The Way of All Means
Author(s): J. M. Borwein and P. B. Borwein
Source: *The American Mathematical Monthly*, Vol. 94, No. 6, (Jun. - Jul., 1987), pp. 519-522
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/2322842
Accessed: 23/05/2008 18:17

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at http://www.jstor.org/page/info/about/policies/terms.jsp. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://www.jstor.org/action/showPublisher?publisherCode=maa.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit organization founded in 1995 to build trusted digital archives for scholarship. We enable the scholarly community to preserve their work and the materials they rely upon, and to build a common research platform that promotes the discovery and use of these resources. For more information about JSTOR, please contact support@jstor.org.

http://www.jstor.org

UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

The Way of All Means

J. M. BORWEIN and P. B. BORWEIN,
Dalhousie University, Halifax, N. S., Canada B3H 3J5

We are interested in presenting some problems concerning the limits of mean iterations. For our current purposes, a *mean* is a continuous real-valued function M of two strictly positive real variables a and b so that, for all such a and b

$$\min(a, b) \leq M(a, b) \leq \max(a, b).$$

This inequality, which ensures that $M(a, b)$ lies between a and b, is the absolutely essential part of the definition. We also assume that means are symmetric and positively homogeneous, that is,

$$M(a, b) = M(b, a) \quad \text{and} \quad M(\lambda a, \lambda b) = \lambda M(a, b) \qquad \lambda \geq 0,$$

though often these assumptions are unnecessary. Three particular classes of means are:

HÖLDER'S MEANS.

$$H_p(a, b) := [(a^p + b^p)/2]^{1/p} \qquad p \neq 0$$

$$G(a, b) := H_0(a, b) := \lim_{p \to 0} H_p(a, b) = \sqrt{(ab)}\,.$$

Then $A := H_1$ is the *arithmetic* mean and G is the *geometric* mean. The mean H_{-1} is called the *harmonic* mean. Hölder's means are sometimes called power means.

LEHMER'S MEANS.

$$L_p(a, b) := (a^p + b^p)/(a^{p-1} + b^{p-1}).$$

Note that $L_1 = A$, $L_{1/2} = G$ and $L_0 = H_{-1}$. These are the only means that are both Lehmer means and Hölder means [7].

STOLARSKY'S MEANS.

$$S_p(a, b) := [(a^p - b^p)/(pa - pb)]^{1/(p-1)} \qquad p \neq 0, 1.$$

The limiting cases ($p = 0, 1$) give the *logarithmic* and *identric* means, respectively.

Thus

$$S_0(a,b) := \lim_{p\to 0} S_p(a,b) = (a-b)/(\log a - \log b)$$

and

$$S_1(a,b) := \lim_{p\to 1} S_p(a,b) = e^{-1}[a^a/b^b]^{1/(a-b)}.$$

Note that $S_2 = A$ and $S_{-1} = G$.

A (Gaussian) *mean iteration* associated with two means M and N is the two-term iteration

$$a_{n+1} := M(a_n, b_n) \quad \text{and} \quad b_{n+1} := N(a_n, b_n)$$

with initial values $a_0 := a$ and $b_0 := b$. The common limit of $\{a_n\}$ and $\{b_n\}$ when it exists, is called the (Gaussian) *compound* of M and N and is denoted by $M \otimes N := M \otimes N(a,b)$. The compound $M \otimes N$ exists under very general assumptions on M and N. For any pair of means from the three classes above $M \otimes N(1,z)$ is an analytic function in a neighborhood of 1 and the underlying convergence is quadratic—in the sense that $|a_{n+1} - b_{n+1}| = O(|a_n - b_n|^2)$, so that n iterations typically give c^n digits of $M \otimes N$ for some $c > 1$. (See [3] or [6] for the reasons behind these assertions.)

In certain cases it is easy to identify the limit function. For example

$$H_p(a,b) \otimes H_{-p}(a,b) = \sqrt{(ab)},$$

which follows from the fact that

$$[(a^p + b^p)/2]^{1/p}[(a^{-p} + b^{-p})/2]^{-1/p} = ab.$$

The *Gaussian arithmetic-geometric mean iteration* (AGM) is given by

$$a_{n+1} := (a_n + b_n)/2 \quad \textit{and} \quad b_{n+1} := \sqrt{(a_n b_n)}.$$

In our notation this is the compounding of A and G and the common limit is $A \otimes G$. The remarkable fact is that this has a closed form in terms of complete elliptic integrals, namely

$$A \otimes G(1,z) = (\pi/2)\Big/\int_0^{\pi/2}[1 - (1 - z^2)\sin^2 t]^{-1/2}\,dt.$$

The AGM sits at the heart of the most rapid algorithms for the extended precision algorithms for π and all the elementary functions. (See [2], [3], [4], [8], and [9].) The reason for this is that, up to trivial changes of variable, this is the only known quadratically convergent algebraic iteration with an identifiable nonelementary limit. Thus $H_p \otimes G$ are the only compounds of two means from the above three classes where we know that we get an explicitly identifiable nontrivial limit.

This leads to the first question.

