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RAMANUJAN'S RATIONAL AND ALGEBRAIC SERIES FOR $1/\pi$

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[Dedicated to the memory of Srinivasa Ramanujan]

1. Introduction. In [7, § 13] Ramanujan sketches the genesis of 3 remarkable series for $1/\pi$. In § 14, with essentially no explanation, he gives 14 more remarkable series. Hardy [3], quoting Mordell, observes that "it is unfortunate that Ramanujan has not developed in detail the corresponding theories."

In this paper we construct various general classes of hypergeometric-like power series for $1/\pi$, and for several related quantities. In each case the power is a well-known invariant from elliptic function theory and the coefficients involve a similar invariant. In particular, we recover all but 2 of Ramanujan's series and largely explain Ramanujan's "corresponding theories". A complete treatment appears in [1] which we follow closely in the development of the material and which explains the two missing series.

Recently Shanks, [8], and Newman and Shanks [5] have studied series for π in which the power is again an invariant. Their remarkable series while very rapid are not entirely algebraic since they commence with a logarithmic term. Moreover, the coefficients of their series are not entirely explicit.

The most recent record setting calculations of digits of pirely on methods that trace their genesis to this material. Details of the calculations of Gosper, Bailey, Tamura and Kanada, and Kanada may be found in [1].

2. Preliminary Results. Recall that the *hypergeometric function*, $_2F_1$, is defined by

$$ {}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\frac{x^n}{n!}, \tag{2.1}$$

for appropriate values of the variables. Here

$$(a)_n := \Gamma(a+n)/\Gamma(n) = a(a+1)(a+2)\dots(a+n-1).$$

Similarly the generalized hypergeometric function, ${}_3F_2$, is defined by

$$ {}_3F_2\,(a,\,b,\,c;\,d,\,e;\,x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n}\frac{x^n}{n!}, \tag{2.2}$$

again where appropriate [6], [9]. We define the *generalized complete elliptic integrals of the first and second kind by*

$$K_s(k) := \frac{\pi}{2}\cdot {}_2F_1\left(\frac{1}{2}-s,\,\frac{1}{2}+s;\,1;\,k^2\right), \tag{2.3}$$

and

$$E_s(k); = \frac{\pi}{2}\cdot {}_2F_1\left(-\frac{1}{2}-s,\,\frac{1}{2}+s;\,1;\,k^2\right), \tag{2.4}$$

for $|s| < \frac{1}{2}$ and $0 \leqslant k < 1$. We denote the *complementary modulus* $k' := \sqrt{1-k^2}$ and write $K_s'(k) := K_s(k')$, $E_s'(k) := E_s(k)$. Now $K := K_0$ and $E := E_0$ are the classical elliptic integrals, and each K_s, E_s admits many integral representations [2, § 2.12]. Moreover, one has

$$E_s = k'^2 K_s + \frac{kk'^2}{1+2s}\dot{K}_s. \tag{2.5}$$

This may be verified directly or by using [2, § 2.8]. $\left(\text{Here } \dot{K}_s = \frac{d}{dk}K_s.\right)$ Similarly, using [2, (13) p. 85] we have

$$E_sK_s' + K_sE_s' - K_sK_s' = \frac{\pi}{2}\frac{\cos(\pi s)}{1+2s}. \tag{2.6}$$

When $s := 0$, this is *Legendre's relation* [1], [11]. The following relationships will be helpful.

PROPOSITION 2.1. *For* $0 \leqslant h < \frac{1}{\sqrt{2}}$ *we have*

(a) $\frac{2}{\pi}K_s(h) = {}_2F_1\left(\frac{1}{4}-\frac{s}{2},\,\frac{1}{4}+\frac{s}{2};\,1;\,(2hh')^2\right),$

(b) $$\left[\frac{2}{\pi}K_s(h)\right]^2 = {}_3F_2\left(\frac{1}{2}-s, \frac{1}{2}+s, \frac{1}{2}; 1, 1: (2hh')^2\right).$$

Proof. (a) is a special case of *Kummer's identity* given in Rainville [6, p. 67] or in [2, (2) § 2.11]. It may be verified by showing that both sides satisfy the appropriate hypergeometric differential equation (and agree at zero). (b) is a special case of *Clausen's product formula* for hypergeometric functions given by Slater [9, p. 75]. □

