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Padé Approximants for the q-Elementary Functions

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Abstract. We give a simple construction of the Padé approximants to q analogues of exp and log. The construction is based on the functional relations they satisfy. The Padé approximants for the ordinary exp and log are then limiting cases.

Introduction

There are a few particular functions whose properties under rational approximation have received special scrutiny. Exp and log are probably the central examples. This stems both from the fact that we can actually work out the details, though by no means trivially (see, for example, [9] or [11]) and from the pivotal role of these functions in applied analysis. It is also the case that almost all the known results concerning the measure of transcendence of e and π are tied into rational approximations to exp or log [4], [8].

Our intention is to show how to construct Padé and related approximants to functions that satisfy particularly simple functional relations. Two examples for which this method works are the q analogues of exp and log. The q analogues of exp and log are functions parametrized by q that, in some sense, naturally reduce to exp and log on letting q tend to one. (See Sections 1 and 3.) The introduction of the q variable allows us to construct the Padé approximants from functional relations rather than the more usual use of the differential equations. An alternate route to some of these constructions, based on the $q-d$ algorithm, is given by Wynn [13]. Most notably, Wynn derives $m \geq n-1$ forms of the approximants of Theorems 2 and 3. (See also [12].)

The construction of q analogues of hypergeometric functions appears to be a profitable endeavor [1], [2], [5]. We might view the partial theta functions

$$T_q(x) := \sum_{n=0}^{\infty} q^{n(n-1)/2} x^n$$

as a q analogue of $(1-x)^{-1}$. Lubinsky and Saff have examined Padé approximants to T_q in some detail and proved some surprising convergence results (see Section 5).

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Considerable number theoretic information is also deducible from the approximations to T_q and the approximations to the q exponential function. (See Section 5 and [12].)

Section 1 introduces two q analogues to the exponential function and also the basic notations. Section 2 shows how to construct Padé approximants to entire functions satisfying certain function relations. Sections 3 and 4 then give the construction of the Padé approximants to the q analogues of exp and log, respectively. The last two sections discuss briefly the partial theta function case and a higher-order case.

1. The q-Exponential Function

We need the standard q analogues of factorials and binomial coefficients. The *q-factorial* is

$$[n]_q! := [n]! := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q)}{(1-q)(1-q)\cdots(1-q)}, \tag{1.1}$$

where $[0]_q! := 1$. Since $(1-q^n)/(1-q) = 1+\cdots+q^{n-1}$ it is clear that

$$\lim_{q\to 1} [n]_q! = n!. \tag{1.2}$$

The *q-binomial coefficient* (or Gaussian binomial coefficient) is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[n-k]!\,[k]!} \tag{1.3}$$

and as above

$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}. \tag{1.4}$$

The *q-binomial theorem* (or Cauchy binomial theorem) is

$$\sum_{k=0}^{n} x^k q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} = \prod_{k=1}^{n} (1+xq^k). \tag{1.5}$$

This is all standard and may be found in [5]. We give two versions of the *q-exponential* due to F. H. Jackson [5], [6].

$$E_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \tag{1.6}$$

and

$$E_q^*(x) := \sum_{n=0}^{\infty} \frac{x^n q^{n(n-1)/2}}{[n]!}. \tag{1.7}$$

In both cases, as $q \to 1$, the functions reduce to the ordinary exponential. Note that E_q is entire (in x) if $|q| > 1$ while it has radius of convergence $|1-q|^{-1}$ if $|q| < 1$. The function E_q^* is entire for $|q| < 1$. It is related to E_q by inverting the base q, that is $E_q^* = E_{1/q}$. If we set $x := (1-q)q^{-1}x$ in (1.5) and let n tend to ∞ we deduce that

$$E_q^*(x) = \prod_{k=0}^{\infty} (1 + xq^k(1-q)), \qquad |q| < 1, \tag{1.8}$$

and on setting $q := 1/q$

$$E_q(x) = \prod_{k=0}^{\infty} \left(1 + xq^{-k}\left(1 - \frac{1}{q}\right)\right), \qquad |q| > 1. \tag{1.9}$$

The Cauchy product of (1.6) and (1.7) can be simplified using the q-binomial theorem to give

$$E_q(-x)E_q^*(x) = 1, \tag{1.10}$$

which provides for the analytic continuation of both E_q and E_q^* where necessary. From (1.8) $E_q^*(x)$ satisfies the functional relation

$$E_q^*(x) = M_m(q, x)E_q^*(xq^m), \tag{1.11}$$

where $M_m(q, x) = \prod_{k=0}^{m-1} (1 + xq^k(1-q))$.