Question 1. Can one identify, in closed form, either

a] $$A(a,b)\otimes H_2(a,b)$$

or

b] $$A(a,b)\otimes L_2(a,b)?$$

These are perhaps the two most tantalizing of the unknown mean iterations and almost any information about them would be of some interest. Case *b*] is analyzed in some detail by Lehmer [7], where various expansions are provided. What constitutes a closed form answer is not always clear. We can be more precise. A function is said to be *hypertranscendental* if it satisfies no algebraic differential equation of any order (the solutions of an equation like $(f\cdot f'''' + 7xf')^2 = x^2f^3$ are thus not hypertranscendental). Virtually all the special functions are not hypertranscendental, precisely because they arise as solutions of D.E.s. An exception is the Gamma function, which was shown to satisfy no algebraic differential equation by Hölder.

Question 2. Are either

a] $$A(a,b)\otimes H_2(a,b)$$

or

b] $$A(a,b)\otimes L_2(a,b).$$

hypertranscendental?

An easier problem might be to determine whether either of the above functions is hypergeometric. We say that a function is hypergeometric if it has a power series expansion (around some point x_0) with coefficients $\{c_n\}$ that satisfy $c_n/c_{n-1} = R(n)$, where R is a fixed rational function. Many familiar transcendental functions are hypergeometric including $\exp(x_0 := 0$ and $R(x) := 1/x)$, $\log(x_0 := 1$ and $R(x) := (1-x)/x)$, and the complete elliptic integrals.

One can show that none of the most elementary of the transcendental functions (exp, log, sin, etc.) can be Gaussian compounds of algebraic means. This is done in [3]. Some of these functions are, nonetheless, limits of two term iterations.

For example

$$a_{n+1} := \left(a_n + \sqrt{(a_nb_n)}\right)/2 \quad\text{and}\quad b_{n+1} := \left(b_n + \sqrt{(a_nb_n)}\right)/2$$

converge to $(b_0 - a_0)/(\log b_0 - \log a_0)$. This, however, isn't a Gaussian mean iteration by our definition (because of the lack of symmetry) and the convergence isn't quadratic.

The class of possible compounds of *rational* means (means which are also rational functions) is probably more structured. We show in [3] that the only algebraic functions in this class are pth roots of rational functions. In fact, $A \otimes L_2$ is not algebraic while $H_1 \otimes H_{-1}$ clearly is. This leads to

Question 3. Characterize (or say something interesting about)

a] $F_R := \{M \otimes N(1,z) \mid$ where M and N are rational means$\}$.

b] $F_A := \{M \otimes N(1, z) |$ where M and N are algebraic means$\}$.
c] $F_H := \{M \otimes N(1, z) |$ where M and N are analytic means$\}$.

Here rational, algebraic, or analytic means are means which are, respectively, also rational, algebraic, or analytic functions of each variable (in some neighborhood of 1). Thus A and H_{-1} are rational means; G and H_2 are algebraic means; and S_0 is an analytic mean. Of course, A and H_{-1} are also algebraic and analytic means, as is any rational mean.

Question 4. Do F_R or F_A contain any of the special functions other than algebraic functions or complete elliptic integrals (or some essentially trivial variant thereof)?

There are many multidimensional quadratically convergent analogs of compounding due to Borchardt and others. (See **[1]** and **[3]**.) To our knowledge none are known to lead to new explicit transcendental limits.

Question 5. What can be said about multidimensional compounds?

An example of a three-dimensional compound is the following. Let

$$a_{n+1} := M_1(a_n, b_n, c_n) := (a_n + b_n + c_n)/3$$
$$b_{n+1} := M_2(a_n, b_n, c_n) := (a_n^2 + b_n^2 + c_n^2)/(a_n + b_n + c_n)$$
$$c_{n+1} := M_3(a_n, b_n, c_n) := \left[(a_n^2 + b_n^2 + c_n^2)/3\right]^{1/2}.$$

Then, if a_0, b_0, and c_0 are positive, the three sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ converge quadratically to a common limit. What can be said about the limit function?

The fourth question may be the most interesting. A positive answer would be of astonishing consequence. A negative answer would explain the central role of the AGM.

Additional related material is developed in **[1]**, **[3]**, **[10]**, and **[11]**.

REFERENCES

1. J. Arazy, J. Claesson, S. Janson, and J. Peetre, Means and their iterations, preprint.
2. J. M. Borwein and P. B. Borwein, The arithmetic-geometric mean and fast computation of the elementary functions, SIAM Review, 95 (1985) 351–365.
3. J. M. Borwein and P. B. Borwein, Pi and the AGM—A Study in Analytic Number Theory and Computational Complexity, Wiley, New York, 1986.
4. R. P. Brent, Fast multiple-precision evaluation of elementary functions, J. Assoc. Comput. Mach., 23 (1976) 242–251.
5. B. C. Carlson, Algorithms involving arithmetic and geometric means, this MONTHLY, 78 (1971) 486–505.
6. D. M. E. Foster and G. M. Phillips, A generalization of the Archimedean double sequence, J. Math Anal. Appl., 101 (1984) 575–581.
7. D. H. Lehmer, On the compounding of certain means, J. Math. Anal. Appl., 36 (1971) 183–200.
8. D. J. Newman, A simplified version of the fast algorithms of Brent and Salamin, Math. Comput., 42 (1985) 207–210.
9. E. Salamin, Computation of π using arithmetic-geometric mean, Math. Comput., 30 (1976) 565–570.
10. K. B. Stolarsky, The power and generalized logarithmic means, this MONTHLY, 87 (1980) 545–548.
11. J. Wimp. Computation with Recurrence Relations, Pitman, London, 1984.