In the sequel it will be convenient to isolate the following *invariants* used by Ramanujan [7],

$$G := (2kk')^{-1/12},\ g := (2k/k'^2)^{-1/12} \tag{2.7}$$

and

$$2^{1/4}gG = (k^2/2k')^{-1/12}.$$

In Weber's terms [10] $2^{1/4}G = f$, $2^{1/4}g = f_1$, and $gG = f_2^{-1}$. We also need Klein's *absolute invariant* J which is

$$J := \frac{(4G^{24}-1)^3}{27G^{24}} = \frac{(4g^{24}+1)^3}{27g^{24}}. \tag{2.8}$$

Ramanujan talks about "corresponding theories" for $K_s\left(s := \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right)$ to that for K. For $s := \frac{1}{3}, \frac{1}{4}$ this is explained by the next result.

PROPOSITION 2.2.

(a) $K_{1/4}(h) = (1+k^2)^{1/2}K(k)$

if $2hh' = \left(\frac{g^{12}+g^{-12}}{2}\right)^{-1}$ and $0 \leqslant h < \frac{1}{\sqrt{2}}$, $0 < k < \sqrt{2}-1$

(b) $K_{1/3}(h) = (1-(kk')^2)^{1/4}K(k)$

if $2hh' = J^{-1/2}$ *and* $0 \leqslant h < \frac{1}{\sqrt{2}}$, $0 \leqslant k < \frac{1}{\sqrt{2}}$.

Proof. These were discovered by piecing together the quadratic and cubic transformations given in [2, § 2.11] They may be verified by establishing that both sides satisfy the same differential equation (derived from the appropriate hypergeometric differential equation), and both functions involved have the same finite value at zero. □

There is a corresponding relation for $K_{1/6}$ although it is less concise than those for $K_{1/4}$, $K_{1/3}$ [1, Ch. 5]. Combining these last two propositions leads to a variety of alternate hypergeometric expressions for K and K^2

THEOREM 2.3

(*a*)

(*i*) $$\frac{2K}{\pi}(k) = {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; [2kk']^2\right), \qquad \left(0 \leqslant k \leqslant \frac{1}{\sqrt{2}}\right)$$

(*ii*) $$= k^{-1/2}\, {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; -[2k/k'^2]^2\right), \qquad (0 \leqslant k \leqslant \sqrt{2}-1)$$

(*iii*) $$= k^{-1/2}\, {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; -[k^2/2k']^2\right), \qquad (0 \leqslant k^2 \leqslant 2\sqrt{2}-2)$$

(*b*)

(*iv*) $$\frac{2K}{\pi}(k) = (1+k^2)^{-1/2}\, {}_2F_1\left(\frac{1}{8}, \frac{3}{8}; 1; \left[\frac{g^{12}+g^{-12}}{2}\right]^{-2}\right)$$
$$(0 \leqslant k \leqslant \sqrt{2}-1)$$

(*v*) $$= (k'^2-k^2)^{-1/2}\, {}_2F_1\left(\frac{1}{8}, \frac{3}{8}; 1; -\left[\frac{G^{12}-G^{-12}}{2}\right]^{-2}\right)$$
$$\left(0 \leqslant k \leqslant \frac{2^{1/4}-\sqrt{2-\sqrt{2}}}{2}\right)$$

(*c*)

(*vi*) $$\frac{2K}{\pi}(k) = (1-(kk')^2)^{-1/4}\, {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; J^{-1}\right)$$
$$\left(0 \leqslant k \leqslant \frac{1}{\sqrt{2}}\right).$$

Proof. (a) We let $s := 0$ above to deduce (i). Then (ii) follows on replacing q by $-q$ in the theta-function representations of K and $(2kk')^2$. This is Jacobi's imaginary transformation [11]. We derive (iii) from (ii) by replacing k by $k_1 := (1-k')/(1+k')$ and using the quadratic transformation $K(k_1) = \left(\frac{1+k'}{2}\right) K(k)$ [11].

(b) (iv) comes from letting $s := \frac{1}{4}$ above. Then (v) again follows from Jacobi's imaginary transformation.