This may be recast on using (1.10) as

$$E_q(xq^m) = M_m(q, -x)E_q(x). \tag{1.12}$$

This functional relation allows for the construction of the Padé approximation to E_q. The method is presented in the next section.

2. The Basic Method for Constructing the Padé Approximants

We collect together, in convenient form, the pieces that allow for the construction of Padé (and related approximants) for entire functions satisfying functional relations like (1.11).

Theorem 1. *Suppose $F_q(x)$ is an entire function of x (analytic in some open connected set containing $0, 1, q, \ldots, q^n$ in fact suffices). Let*

$$I(x) := \frac{1}{2\pi i} \int_{C_\infty} \frac{F_q(xt)\, dt}{(t-q^0)(t-q^1)\cdots(t-q^n)t^{m+1}}, \tag{2.1}$$

where C_∞ is a circular contour containing $0, q^0, \ldots, q^n$. Let

$$A(x) := \sum_{i=0}^{n} \frac{F_q(q^i x)/F_q(x)}{\left[\prod_{h=0, h\neq i}^{n} (q^i - q^h)\right] q^{i(m+1)}} \tag{2.2}$$

and

$$B(x) := \frac{1}{m!}\left[\frac{d}{dt}\right]^m \left\{\frac{F_q(xt)}{(t-q^0)\cdots(t-q^n)}\right\}_{t=0}. \tag{2.3}$$

Then:

(a) $I(x) = A(x)F_q(x) + B(x)$.
(b) $I(x) = O(x^{n+m+1})$.
(c) $B(x)$ *is a polynomial of degree less than or equal to m in x.*
(d) *If, for each* $i \leq n$, F_q *satisfies a function relation of the form*

$$F_q(xq^i) = N_i(x, q)F_q(x)$$

where N_i *is a polynomial of degree i in x, then*

$$A(x) := \sum_{i=0}^{n} \frac{N_i(x, q)}{\left(\prod_{h=0, h\neq i}^{n} (q^i - q^h)\right) q^{i(m+1)}} \tag{2.4}$$

is a polynomial of degree n in x. Furthermore, the (m, n) *Padé approximant to* F_q *is just* $-B(x)/A(x)$.

Proof. Part (a) is just the evaluation of I by the residue theorem.

Part (b) is easily deduced. We observe that the denominator in I grows like t^{n+m+2}. Thus any terms of the expansion of $F_q(xt)$ of order less than $n+m+2$ vanish on integration.

Part (c) is obvious, while part (d) is just a substitution with the final observation following from (b). ■

Construction of Padé-type approximants based on contour integrals like I is a classical and familiar technique, particularly for functions satisfying simple differential equations, like exp. (See [4] and [8].)

We note that

$$\prod_{h=0, h\neq i}^{n} (q^i - q^h) = q^{i(2n-i-1)/2}[n-i]!\,[i]!\,(1-q)^n(-1)^i. \tag{2.5}$$

3. The Padé Approximants to E_q

We apply Theorem 1 to derive the Padé approximants for E_q.

Theorem 2. *Suppose* $|q| \neq 1$. *Let*

$$Q_{m,n}(x) := \frac{(-1)^n q^{n(n+2m+1)/2}}{(1-q)^n} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^i \times \left\{ \prod_{k=0}^{i-1} (1 + xq^k(1-q)) \right\} q^{i(i-1)/2 - i(m+n)} \tag{3.1}$$

and let

$$P_{m,n}(x) := \left\{ \frac{\prod_{k=1}^{n} (1 - q^{m+k})}{\prod_{k=1}^{m} (1 - q^{n+k})} \right\} \cdot \frac{1}{(1-q)^n} \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix} (-1)^i \times \left\{ \prod_{k=0}^{i-1} (1 - xq^{-k})(1 - q^{-1}) \right\} q^{i(i+1)/2 + in}. \tag{3.2}$$

(The empty product is identically unity.) Then $Q_{m,n}(x)/P_{m,n}(x)$ is the (n, m) Padé approximant to $E_q^*(x)$, while $P_{m,n}(-x)/Q_{m,n}(-x)$ is the (m, n) Padé approximant to $E_q(x)$. $P_{n,n}$ and $Q_{n,n}$ are polynomials of degree n in x and of degree $n(3n-1)/2$ in q with integer coefficients and with constant coefficient one.

