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CONSTRUCTIVE APPROXIMATION

Quadratic Hermite–Padé Approximation to the Exponential Function

Peter B. Borwein

Abstract. Approximation to exp of the form

$$\mathscr{E}_m(z) := p_m(z)\,e^{-2z} + q_m(z)\,e^{-z} + r_m(z) = O(z^{3(m+1)-1}),$$

where p_m, q_m, and r_m are polynomials of degree at most m and p_m has lead coefficient 1 is considered. Exact asymptotics and explicit formulas are obtained for the sequences $\{\mathscr{E}_m\}$, $\{p_m\}$, $\{q_m\}$, and $\{r_m\}$. It is observed that the above sequences all satisfy the simple four-term recursion:

$$T_{m+3} = \frac{1}{3m+4}[(-6m-14)z^3 T_m + (9m+15)(z^2+(3m+4)(3m+7))T_{m+1} + 3zT_{m+2}].$$

It is also observed that these generalized Padé-type approximations can be used to asymptotically minimize expressions of the above form on the unit disk.

1. Introduction

We wish to consider approximations to e^{-x} generated by finding polynomials p_m, q_m, and r_m so that

$$(1.1) \qquad \mathscr{E}_m(x) := p_m(x)\,e^{-2x} + q_m(x)\,e^{-x} + r_m(x) = O(x^{3(m+1)-1}),$$

with $p_m, q_m, r_m \in \pi_m$ (the algebraic polynomials of degree at most m). The approximation to e^{-x} given by

$$\delta_m(x) := \frac{-q_m(x) + \sqrt{q_m^2(x) - 4r_m(x)p_m(x)}}{2p_m(x)}$$

is the natural quadratic generalization of the main diagonal Padé approximant $-Q_m/P_m$ which satisfies

$$(1.2) \qquad \gamma_m(x) := P_m(x)\,e^{-x} + Q_m(x) = O(x^{2(m+1)-1}).$$

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Our primary aim is to derive exact asymptotic formulas and recursions for $\{p_m\}$, $\{q_m\}$, $\{r_m\}$, $\{\mathscr{E}_m\}$, and $\{\delta_m\}$ and to treat some minimization problems concerning related approximations on the unit disk in $\mathbf{C}$.

Hermite, who was Padé's thesis supervisor, considered expressions of the form

$$t_k(x)\,e^{s_k x}+t_{k-1}(x)\,e^{s_{k-1}x}+\cdots+t_1(x)\,e^{s_1 x}=O(x^h), \tag{1.3}$$

where $t_1,\ldots,t_k$ are polynomials, of specified degrees, chosen so that h is as large as possible [3]. Included, of course, in expression of type (1.3) are both the Padé approximations (1.2) and the quadratic Hermite-Padé approximations (1.1). Hermite viewed his approximants as an algebraic generalization of the continued fraction for exp. Some forty years later Mahler [4] showed how Hermite's approximations could be used to prove the transcendence of e and π and, still later [5], showed how to derive an irrationality measure for π from the approximations. This is computationally difficult, but very natural, and is different from Hermite's original approach to the transcendence of e. Since we do not treat approximations of the generality of (1.3) we are able to provide exact asymptotics rather than the estimates included in [3], [4], and [5].

The general problem of Hermite-Padé approximation is the following: given functions $f_1,\ldots,f_n$ and integers $d_0,\ldots,d_n$, find polynomials $p_0,\ldots,p_n$ ($\deg(p_i)\le d_i$) so that

$$p_0(x)+p_1(x)f_1(x)+\cdots+p_n(x)f_n(x)=O(x^h), \tag{1.4}$$

where h is as large as possible. Particularly interesting cases arise where the f_i are related, for example f_i is the ith power (or ith derivative) of a fixed f. The $i=1$ case is the familiar Padé case. As one might expect, the already complex Padé questions concerning the nature and region of convergence become more complicated. The substantial body of research in this area, due to, among others, Baker, Chudnovsky, Della-Dora, Mahler, Nuttall, and Wallin can be accessed through the extensive bibliography in [1].