(c) (vi) comes from letting $s := \frac{1}{3}$ above. □

Similarly

THEOREM 2.4. *For $0 < k \leqslant \dfrac{1}{\sqrt{2}}$ restricted as in Theorem 2.3.*

(*a*)

$$(i)\quad \left[\frac{2K}{\pi}(k)\right]^2 = {}_3F\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};\,1,1;\,[2kk']^2\right),$$

$$(ii)\quad = k'^{-2}\,{}_3F_2\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};\,1,1;\,-[2k/k'^2]^2\right),$$

$$(iii)\quad = k'^{-1}\,{}_3F_2\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};\,1,1;\,-[k^2/2k']^2\right)$$

(*b*)

$$(iv)\quad \left[\frac{2K}{\pi}(k)\right]^2 = (1+k^2)^{-1}\,{}_3F_2\left(\frac{1}{4},\frac{3}{4},\frac{1}{2};\,1,1;\,\left[\frac{g^{12}+g^{-12}}{2}\right]^{-2}\right);$$

$$(v)\quad = (k'^2-k^2)^{-1}\,{}_3F_2\left(\frac{1}{4},\frac{3}{4},\frac{1}{2};\,1,1;\,-\left[\frac{G^{12}-G^{-12}}{2}\right]^{-2}\right).$$

(*c*)

$$(vi)\quad \left[\frac{2K}{\pi}(k)\right]^2 = (1-(kk')^2)^{-1/2}\,{}_3F_2\left(\frac{1}{6},\frac{5}{6},\frac{1}{2};\,1,1;\,J^{-1}\right).$$

Proof. We combine Theorem 2.3 with Proposition 2.1. □

Thus we have provided series for K and K^2 in terms of each of the six invariants. One can produce other such formulae by further use of transformation identities. For example, Bailey's formula [2, p. 84(2)] with $a := \frac{1}{2}$, $b := 1$ gives

$$\left[\frac{2K}{\pi}\right]^2(k) = (1-4(2kk')^2)^{-1/2}\,{}_3F_2\left(\frac{1}{6},\frac{5}{6},\frac{1}{2};\,1,1;\,\frac{-27\,(2kk')^2}{[1-4(2kk')^2]^3}\right)$$

$$= (k'^4+16k^2)^{-1/2}\,{}_3F_2\left(\frac{1}{6},\frac{5}{6},\frac{1}{2};\,1,1;\,\frac{27g^{48}}{(g^{24}+4)^3}\right). \qquad (2.9)$$

Note also that we may use (2.5) with $s := 0$ and Theorem 2.3 to produce similar series for E.

3. The General Identities. The *singular value function* is defined by solution of

$$\frac{K'}{K}(k(N)) = \sqrt{N} \tag{3.1}$$

for positive N. The uniquely defines k on $[0, \infty]$ as a decreasing function with $k(0) = \infty$, $k(1) = \frac{1}{\sqrt{2}}$, $k(\infty) = 0$. It is known that k is algebraic when N is rational [1]. Various values are tabulated below (§ 4). Moreover, for some N one or more of our invariants becomes very simple. In [1] a corresponding function α (a *singular value* of *the second kind*) was studied. It is defined by

$$\alpha(N) = \left(\frac{E'}{K} - \frac{\pi}{4K^2}\right)(k(N)) \tag{3.2}$$

and also is algebraic at rational values with $\alpha(1) = \frac{1}{2}$, $\alpha(\infty) = \frac{1}{\pi}$. Moreover, α satisfies remarkable recursions which allow one to compute it at many values both numerically and explicitly. For example,

$$\alpha(4N) = \frac{4\alpha(N) - 2\sqrt{N}\,k^2(N)}{[1+k'(N)]^2}, \tag{3.3}$$

and

$$k(4N) = \frac{1-k'(N)}{1+k'(N)}. \tag{3.4}$$

This leads to high-order iterations for $1/\pi$, [1]. Values of α are also given below (§ 4). From the definition of α we may derive that

$$\frac{1}{\pi} = \sqrt{N}\,kk'^2\,\frac{4K\dot{K}}{\pi^2} + (\alpha(N) - \sqrt{N}\,k^2)\,\frac{4K^2}{\pi^2}\ (k := k(N)). \tag{3.5}$$

This follows from using Legendre's identity (2.6) to write

$$\alpha(N) = \pi/(4K^2) - \sqrt{N}(E/K - 1)$$

and then using (2.5) to replace E by $\dot{K}$.