Proof. The derivation of (3.1) is just part (d) of Theorem 1 coupled with the functional equation for $E_q(-x)$, (1.12). The simplification requires using (2.5). We have normalized so that $Q_{n,n}$ and $P_{n,n}$ have integer coefficients in q and in x and have constant coefficient one. This is done by multiplying by $[n]!\,(q^{n(n+2m+1)/2})$.

For (3.1) we observe that by (1.10) the (n, m) Padé approximant to $E_q(-x)$ is just the (m, n) Padé approximant to $E_q^*(x)$. In particular on setting $q := 1/q$, $m := n$, and $x := -x$ in (3.1) we get

$$(3.3) \qquad P_{m,n}(x) = c(q) \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix} (-1)^i \left\{ \prod_{k=0}^{i-1} \left(1 - xq^{-k}\left(1 - \frac{1}{q}\right)\right)\right\} q^{i(i+1)/2+in}$$

for some function $c(q)$ that is independent of x. Here we have used the fact that

$$\begin{bmatrix} n \\ i \end{bmatrix}_{1/q} = \begin{bmatrix} n \\ i \end{bmatrix}_q q^{-i(n-i)}.$$

If we set $x := 0$ in (3.3) we get

$$P_{m,n}(0) = c(q) \sum_{i=0}^{n} \begin{bmatrix} m \\ i \end{bmatrix} (-1)^i q^{i(i+1)/2+in}$$

$$= c(q) \prod_{k=1}^{m} (1 - q^{n+k})$$

on using the q-binomial theorem (with $x := -q^n$). Similarly,

$$Q_{m,n}(0) = \frac{(-1)^n q^{n(n+2m+1)/2}}{(1-q)^n} \prod_{k=1}^{n} (1 - q^{-(m+n+1)+k}).$$

And since $Q_{m,n}(0)/P_{m,n}(0) = 1$ we deduce that

$$c(q) := \frac{(-1)^n q^{n(n+2m+1)/2} \prod_{k=1}^{n} (1 - q^{-(m+n+1)+k})}{(1-q)^n \prod_{k=1}^{m} (1 - q^{n+k})}$$

$$= \frac{1}{(1-q)^n} \frac{\prod_{k=1}^{n} (1 - q^{m+k})}{\prod_{k=1}^{m} (1 - q^{n+k})}.$$

Most of the final part of the conclusion is just a check. The divisibility of the sums by $(1-q)^n$, however, requires a little effort ■

As Andrews observes (Math. Reviews, 86i: 11036) in the main diagonal case, $Q_{n,n}$ has a closed ${}_2\varphi_1$ representation, namely

$$Q_{n,n}(-x)=q^{-n^2-n}x^n{}_2\varphi_1\left(\begin{matrix}q^{-n} & q^n \\ & 0\end{matrix};\ \frac{1}{x(1-q)}\right),$$

while

$$P_{n,n}(-x)=x^n(-1)^{n+1}q^{-[(n+1)/2]}\lim_{\tau\to 0}{}_2\varphi_1\left(\begin{matrix}q^{-n} & q^n \\ & \tau^{-1}\end{matrix};\ \frac{1}{\tau x(1-q)}\right).$$

See [2] for definitions. In the limiting case $(q\to 1)$, $P_{m,n}(-x)$ and $Q_{m,n}(-x)$ reduce to the usual Padé numerators and denominators for the exponential function.