Exact results concerning best rational approximation to exp, particularly the Meinardus conjecture, have attracted much attention ([2], [6]). Proposition 5 can be viewed as a quadratic version of this conjecture on the disk. A linear version, due to Trefethen appears in [9]. Virtually no completely worked out examples for higher-dimensional approximations exist and exp is a natural candidate for such a complete analysis. This analysis constitutes the thrust of this paper.

2. Explicit Formulas

We derive the quadratic approximations from the Padé approximation as follows. With the notation of (1.1) observe that

$$[p_m(x)\,e^{-2x}+q_m(x)\,e^{-x}+r_m(x)]^{(m+1)}=s_m(x)\,e^{-2x}+t_m(x)\,e^{-x}=O(x^{2m+1}),$$

where s_m and $t_m\in\pi_m$. In particular, $-t_m/s_m$ must be the (m,m) Padé approximant to e^{-x}. This, coupled with the observation that if

$$g(x)=\frac{1}{m!}\int_0^x (x-t)^m f(t)\,dt$$

then

$$g^{(m+1)}(x)=f(x),$$

leads to the formulas for p_m and q_m ((2.4) and (2.7)) on which our analysis rests.

We commence the detailed development of formulas for $\{p_m\}$, $\{q_m\}$, and $\{r_m\}$. Let

$$P_m(x):=\frac{1}{m!}\int_0^\infty t^m(t+x)^m e^{-t}\,dt=\sum_{k=0}^{m}\frac{(2m-k)!}{(m-k)!\,k!}x^k \tag{2.1}$$

and

$$\begin{aligned}Q_m(x)&:=\frac{-1}{m}\int_0^\infty t^m(t-x)^m e^{-t}\,dt=-\sum_{k=0}^{m}\frac{(2m-k)!}{(m-k)!\,k!}(-x)^k\\&=-P_m(-x).\end{aligned} \tag{2.2}$$

Then $-Q_m/P_m$ is the (m, m) Padé approximant to e^{-x}. Also, P_m and Q_m satisfy (1.2) (see, for example, [6] or [7]) and

$$P_m(x)\,e^{-x}+Q_m(x)=(-1)^{m+1}\frac{x^{2m+1}}{m!}\int_0^1(1-u)^m u^m e^{-ux}\,du. \tag{2.3}$$

Now let

$$\begin{aligned}p_m(x)&:=\frac{e^{2x}2^{m+1}}{m!}\int_x^\infty (t-x)^m P_m(t)\,e^{-2t}\,dt\\&=\frac{e^{2x}2^{m+1}}{m!}\int_x^\infty (t-x)^m e^{-2t}\sum_{k=0}^{m}\frac{(2m-k)!}{(m-k)!\,k!}t^k\,dt\\&=\frac{2^{m+1}}{m!}\int_0^\infty u^m e^{-2u}\sum_{k=0}^{m}\frac{(2m-k)!}{(m-k)!\,k!}(u+x)^k\,du \qquad (u=t-x)\\&=\frac{2^{m+1}}{m!}\sum_{k=0}^{m}\sum_{j=0}^{k}\frac{(2m-k)!\,(m+k-j)!\,x^j}{(m-k)!\,j!\,(k-j)!\,2^{m+k+1-j}}\\&=m!\sum_{j=0}^{m}\frac{x^j}{j!}\sum_{k=j}^{m}\binom{2m-k}{m}\binom{m+k-j}{m}2^{j-k}.\end{aligned} \tag{2.4}$$

If we set

$$c_j:=\sum_{k=0}^{m-j}\binom{2m-(k+j)}{m}\binom{m+k}{m}\frac{1}{2^k}, \tag{2.5}$$

then

$$p_m(x)=m!\sum_{j=0}^{m}\frac{c_jx^j}{j!}. \tag{2.6}$$

Note that p_m is a polynomial of degree m with positive integer coefficients and with highest coefficient 1. Let

$$(2.7)\qquad q_m(x):=\frac{e^x2^{m+1}}{m!}\int_x^\infty (t-x)^mQ_m(t)\,e^{-t}\,dt$$
$$=-2^{m+1}m!\sum_{j=0}^m\frac{x^j}{j!}\sum_{k=j}^m\binom{2m-k}{m}\binom{m+k-j}{m}(-1)^k.$$

Thus, if we set

$$d_j:=\sum_{k=0}^{m-j}\binom{2m-(k+j)}{m}\binom{m+k}{m}(-1)^{k-j},$$

then

$$(2.8)\qquad d_j=(-1)^j\binom{3m/2-j/2}{m}\left(\frac{1+(-1)^{m-j}}{2}\right)$$

and

$$(2.9)\qquad q_m(x)=-2^{m+1}m!\sum_{j=0}^m\frac{d_jx^j}{j!},$$

where q_m is also a polynomial of degree at most m with integer coefficients.