Similarly,

$$\frac{1}{K} = \sqrt{N}\,kk'^2\,\frac{4\dot{K}}{\pi} + (\alpha(N) - \sqrt{N}\,k^2)\,\frac{4K}{\pi}\ (k := k(N)). \tag{3.6}$$

Thus given $\alpha(N)$ and $k(N)$ we can combine (3.5) with Theorem 2.4 to produce power series for $\frac{1}{\pi}$. In like fashion we derive series for $\frac{1}{K}$ or for the Gaussian *AGM*, $M(1, k') = \pi/(2K)$, [1]. In each case we have

$$\left(\frac{2K}{\pi}\right)^2(k) = m(k)\, F(\phi(k))$$

for algebraic m and ϕ while $F(\phi)$ has a hypergeometric power-series expansion $\sum_{n=0}^{\infty} a_n\phi^n$. Then

$\frac{4K\dot{K}}{\pi^2} = \frac{1}{2}\dot{m}F + \frac{1}{2}m\dot{\phi}\,\dot{F}(\phi)$. Substitution in (3.5) leads to

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} a_n \left[\frac{\sqrt{N}}{2}\, kk'^2\, \dot{m} + (\alpha(N) - \sqrt{N}\, k^2)\, m + n\frac{\sqrt{N}}{2} m\frac{\dot{\phi}}{\phi}\, kk'^2\, a\right]\phi^n. \tag{3.7}$$

Thus for each rational N the bracketed term is of the form $a + nb$ with a, b algebraic. We specialize this for our six invariants.

Series in G_N^{-12}: $= 2k(N)k'(N)$: For $N > 1$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty}\left[\frac{(1/2)_n}{n!}\right]^3 a_n(N)(G_N^{-12})^{2n} \tag{3.8}$$

where

$$a_n(N) := [\alpha(N) - \sqrt{N}\, k^2(N)] + n\sqrt{N}[(k'^2(N) - k^2(N)].$$

Series in g_N^{-12}: $= 2k(N)/k'^2(N)$: For $N \geqslant 2$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty}(-1)^n\left[\frac{(1/2)_n}{n!}\right]^3 b_n(N)(g_N^{-12})^{2n} \tag{3.9}$$

where

$$b_n(N) := \alpha(N)k_N'^{-2} + n\sqrt{N}\left[\frac{1 + k^2(N)}{1 - k^2(N)}\right].$$

Series in g_{4N}^{-12}: $= (2^{1/4}\, g_N G_N)^{-12} = k^2(N)/(2k'(N))$: For $N \geqslant \frac{1}{2}$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty}(-1)^n\left[\frac{(1/2)_n}{n!}\right]^3 c_n(N)(g_{4N}^{-12})^{2n} \tag{3.10}$$

where

$$c_n(N) := [\alpha(N) - \frac{\sqrt{N}}{2}\, k^2(N)]k'^{-1}(N) + n\sqrt{N}\,[k'(N) + k'^{-1}(N)].$$

Series in x_N: $= \left[\frac{g_N^{12} + g_N^{-12}}{2}\right]^{-1} = \frac{4k(N)k'^2(N)}{[1 + k^2(N)]^2}$: For $N > 2$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n}{(n!)^3} d_n(N) x_N^{2n+1} \tag{3.11}$$

where

$$d_n(N) := \left[\frac{\alpha(N) x_N^{-1}}{1 + k^2(N)} - \frac{\sqrt{N}}{4} g_N^{-12}\right] + n\sqrt{N}\left[\frac{g_N^{12} - g_N^{-12}}{2}\right].$$

Series in y_N: $= \left[\frac{G_N^{12} - G_N^{-12}}{2}\right]^{-1} = \frac{4k(N)k'(N)}{1 - [2k(N)k'(N)]^2}$: For $N \geqslant 4$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n}{(n!)^3} e_n(N) y_N^{2n+1} \tag{3.12}$$

where

$$e_n(N) := \left[\frac{\alpha(N) y_N^{-1}}{k'^2(N) - k^2(N)} + \frac{\sqrt{N}}{2} k^2(N) G_N^{12}\right] + n\sqrt{N}\left[\frac{G_N^{12} + G_N^{-12}}{2}\right].$$

Series in J_N^{-1}: $= \frac{27 G_N^{24}}{(4G_N^{24} - 1)^3} = \frac{27 g_N^{24}}{(4g_N^{24} + 1)^3}$: For $N > 1$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{(\frac{1}{6})_n (\frac{1}{2})_n (\frac{5}{6})_n}{(n!)^3} f_n(N) (J_N^{-1/2})^{2n+1} \tag{3.13}$$

where

$$f_n(N) := \frac{1}{3\sqrt{3}}\left[\sqrt{N}\sqrt{1 - G_N^{-24}} + 2(\alpha(N) - \sqrt{N}k^2(N))(4G_N^{24} - 1)\right]$$
$$+ n\sqrt{N}\frac{2}{3\sqrt{3}}\left[(8G_N^{24} + 1)\sqrt{1 - G_N^{-24}}\right].$$

There are many equivalent rearrangements of the formulae for $a_n(N) - f_n(N)$.