4. The Padé Approximants to a q-Logarithm

We define a q-logarithm by

$$L_q(x):=\sum_{n=1}^{\infty}\frac{x^n}{1+q+\cdots+q^{n-1}}=\sum_{n=1}^{\infty}\left(\frac{1-q}{1-q^n}\right)x^n. \tag{4.1}$$

Then, as $q\to 1$, $L_q(x)\to -\log(1-x)$. We define

$$f_q(x):=\prod_{k=1}^{\infty}(1-xq^{-k})=\left(E_q\left(\frac{-x}{1-q^{-1}}\right)\right)\Big/(1-x) \tag{4.2}$$

and observe that

$$\frac{f_q'(x)}{f_q(x)}=-\sum_{n=1}^{\infty}\frac{q^{-n}}{1-xq^{-n}}=\sum_{n=1}^{\infty}\frac{1}{x-q^n}. \tag{4.3}$$

Note that f_q is entire for $|q|>1$ and

$$f_q(xq^m)=\prod_{k=0}^{m-1}(1-xq^k)f_q(x). \tag{4.4}$$

We also observe that, for $|x|<1$,

$$\begin{aligned} L_q(qx)&=\sum_{n=1}^{\infty}q^n\left(\frac{1-q}{1-q^n}\right)x^n \\ &=\sum_{n=1}^{\infty}\frac{(-1+q^n)(1-q)x^n}{(1-q^n)}+L_q(x) \\ &=\frac{-x(1-q)}{1-x}+L_q(x). \end{aligned} \tag{4.5}$$

So for $|q|<1$

$$L_q(x)=\sum_{n=0}^{N-1}\frac{(1-q)xq^n}{1-xq^n}+L_q(q^Nx) \tag{4.6}$$

$$=(1-q)x\sum_{n=0}^{\infty}\frac{q^n}{1-xq^n}$$

$$=(q-1)\sum_{n=0}^{\infty}\frac{x}{x-q^{-n}}$$

which should be compared with (4.3).

We can now derive the main-diagonal Padé denominators for the q logarithm.

Theorem 3.

(a) *Let* $|q|>1$ *and let*

$$Q_n(x):=q^{n^2}\sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}^2\frac{\prod_{k=0}^{i-1}(1-xq^k)}{q^{i(2n-i)}}. \tag{4.7}$$

Then $Q_n(x)$ *is the denominator of the* (n, n) *Padé approximant to*

$$\sum_{n=1}^{\infty}\frac{x}{x-q^n}. \tag{4.8}$$

(b) *Let* $|q|<1$ *and let*

$$W_n(x):=\sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}^2 q^{i^2}\prod_{k=0}^{i-1}(1-xq^{-k}). \tag{4.9}$$

Then $W_n(x)$ *is the denominator of the* (n, n) *Padé approximant to*

$$L_q(x)-\frac{(q-1)x}{x-1}=(1-q)\sum_{n=1}^{\infty}\frac{xq^n}{1-xq^n}.$$

W_n is a polynomial of degree n in x and degree n^2 in q with integer coefficients and constant coefficient one.

Proof. Let $D_n:=[n]!^2q^{n^2}(1-q)^{2n}$ and consider

$$I:=\frac{D_n}{2\pi i}\int_{C_\infty}\frac{f_q(xt)\,dt}{((t-q^0)\cdots(t-q^n))^2}=O(x^{2n+1}), \tag{4.10}$$

with the last equality as in Theorem 1. Then by the residue theorem

$$I=D_n(A(x)+B(x)),$$

where

$$A(x):=\sum_{i=0}^{n}f_q(xq^i)\left\{\frac{d}{dt}\frac{1}{\prod_{k\neq i}(t-q^k)^2}\right\}_{t=q^i}$$

and

$$B(x) := \sum_{i=0}^{n} xf_q'(xq^i)\frac{1}{\prod\limits_{k\neq i}(q^i-q^k)^2}.$$

From the functional relation for f, (4.4),

$$q^i f_q'(xq^i) = M_i'(x, q)f_q(x) + M_i(x, q)f_q'(x),$$

where $M_m(x, q) := \prod_{k=0}^{m-1}(1-xq^k)$. (Here differentiation is with respect to x.) Thus

$$B(x) := \sum_{i=0}^{n} \frac{xM_i'(x, q)f_q(x) + xM_i(x, q)f_q'(x)}{q^i\left(\prod\limits_{k\neq i}(q^k-q^i)\right)^2}.$$

