $s_n \in \Pi_n$ so that

$$\frac{s_n(x_i)}{1 - (\beta/\alpha)\, x_i^k} = f(x_i), \qquad i = 1, \ldots, n+2. \tag{1}$$

We show that $\beta/\alpha < 1$. Consider

$$(1 - (\beta/\alpha)\, x^k) f(x) = \sum_{i=0}^{\infty} (a_i - (\beta/\alpha)\, a_{i-k})\, x^i.$$

If $\beta/\alpha \geqslant 1$ then, for each $i \geqslant n$, $a_i - (\beta/\alpha)a_{i-k} \leqslant 0$ and (unless $f(x) = 1/(1-x)$) for some $i_0 \geqslant n$, $a_{i_0} - (\beta/\alpha)a_{i_0-k} < 0$. Thus, $[(1 - (\beta/\alpha)x^k) f(x)]^{(n+1)} < 0$ and $s_n(x)/(1 - (\beta/\alpha)x^k)$ could not interpolate $f(x)$ at $n+2$ points. This contradicts (1). Now, if a best approximation to f of the required form were a polynomial p, then the set Y of Theorem 4 would consist of at most $n+2$ points $y_1, y_2, \ldots, y_{n+2}$. By the first part of this proof we can choose $p_n \in \Pi_n$ and $\delta \geqslant -1$ so that

$$\frac{p_n(y_i)}{1 + \delta y_i^k} = f(y_i) \qquad \text{for} \quad i = 1, \ldots, n+2.$$

We see that conditions (a) and (b) of Theorem 4 are satisfied with

$$q = 1, \qquad q_n = 1 + \delta x^k, \qquad A = \Pi_n, \qquad B = \{\lambda_1 + \lambda_2 x^k \mid \lambda_1, \lambda_2 \text{ real}\}$$

and hence, p cannot be a best approximation.

THEOREM 7. *Suppose $f(x) = \sum_{i=0}^{\infty} a_i x^i$, where $a_i \geqslant a_{i+1} \geqslant 0$ for all i. Then, if $0 < b \leqslant 1$,*

$$R_n^{+}(f: [0, b]) = \Pi_n(f: [0, b]).$$

Proof. Suppose $\|f - p/q\| = R_n^+(f: [0, b])$, where $p \in \Pi_n$ and $q(x) = 1 + b_k x^k + \cdots$ with $b_k \neq 0$. Then, since $(q \cdot f)^{(n+1)} > 0$ on $(0, b]$, the set Y of Theorem 4 contains at most $n + 2$ points $y_1, \ldots, y_{n+2}$. As in the proof of Theorem 6 we may find $\delta \geqslant -1$ and $p_n \in \Pi_n$ so that

$$\frac{p_n(y_i)}{1 + \delta y_i^k} = f(y_i) \qquad \text{for} \quad i = 1, \ldots, n + 2$$

and hence, conditions (a) and (b) of Theorem 4 are satisfied by p_n, q, p, $q_n = 1 + \delta x^k$, $A = \Pi_n$, $B = \Pi_n^+$, where we observe that for $0 \leqslant \lambda \leqslant b_k$

$$q(x) + \lambda(1 + \delta x^k) \in \Pi_n^+.$$

Thus p/q is not a best approximation unless $q = 1$.

Bernstein's theorem [4, p. 38] can be stated in the following way:

THEOREM 8. *Suppose that f and g are both $n + 1$ times continuously differentiable on $[a, b]$ and*

$$|f^{(n+1)}(x)| \leqslant g^{(n+1)}(x) \qquad \textit{for} \quad x \in [a, b].$$

(a) *If $p_2 \in \Pi_n$ is the polynomial that interpolates g at the $n + 1$ points $x_1, \ldots, x_{n+1}$ in $[a, b]$, then there exists $p_1 \in \Pi_n$ so that*

$$|f(x) - p_1(x)| \leqslant |g(x) - p_2(x)| \qquad \textit{for} \quad x \in [a, b].$$

(b) $\Pi_n(f: [a, b]) \leqslant \Pi_n(g: [a, b])$.

We can extend Bernstein's theorem to a result concerning approximations from R_n^+.

THEOREM 9. *If f and g are both $n + 1$ times continuously differentiable on $[a, b]$, $a \geqslant 0$ and*

$$|f^{(k)}(x)| \leqslant g^{(k)}(x), \qquad \textit{for} \quad x \in [a, b] \textit{ and } k = 0, 1, \ldots, n + 1.$$

Then

$$R_{n,m}^+(f: [a, b]) \leqslant R_{n,m}^+(g: [a, b]).$$

Proof. Let p/q be a best approximation to g on $[a, b]$ from the class $R_{n,m}^+$. Since q has nonnegative coefficients,

$$|(q(x) \cdot f(x))^{(n+1)}| \leqslant \sum_{k=0}^{n+1} \binom{n+1}{k} |q(x)^{(k)} \cdot f(x)^{(n+k+1)}|$$

$$\leqslant \sum_{k=0}^{n+1} \binom{n+1}{k} q(x)^{(k)} \cdot g(x)^{(n+1-k)} = (q(x) \cdot g(x))^{(n+1)}$$