In similar fashion we may deduce that for all N

$$M(1, k'(N)) = \frac{\pi}{2K(k(N))} = \pi \sum_{n=0}^{\infty} m_n(N) \left(\frac{(\frac{1}{2})_n}{n!}\right)^2 k(N)^{2n} \tag{3.14}$$

where

$$m_n(N) := [\alpha(N) - \sqrt{N}k^2(N)] + n2\sqrt{N}k'^2(N);$$

and for $N > 1$

$$M(1, k'(N)) = \pi \sum_{n=0}^{\infty} (-1)^n \, n_n(N) \left(\frac{(\frac{1}{2})_n}{n!}\right) \left(\frac{k(N)}{k'(N)}\right)^{2n}. \quad (3.15)$$

where

$$n_n(N) := \alpha(N) k_N'^{-1} + n 2\sqrt{N} \, k_N'^{-1}$$

These use (3 6), and (2.3) with $s := 0$. Also using (3.6) and Theorem (2.3) (a) (i) and (ii), leads to, for $N > 1$

$$M(1, k'(N)) = \pi \sum_{n=0}^{\infty} o_n(N) \left[\frac{(\frac{1}{4})_n}{n!}\right]^3 (G_N^{-12})^{2n} \quad (3.16)$$

where

$$o_n(N) := [\alpha(N) - \sqrt{N} \, k^2(N)] + n 2\sqrt{N} \, [k'^2(N) - k^2(N)];$$

and, for $N \geqslant 2$

$$M(1, k'(N)) = \pi \sum_{n=0}^{\infty} (-1)^n \, p_n(N) \left[\frac{(\frac{1}{4})_n}{n!}\right]^3 (g_N^{-12})^{2n} \quad (3.17)$$

where

$$p_n(N) := \alpha(N) k'^{-1}(N) + n 2\sqrt{N} \, [1 + k^2(N)] \, k'^{-1}(N).$$

Obviously similar formulae may be derived in the other invariants.

4. **Specific Examples.** We first list various values of $\alpha(N)$ and $k(N)$ (or equivalently G_N^{-12} or g_N^{-12} whichever is simpler).

Many other values of G_N, g_N may be found in [7], [10] or [1]. Certain values of $k(N)$ are given in [12]. The explanation of the computation of $k(N)$ can be found in [1], [8] and [13]. Many values of $\alpha(N)$ are derived in [1]. For $N := 2, 3, 4, 5, 7$ they are given in [12].

From the information in these tables and the formulae given for π^{-1}, we may explicitly compute all but two of Ramanujan's series. (These two which rely on $K_{1/6}$ are treated in [1].)

Ramanujan gives series of form (3.8) for $N := 3, 7, 15$ and none of form (3.9) or (3.10). He gives series of form (3.11) for $N := 6, 10, 18, 22, 58$ and of form (3.12) for $N := 5, 9, 13, 25, 37$. He gives series of form (3.13) for $N := 3, 7$. In each case manipulation of the formulae yields the desired

Values of G_N^{-12} and $\alpha(N)$ for N odd

N	$2k(N)k'(N) = G_N^{-12}$	$\alpha(N)$
1	1	$1/2$
3	$1/2$	$(\sqrt{3}-1)/2$
5	$\sqrt{5}-2$	$(\sqrt{5}-\sqrt{2\sqrt{5}-2})/2$
7	$1/8$	$(\sqrt{7}-2)/2$
9	$(2-\sqrt{3})^2$	$(3-3^{3/4}\sqrt{2}(\sqrt{3}-1))/2$
13	$5\sqrt{13}-18$	$(\sqrt{13}-\sqrt{74\sqrt{13}-258})/2$
15	$\frac{1}{8}\left(\frac{\sqrt{5}-1}{2}\right)^4$	$(\sqrt{15}-\sqrt{5}-1)/2$
25	$(\sqrt{5}-2)^4$	$5(1-2.5^{1/4}(7-3\sqrt{5}))/2$
27	$(2^{1/3}-1)^4/2$	$3(\sqrt{3}+1-2^{4/3})/2$
37	$(\sqrt{37}-6)^3$	$(\sqrt{37}-(171-25\sqrt{37})(\sqrt{37}-6)^{1/2})/2$