Separating the terms multiplying $f_q(x)$ from those multiplying $f_q'(x)$ gives

$$I = Q_n(x)xf_q'(x) + P_n(x)f_q(x),$$

where

(4.11) $$Q_n(x) := (1-q)^{2n}[n]!^2 q^{n^2} \sum_{i=0}^{n} \frac{M_i(x, q)}{q^i\left(\prod\limits_{k\neq i}(q^k-q^i)\right)^2}$$

and

(4.12) $$P_n(x) := D_n\left\{\sum_{i=0}^{n} M_i(x, q)\left\{\frac{d}{dt}\frac{1}{\prod\limits_{k\neq i}(t-q^k)^2}\right\}_{t=q^i}\right\} + D_n\left\{\sum_{i=0}^{n}\frac{xM_i'(x, q)}{q^i\prod\limits_{k\neq i}(q^k-q^i)^2}\right\}.$$

We observe that P_n and Q_n are both polynomials of degree n and with (4.11) must be the numerator and denominator of the (n, n) Padé approximant to $xf_q'(x)/f_q(x)$. With the aid of (2.5) we can simplify (4.11) to get (4.7). We deduce the form of (4.9) from (4.6) and (4.7) on setting $q := q^{-1}$ in (4.7) and multiplying by q^{n^2} to ensure that the constant coefficient is 1. ∎

Once again we observe that, as $q \to 1$, $W_n(x)$ reduces to the usual Padé denominator for $\log(1-x)$. For real $q > 1$, the function

$$\sum_{n=1}^{\infty}\frac{1}{x-q^n}$$

is a Stieltjes transform of a positive discrete measure and it follows that the polynomials $Q_n(x)$ are a sequence of orthogonal polynomials and have interlacing real zeros. Likewise for the W_n when q is real and $|q| < 1$. (see [3].)

We can explicitly solve for the three-term recursions that W_n and Q_n satisfy. This can easily be done using the coefficients of x^0, x^{n-1}, and x^n.

Theorem 4.

(a) *Let*

$$\bar{Q}_n(x) := \frac{Q_n(x)}{q^{n^2}} = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}^2 \frac{\prod_{k=0}^{i-1} (1 - xq^k)}{q^{i(2n-i)}}. \tag{4.13}$$

Then the three-term recursion that $\bar{Q}_n$ *satisfies is given by*

$$\bar{Q}_{n+1} = (a_n + b_n x)\bar{Q}_n + c_n x^2 \bar{Q}_{n-1}, \tag{4.14}$$

where

$$a_n := \frac{(q^{2n+1}-1)(q^{n+1}+1)}{(q^{n+1}-1)q^{2n+1}}, \tag{4.15}$$

$$b_n := \frac{-2(q^{2n+1}-1)}{(q^n+1)(q^{n+1}-1)q^{n+1}},$$

and

$$c_n := \frac{-(q^{n+1}+1)(q^n-1)}{(q^n+1)(q^{n+1}-1)q^{2n+1}}.$$

(b) *Let*

$$W_n(x) := \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}^2 q^{i^2} \prod_{k=0}^{i-1} (1 - xq^{-k}). \tag{4.16}$$

Then the three-term recursion that W_n *satisfies is given by*

$$W_{n+1} = (a_n^* + b_n^* x) W_n + c_n^* x^2 W_{n-1}, \tag{4.17}$$

where

$$a_n^* := \frac{(q^{2n+1}-1)(q^{n+1}+1)}{(q^{n+1}-1)}, \tag{4.18}$$

$$b_n^* := \frac{-2(q^{2n+1}-1)q^{n+1}}{(q^n+1)(q^{n+1}-1)},$$

and

$$c_n^* := \frac{-(q^{n+1}+1)(q^n-1)q^{2n+1}}{(q^n+1)(q^{n+1}-1)}.$$

At $x = 0$ the recursions both reduce to products and we deduce the following amusing corollary.

Corollary 1.

(a)
$$\sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}^2 q^{-i(2n-i)} = \frac{1}{q^{n^2}}\begin{bmatrix} 2n \\ n \end{bmatrix},$$

(b)
$$\sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}^2 q^{i^2} = \begin{bmatrix} 2n \\ n \end{bmatrix}.$$

Both (a) and (b) reduce to the familiar summation

$$\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{2n}{n}$$

when $q := 1$.