We deduce from Theorem 5 that $p - q \cdot g$ must change sign at least $n + 1$ times and hence, by Theorem 8, there exists $p_1 \in \Pi_n$ so that for each $x \in [a, b]$,

$$| p_1(x) - q(x) f(x)| \leqslant | p(x) - q(x) g(x)|.$$

Thus,

$$R_{n,m}^+(f: [a, b]) \leqslant R_{n,m}^+(g: [a, b]).$$

COROLLARY 1. *Suppose* $p_{n+1} \in \Pi_{n+1}^+$; *then*

$$R_n^+(p_{n+1}: [0, 1]) = \Pi_n(p_{n+1}: [0, 1]).$$

Proof. Suppose $p_{n+1}(x) = \alpha x^{n+1} + \cdots$. We observe that

$$\Pi_n(p_{n+1}: [0, 1]) \geqslant R_n^+(p_{n+1}: [0, 1]).$$

By Theorem 1,

$$R_n^+(\alpha x_{n+1}: [0, 1]) = \Pi_n(\alpha x_{n+1}: [0, 1]) = \Pi_n(p_{n+1}: [0, 1]).$$

By Theorem 9, since p_{n+1} has positive coefficients,

$$R_n^+(p_{n+1}: [0, 1]) \geqslant R_n^+(\alpha x^{n+1}: [0, 1]).$$

Thus,

$$\Pi_n(p_{n+1}: [0, 1]) \geqslant R_n^+(p_{n+1}: [0, 1]) \geqslant \Pi_n(p_{n+1}: [0, 1]).$$

POLYNOMIAL APPROXIMATIONS WITH POSITIVE COEFFICIENTS

Let $\Gamma(\alpha) = \{f \mid \alpha f^{(n)}(x) \geqslant f^{(n+1)}(x) \geqslant 0$ for all $x \in [0, 1]$ for all $n\}$. We show that if $f \in \Gamma(1)$, then the best uniform polynomial approximation to f on $[0, 1]$ has positive coefficients. This result, combined with the observation that if $f \in \Gamma(1)$ then f satisfies the conditions of Theorem 7, establishes Theorem 3. We need the following straightforward lemma.

LEMMA 1. *Suppose* $\alpha > 0$.

(a) *If* $f \in \Gamma(\alpha)$ *then* $f^{(n)} \in \Gamma(\alpha)$ *for all* n.

(b) $f \in \Gamma(\alpha)$ *iff* $f(x) = \sum_{n=0}^{\infty} a_n x^n$, *where* $a_n \geqslant ((n + 1)/\alpha)\, a_{n+1} \geqslant 0$.

(c) *If* $f \in \Gamma(\alpha)$ *then* $f^{(n)}(1) \leqslant e^{\alpha} f^{(n)}(0)$.

THEOREM 10. *If* $f \in \Gamma(1)$ *then*

$$\Pi_n(f: [0, 1]) = \Pi_n^+(f: [0, 1]).$$

Proof. Suppose $f(x) = \sum_{n=0}^{\infty} b_n x^n$ and suppose that $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is the best approximation to f on $[0, 1]$ from Π_n. Chebyshev's alternation theorem [1, p. 75] guarantees that $f - p$ has at least $n + 1$ zeros on $[0, 1]$ and hence, $f^{(n+1-k)} - p^{(n+1-k)}$ has at least k zeros on $[0, 1]$. Since $f^{(n+1)} > 0$ on $(0, 1]$ (if f is not a polynomial of degree n) it follows that $f^{(n+1-k)} - p^{(n+1-k)}$ has exactly k zeros at $x_{k,1}, \ldots, x_{k,k}$ and by the Lagrange interpolation formula [2, p. 56],

$$f^{(n+1-k)}(x) - p^{(n+1-k)}(x) = \frac{(x - x_{k,1}) \cdot \cdots \cdot (x - x_{k,k})}{k!} f^{(n+1)}(\zeta), \qquad (1)$$

where $\min\{x, x_{k,1}\} < \zeta < \max\{x, x_{k,k}\}$. We note that $f^{(n+1)}(\zeta) \leqslant f^{(n+1)}(1) \leqslant f^{(n+1-k)}(1) \leqslant e f^{(n+1-k)}(0)$ and hence, by (1),

$$f^{(n+1-k)}(0) - (n + 1 - k)!\, a_{n+1-k} \leqslant \frac{e}{k!} f^{(n+1-k)}(0). \qquad (2)$$