Values of g_N^{-12} and $\alpha(N)$ for N even

N	$2k(N)/k'(N)^2 = g_N^{-12}$	$\alpha(N)$
2	1	$(\sqrt{2}-1)$
4	$1/2\sqrt{2}$	$2(\sqrt{2}-1)^2$
6	$(\sqrt{2}-1)^2$	$(\sqrt{3}-\sqrt{2})(2-\sqrt{3})(3-\sqrt{2})(\sqrt{2}+1)$
10	$(\sqrt{5}-2)^2$	$(7+2\sqrt{5})(\sqrt{10}-3)(\sqrt{2}-1)^2$
18	$(\sqrt{3}-\sqrt{2})^4$	$3(\sqrt{3}+\sqrt{2})^4(\sqrt{6}-1)^2(7\sqrt{2}-5-2\sqrt{6})$
22	$(\sqrt{2}-1)^6$	$(\sqrt{2}+1)^6(33-17\sqrt{2})(3\sqrt{22}-7-5\sqrt{2})$
58	$\left(\frac{\sqrt{29}-5}{2}\right)^6$	$\left(\frac{\sqrt{29}+5}{2}\right)^6(99\sqrt{29}-444)(99\sqrt{2}-70-13\sqrt{29})$

result. (The algebra getting harder as N increases.) In fact $\alpha(37)$ and $\alpha(58)$ were calculated by obtaining $d_0(58)$ and $e_0(37)$ to high precision numerically and then solving for α. Given the algebraic nature of α this ultimately sufficies to verify the values. In fact, we have

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n}{(n!)^3} \left[\frac{1123 + n21460}{4}\right]\left[\frac{1}{882}\right]^{2n+1} \tag{4.1}$$

using (3.12) for $N: = 37$; and using (3.11) for $N: = 58$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n}{(n!)^3} [2\sqrt{2}\,(1103 + n26390)] \left(\frac{1}{99^2}\right)^{2n+1} \tag{4.2}$$

Since $k^2(N)$ behaves like $16 \exp(-\pi\sqrt{N})$ [1] it is very easy to estimate the number of digits added in each series. For N at all large, the convergence while linear is most impressive. Not surprisingly Ramanujan has given most of the special cases of (3.8), (3.11), (3.12) for which the power is rational. We do add a few examples below. We have

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \left[\frac{(\frac{1}{2})_n}{n!}\right]^3 \sqrt{2}\, 3^{1/4} \left[\frac{3-\sqrt{3}}{2} + 6n\right] [2-\sqrt{3}]^{4n+1} \tag{4.3}$$

using $N: = 9$ in (3.8); and using $N: = 18$ in (3.9)

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} (-1)^n \left[\frac{(\frac{1}{2})_n}{n!}\right]^3 \left[\frac{21 - 6\sqrt{6} + 84n}{2}\right] (\sqrt{3}-\sqrt{2})^{8n+2}. \tag{4.4}$$

Similarly using $N: = 15$ in (3.10) we find that

$$c_n(15) = \frac{\sqrt{2}}{32}(-1009 + 591\sqrt{3} - 445\sqrt{5} + 255\sqrt{15}) + n\,\frac{3\sqrt{2}}{16}(-395 + 245\sqrt{3} - 175\sqrt{5} + 101\sqrt{15}) \tag{4.5}$$

while

$$\frac{k^2(15)}{2k'(15)} = \frac{\sqrt{2}}{32}(305 - 177\sqrt{3} + 141\sqrt{5} - 81\sqrt{15}) \tag{4.6}$$

and the associated series gains more than 8 digits a term. Also (3.13), with $N := 4$ gives

$$f_n(4) = (63n + 5)\sqrt{6}/3;\ J_4^{-1} = (2/11)^3, \tag{4.7}$$

and with $N := 2$

$$f_n(2) = (28n + 3)\sqrt{3}/9;\ J_2^{-1} = (3/5)^3. \tag{4.8}$$

The vigorous reader will be able to compute many more such series. We now give four mean series. From (3.16) with $N := 1$ we derive

$$M\left(1, \frac{1}{\sqrt{2}}\right) = \pi \sum_{n=0}^{\infty} n \left(\frac{(\frac{1}{2})_n}{n!}\right)^2 2^{-n}. \tag{4.9}$$