We define r_m in terms of p_m by

$$(2.10)\qquad r_m(x):=(-1)^m p_m(-x),$$

with p_m given by (2.6) or (2.3).

Finally, let

$$(2.11)\quad \mathscr{E}_m(x):=\frac{2^{m+1}}{m!\,m!}\int_0^x(x-t)^m\,e^{-t}t^{2m+1}\int_0^1(1-u)^mu^m\,e^{-ut}\,du\,dt$$
$$=\frac{2^{m+1}x^{3m+2}}{m!\,m!}\int_0^1\int_0^1(1-v)^m(1-u)^mv^{2m+1}u^m\,e^{-uvx}\,e^{-vx}\,du\,dv.$$

We may now establish the basic proposition.

Proposition 1.

$$\mathscr{E}_m(x)=p_m(x)\,e^{-2x}+q_m(x)\,e^{-x}+r_m(x)=O(x^{3(m+1)-1}),$$

where $\mathscr{E}_m$, p_m, q_m, *and* r_m *are given by* (2.11), (2.6), (2.9), *and* (2.10), *respectively.*

Proof. By (2.11) and (2.3)

$$\mathscr{E}_m(x)=\frac{-2^{m+1}}{m!}\int_0^x(t-x)^m\,e^{-t}[P_m(t)\,e^{-t}+Q_m(t)]\,dt$$
$$=\frac{2^{m+1}}{m!}\int_x^\infty(t-x)^mP_m(t)\,e^{-2t}\,dt+\frac{2^{m+1}}{m!}\int_x^\infty(t-x)^mQ_m(t)\,e^{-t}\,dt$$
$$-\frac{2^{m+1}}{m!}\int_0^\infty(t-x)^m\,e^{-t}[P_m(t)\,e^{-t}+Q_m(t)]\,dt.$$

By (2.4) and (2.7) the first two integrals of the last equation are $e^{-2x}p_m(x)$ and $e^{-x}q_m(x)$ while the last integral is a polynomial of degree m in x, say $w_m(x)$. It remains to show that $w_m(x)=(-1)^m p_m(-x)$. We have deduced, with (2.11), that

$$p_m(x)\,e^{-x}+q_m(x)+w_m(x)\,e^{x}=O(x^{3(m+1)-1}). \tag{2.12}$$

Repeated differentiation shows that an expression of the form

$$s(x)\,e^{-x}+t(x)+u(x)\,e^{x}, \tag{2.13}$$

where s, t, and $u\neq 0$ are polynomials of degree at most n_1, n_2, and n_3 can have a zero of order at most $n_1+n_2+n_3+2$. Thus, on replacing x by $-x$ in (2.12) we deduce that $w_m(x)=(-1)^m p_m(-x)$. (Otherwise we could construct a form from (2.12) with a zero of too high a degree at zero. Note that $q_m(x)=(-1)^m q_m(-x)$.) ∎

3. Asymptotics

We now turn to asymptotic estimates for $\{p_m\}$, $\{q_m\}$, $\{r_m\}$, and $\{\mathscr{E}_n\}$. (As usual, $a_m\sim b_m$ means $a_m/b_m\to 1$. Throughout this paper the asymptotics are in the variable m.)

Proposition 2.

$$\mathscr{E}_m(x)\sim\frac{2^{m+1}m!\,x^{3m+2}\,e^{-x}}{(3m+2)!}.$$

(The asymptotic is uniform on bounded subsets of **C**.)