Since $e/k! < 1$ for $k \geqslant 3$, we have $a_{n+1-k} > 0$ for $k \geqslant 3$. By Descartes' rule of signs the coefficients of $f - p$ must have $n + 1$ sign changes and it follows that $a_n, a_{n-2}, a_{n-4}, \ldots$ are all positive. To complete the proof we must show that $a_{n-1} \geqslant 0$. By (1),

$$f^{(n+1-2)}(x) - n!\, a_n x - (n - 1)!\, a_{n-1} = \frac{(x - x_{2,1})(x - x_{2,2})}{2} f^{(n+1)}(\zeta). \quad (3)$$

We observe that $f^{(n)}(x_{1,1}) = p^{(n)}(x_{1,1}) = n!\, a_n$ and that $x_{2,1} < x_{1,1} < x_{2,2}$. Hence,

$$f^{(n-1)}(x_{1,1}) - x_{1,1} f^{(n)}(x_{1,1}) - (n - 1)!\, a_{n-1} < 0 \qquad (4)$$

and since $f \in \Gamma(1)$, a_{n-1} is positive.

We can derive precise estimates of $\Pi_n(f: [0, 1])$ for $f \in \Gamma(\alpha)$. We use a method developed by G. Meinardus;

THEOREM 11 (Meinardus [5]). *Suppose* $f \in C^{n+2}[0, 1]$ *and suppose there exist* γ_1 *and* γ_2 *so that*

$$0 \leqslant f^{(n+1)}(1/2) + (f^{(n+1)}(1/2))(x - 1/2) + \gamma_1(x - 1/2)^2$$
$$\leqslant f^{(n+1)}(x) \leqslant f^{(n+1)}(1/2) + (f^{(n+2)}(1/2))(x - 1/2) + \gamma_2(x - 1/2)^2.$$

Then, if $f^{(n+1)}(1/2) > 0$ *and* $1/2\,|f^{(n+2)}(1/2)| \leqslant (n+2)f^{(n+1)}(1/2)$,

$$\Pi_n(f: [0,1]) \geqslant \frac{f^{(n+1)}(1/2)}{2^{2n+1}(n+1)!}\left|1 + \frac{\gamma_1}{(f^{(n+1)}(1/2))\,8(n+2)}\right|$$

and

$$\Pi_n(f: [0,1]) \leqslant \frac{f^{(n+1)}(1/2)}{2^{2n+1}(n+1)!}\left|1 + \frac{(f^{(n+2)}(1/2))^2(n+1)}{(f^{(n+1)}(1/2))^2\,n(n+2)^2 4} + \frac{|\gamma_2|\,(n+5)}{f^{(n+1)}(1/2)\,8(n+2)(n+3)}\right|.$$

THEOREM 12. *If* $f \in \Gamma(\alpha)$, $\alpha > 0$, *then*

$$\Pi_n(f: [0,1]) \geqslant \frac{f^{(n+1)}(1/2)}{2^{2n+1}(n+1)!}$$

and

$$\Pi_n(f: [0,1]) \leqslant \frac{f^{(n+1)}(1/2)}{2^{2n+1}(n+1)!}\left(1 + \frac{\alpha^2/(n+2) + \alpha e^{\alpha}/2}{4n}\right).$$

Proof. Since $f^{(n+1)}$ is convex,

$$f^{(n+1)}(x) \geqslant f^{(n+1)}(1/2) + f^{(n+2)}(1/2)\cdot(x - 1/2).$$

Also, since $f^{(n+2)}$ is convex,

$$f^{(n+2)}(x) \leqslant f^{(n+2)}(1/2) + 2f^{(n+2)}(1)\cdot(x - 1/2), \qquad \text{for} \quad x \in [1/2, 1]$$

and

$$f^{(n+2)}(x) \geqslant f^{(n+2)}(1/2) + 2f^{(n+2)}(1)\cdot(x - 1/2), \qquad \text{for} \quad x \in [0, 1/2].$$

Hence,

$$f^{(n+1)}(x) \leqslant f^{(n+1)}(1/2) + f^{(n+2)}(1/2)\cdot(x - 1/2) + f^{(n+2)}(1)\cdot(x - 1/2)^2.$$

We now apply Theorem 11 with $\gamma_1 = 0$ and $\gamma_2 = f^{(n+2)}(1)$. We note that by (c) of Lemma 1, $f^{(n+2)}(1) \leqslant e^{\alpha} f^{(n+2)}(1/2) \leqslant \alpha e^{\alpha} f^{(n+1)}(1/2)$. Thus,

$$\begin{aligned}\Pi_n(f: [0,1]) &\leqslant \frac{f^{(n+1)}(1/2)}{2^{2n+1}(n+1)!}\left(1 + \frac{\alpha^2(n+1)}{n(n+2)^2 4} + \frac{\alpha e^{\alpha}(n+5)}{8(n+2)(n+3)}\right)\\ &\leqslant \frac{f^{(n+1)}(1/2)}{2^{2n+1}(n+1)!}\left(1 + \frac{\alpha^2/(n+2) + \alpha e^{\alpha}/2}{4n}\right)\end{aligned}$$

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