From (3.15), with $N := 3$ we derive

$$M\left(1, \frac{\sqrt{3}+1}{2\sqrt{2}}\right) = \frac{\pi}{4} \sum_{n=0}^{\infty} (12n + 1) \left(\frac{(\frac{1}{4})_n}{n!}\right)^2 4^{-n} \tag{4.10}$$

and with $N := 7$ we derive

$$M\left(1, \frac{\sqrt{7}+3}{4\sqrt{2}}\right) = \frac{\pi}{16} \sum_{n=0}^{\infty} (84n + 5) \left(\frac{(\frac{1}{2})_n}{n!}\right)^2 64^{-n}. \tag{4.11}$$

Finally, we use (3.17) with $N := 4$ and the fact that

$$M\left(1, \frac{1}{\sqrt{2}}\right) = \left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right) M(1, k'(4)) \text{ to deduce that}$$

$$M\left(1, \frac{1}{\sqrt{2}}\right) = 2^{1/4}\, \frac{\pi}{4} \sum_{n=0}^{\infty} (-1)^n (12n + 1) \left(\frac{(\frac{1}{4})_n}{n!}\right)^2 8^{-n}. \tag{4.12}$$

It is known [1], [12], [13] that $K(k(N))$ and $M(1, k(N))$ can be evaluated in terms of Γ-values for integral N. For example,

$$M\left(1, \frac{\sqrt{7}+3}{4\sqrt{2}}\right) = \frac{7^{1/4}\, 2}{\Gamma(\frac{1}{7})\,\Gamma(\frac{2}{7})\,\Gamma(\frac{4}{7})}\, \pi^2 \text{ and } M\left(1, \frac{1}{\sqrt{2}}\right) = \frac{2}{\Gamma(\frac{1}{4})^2}\, \pi^{3/2}.$$

Thus, all these series have, in principle, closed forms in terms of Γ-values, such as

$$\left(\frac{15}{7}\right)^{1/4} \frac{\Gamma(\frac{1}{7})\,\Gamma(\frac{2}{7})\,\Gamma(\frac{4}{7})}{4\pi^2} = {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \left(\frac{4}{85}\right)^3\right), \tag{4.13}$$

on using Theorem 2.3 (vi).

Note also that the asymptotic for $k(N)$ leads to logarithmic approximations for π, [1], [7], [8]. For instance

$$\pi \cong \frac{2}{\sqrt{N}} \log (16/x_N), \tag{4.14}$$

which for $N := 58$ yields

$$\pi \simeq \frac{4}{\sqrt{58}} \log (396)$$

which is 3.14159265342... .

References

1. BORWEIN, J.M. and P.B. BORWEIN, Pi and the AGM: *A Study in Analytic Number Theory und Computational Complexity*, Wiley, 1987.
2. ERDEYLI, A. et al., *Higher Transcendental Functions*, McGraw-Hill, 1953.
3. HARDY, G.H. *Ramanujan's Collected Papers*, Chelsea Publishing, 1962.
4. MAGNUS, W. Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag, 1966.
5. NEWMAN, M. and D. SHANKS, On a sequence arising in series for π, *Math. of Computation 42* (1984), 199–217.
6. RAINVILLE, E.D. *Special functions*, McMillan New York, 1960.
7. RAMANUJAN, S. Modular equations and approximations to π, *Quart J. Math. Oxford, 45* (1914), 350–372.
8. SHANKS, D. Dihedral quartic approximations and series for π, *J. Number Theory, 14* (1982), 397–423.
9. SLATER, L J. *Generalized hypergeometric functions*, Cambridge University Press, 1966.
10. WEBER, H. *Lehrbuch der Algebra*, Vol III, Brauschweig, 1908.
11. WHITTAKER, E.T. and G.N. WATSON, *Modern Analysis*, 4th ed., Cambridge University Press, 1927.
12. ZUCKER, I.J. The summation of series of hyperbolic functions, *SIAM J. Math. Anal., 10* (1979), 192–206.

13. ZUCKER, I.J. The evaluation in terms of Γ-functions of the periods of elliptic curves admitting complex multiplication, *Proc. Camb. Phil. Soc. 82* (1977), 111–118.

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