5. Partial Theta Functions

The *partial theta function* T is defined by

$$T_q(x) := \sum_{k=0}^{\infty} q^{k(k-1)/2} x^k. \tag{5.1}$$

(This is just θ_3-principal part (θ_3)). The denominator of the (m, n) Padé approximant is

$$R_{m,n}(x) = \sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix} q^{mi}(-x)^i \qquad (m \geq n-1 \geq 0). \tag{5.2}$$

These are essentially the Rogers-Szegö polynomials (we need just substitute $y := -xq^m$) [7], [10]. Recently, Lubinsky and Saff have investigated the convergence properties, or equivalently the distribution of the zeros of $R_{m,n}$, in some detail [7]. They show, in particular, that if $q = e^{i\tau}$, $\tau/(2\pi)$ irrational, then no subsequence of any Padé row $(n \geq 2)$ can converge locally uniformly in $|z| < 1$.

We can view $T_q(x)$ as a q analogue of $1/(1-x)$. The functional relation for T_q is

$$x^m q^{m(m-1)/2} T_q(q^m x) = T_q(x) - \sum_{k=0}^{m-1} q^{k(k-1)/2} x^k. \tag{5.3}$$

We can derive (5.2) from (5.3) and the form

$$I := \frac{[n]!\,(1-q)^n q^{nm}}{2\pi i} \int_{C_\infty} \frac{x^n T_q(xt)\, dt}{(t-q^0)(t-q^1)\cdots(t-q^n)t^{m-n+1}} \tag{5.4}$$

roughly as in Theorem 1, with a little additional care for the summation term in (5.3). Then

$$I = R_{m,n}(x) T_q(x) + S_{m,n}(x) = O(x^{n+m+1}) \tag{5.5}$$

with $R_{m,n}$ as in (5.2) and $S_{m,n}$ a polynomial of degree m.

The polynomials $R_{n,n}$ and $S_{n,n}$ are both polynomials of degree n^2 in q and both have integer coefficients in x and in q. Furthermore, in this case $(n = m)$

$$I = O(q^{(3n^2+n)/2}).$$

Thus, $S_{n,n}/R_{n,n}$ is also a high-order approximation in q. This can be used to show that $T_q(x)$ is irrational for $1/q$ and $1/x$ sufficiently large integers and to derive rather good irrationality estimates. For example, for $\varepsilon > 0$ and integers $\alpha, \beta \geq N_\varepsilon$

$$\left| T_{1/\alpha}\left(\frac{1}{\beta}\right) - \frac{p}{q} \right| \geq \frac{1}{q^{3+\varepsilon}}$$

for all integral p and q sufficiently large. Note that

$$\lim_{q\to 1} R_{m,n}(x) = \sum_{i=0}^{n} \binom{n}{i} (-x)^i = (1-x)^n.$$

The Padé approximants to the q-exponential converge rapidly enough to prove that $E_{1/q}(x)$ is irrational for q a positive integer and x rational. This is shown in [12].

6. A Hermite–Padé Approximation

Consider the form

$$I := \frac{(1-q)^{2n}[n]!^2}{2\pi i} \int_{C_\infty} \frac{f_q(xt)\, dt}{((t-q^0)\cdots(t-q^n))^2 t^{n+1}} \tag{6.1}$$

with f_q as in (4.2). Then there are polynomials p_n, q_n, and r_n of degree n so that

$$I = p_n f_q + q_n x f_q' + r_n = O(x^{3n+2}). \tag{6.2}$$

The polynomials q_n can be derived as in Theorem 3. We get

$$q_n(x) := \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}^2 \frac{\prod_{k=0}^{i-1} (1-xq^k)}{q^{i(3n+1-i)}}. \tag{6.3}$$

We note that (6.2) can be written as

$$p_n + q_n \left(\frac{x f_q'}{f_q} \right) + \frac{r_n}{f_q} = O(x^{3n+2}), \tag{6.4}$$

that

$$\frac{x f_q'(x)}{f_q(x)} = \frac{1}{q^{-1}-1} L_{1/q}(x) - \frac{x}{x-1}, \tag{6.5}$$

and that

$$\frac{1}{f_q(x)} = \frac{(1-x)}{E_q(-x/(1-q^{-1}))} \tag{6.6}$$

so that we are constructing a mixed Hermite-Padé approximation to the q-exponential and q-logarithm (these are also called Latin-polynomials for this pair of functions).

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