Proof. From (2.11)

$$\begin{aligned}\mathscr{E}_m(x)&=\frac{2^{m+1}x^{3m+2}}{m!\,m!}\int_0^1\int_0^1(1-v)^m(1-u)^m v^{2m+1}u^m\,e^{-uvx}\,e^{-vx}\,du\,dv\\&\sim\frac{2^{m+1}x^{3m+2}}{(2m+1)!}\int_0^1(1-v)^m v^{2m+1}\,e^{-3/2vx}\,dv\\&\sim\frac{2^{m+1}m!\,x^{3m+2}\,e^{-x}}{(3m+2)!}.\end{aligned}$$

Both of the above asymptotics follow from the elementary relation

$$\begin{aligned}\int_0^1(1-t)^{\alpha m}t^{\beta m}\,e^{-\gamma t}\,dt&\sim e^{-\beta\gamma/(\alpha+\beta)}\int_0^1(1-t)^{\alpha m}t^{\beta m}\,dt\\&=e^{-\beta\gamma/(\alpha+\beta)}\frac{(\alpha m)!\,(\beta m)!}{((\alpha+\beta)m+1)!},\end{aligned}$$

which in turn follows since $((1-t)^\alpha t^\beta)^m$ behaves like a "spike" at $\beta/(\alpha+\beta)$ (see [9]). It is easy to check directly from (2.11) that $\mathscr{E}_m(x)/[(2^{m+1}m!\,x^{3m+2}\,e^{-x})/(3m+2)!]$ is uniformly bounded on compact subsets of **C**. The uniformity of the asymptotic now follows from Vitali's theorem. ∎

Proposition 3. *Let* $D_m := 3m(3m-2)\cdots(m+2)$.

(a) $p_m(x) \sim D_m e^{(1-1/\sqrt{3})x}$.
(b) $q_m(x) \sim (-1)^{m+1} D_m [e^{x/\sqrt{3}} + (-1)^m e^{-x/\sqrt{3}}]$.
(c) $r_m(x) \sim (-1)^m D_m e^{-(1-1/\sqrt{3})x}$.

(The asymptotic is uniform on compact subsets of $\mathbf{C} - \{\pm ki\sqrt{3}\pi/2 \mid k = \pm 1, \pm 2, \ldots\}$.)

Proof. In all cases the proof comes from inspection of the coefficients of the expansions. For example, from (2.9),

$$q_m(x) = -2^{m+1} m! \sum_{j=0}^{m} \frac{d_j x^j}{j!},$$

where for fixed j, as $m \to \infty$,

$$d_{j+2} = \frac{-(m/2 - j/2)}{(3m/2 - j/2)} d_j \sim (-\tfrac{1}{3}) d_j.$$

Since these terms dominate the sum, (b) follows. This is readily made rigorous. For part (a) (and (c)) the estimates are more complicated since we must estimate the c_j given by (2.5),

$$c_j := \sum_{k=0}^{m-j} \binom{2m-(k+j)}{m} \binom{m+k}{m} \frac{1}{2^k}. \tag{3.1}$$

With the aid of some elementary calculus we can show, for fixed j, that as $m \to \infty$

$$c_j \sim \left(1 - \frac{1}{\sqrt{3}}\right)^j c_0$$

and (a) follows. We use the calculus to show that as $j \to 0$ the maximum terms in (3.1) occur for k near $(2-\sqrt{3})m$. The details are reasonably straightforward. Once again the uniformity follows from Vitali's theorem and direct calculation of uniform boundedness (using (2.4) and (2.9)). We must avoid the zeros of $e^{x/\sqrt{3}} \pm e^{-x/\sqrt{3}}$, namely the set $\{\pm ki\sqrt{3}\pi/2\}$. ∎

Proposition 4. *Let*

$$\alpha_m(x) = \frac{-q_m(x) + \sqrt{q_m^2(x) - 4p_m(x)r_m(x)}}{2p_m(x)},$$

where the square root is the principal branch. Then, for odd m and for $|x| < \sqrt{3}\pi/2$

$$e^{-x} - \alpha_m(x) \sim \frac{2^{m+1} m!\, x^{3m+2} e^{-x}}{(3m+2)!\, D_m [e^{x/\sqrt{3}} + e^{-x/\sqrt{3}}]}.$$

(The estimate is uniform on any compact subset of $\{|x| < \sqrt{3}\pi/2\}$.)

Proof. Note that $y=\alpha_m(x)$ is a root of the quadratic equation

$$E_m(y):=p_m(x)y^2+q_m(x)y+r_m(x). \tag{3.2}$$

Now observe that

$$\frac{E_m(y)-E_m(e^{-x})}{y-e^{x}}=p_m(x)(y+e^{-x})+q_m(x) \tag{3.3}$$

and

$$E_m(y)-E_m(e^{-x})=-\mathscr{E}_m(x).$$

We verify directly from Proposition 3 that $y\sim e^{-x}$ (as $m\to\infty$) and so (3.3) yields

$$e^{-x}-y\sim\frac{\mathscr{E}_m(x)}{p_m(x)2e^{-x}+q_m(x)} \tag{3.4}$$

and the result follows from the estimates of Propositions 2 and 3. The uniformity follows from the uniformity estimates in Propositions 2 and 3 applied to (3.4). We must avoid the zeros of $e^{x/\sqrt{3}}+e^{-x/\sqrt{3}}$ since α branches at these points. (See the comments following this proof.) ∎

Note, for odd m

$$q_m^2(x)-4p_m(x)r_m(x)\sim D_m^2[e^{x/\sqrt{3}}+e^{-x/\sqrt{3}}]^2,$$

while, for even m

$$q_m^2(x)-4p_m(x)r_m(x)\sim D_m^2[e^{x/\sqrt{3}}-e^{-x/\sqrt{3}}]^2.$$

Thus, while the principal branch of the square root works for the definition of $\alpha_m(x)$ for odd m, we must contend with a branch point near $x=0$ for even m. Hence, an asymptotic like Proposition 3, for even m is more complicated. In fact, if

$$\beta_m(x):=\frac{-q_m(x)-\sqrt{q_m^2(x)-4p_m(x)r_m(x)}}{2p_m(x)},$$

then, for even m,

$$\alpha_m(t)\sim e^{-t},\qquad t\in[-1,0)$$

and

$$\beta_m(t)\sim e^{-t},\qquad t\in(0,1].$$

4. An Exact Minimization

We wish to uniformly minimize over $D:=\{z\in\mathbf{C}:|z|\le 1\}$,

$$w_m(z):=s_m(z)\,e^{-2z}+t_m(z)\,e^{-z}+u_m(z), \tag{4.1}$$

where $s_m, t_m, u_m\in\pi_m$ and s_m has highest coefficient 1.

Lemma 1. *For* $|z|=1$

$$\left|\mathscr{E}_m\left(z+\frac{1}{3m+2}\right)\right| \sim \frac{2^{m+1}m!}{(3m+2)!}.$$

Proof. This follows from Proposition 2 and the observation that

$$\left(z+\frac{1}{3m+2}\right)^{3m+2} \sim z^{3m+2}\, e^{1/z}$$ ■

Let

(4.2) $$p_m^*(z), \qquad q_m^*(z) \quad \text{and} \quad r_m^*(z),$$

respectively denote

$$p_m\left(z+\frac{1}{3m+2}\right), \qquad q_m\left(z+\frac{1}{3m+2}\right) \quad \text{and} \quad r_m\left(z+\frac{1}{3m+2}\right).$$

Let $\|\cdot\|_D$ denote the supremum norm on D.

Proposition 5.

(a) $$\|p_m^*(z)\, e^{-2z}+q_m^*(z)\, e^{-z}+r_m^*(z)\|_D \sim \frac{2^{m+1}m!}{(3m+2)!}.$$

(b) *Let*

$$w_m^* := \min_{\substack{s,t,u\in\pi_m\\ s=z^m+\cdots}} \|s(z)\, e^{-2z}+t(z)\, e^{-z}+u(z)\|_D.$$

Then

$$w_m^* \sim \frac{2^{m+1}m!}{(3m+2)!}.$$

Proof. Part (a) is just a restatement of Lemma 1. Observe that p_m^* has lead coefficient 1.

To prove part (b) we use the fact that a nonzero expression of the form

(4.3) $$v_1(z)\, e^{-2z}+v_2(z)\, e^{-z}+v_3(z),$$

where v_1, v_2, and v_3 are polynomials, the sum of whose degrees is h, can have at most $h+2$ zeros in D. This is winding number argument and is proved in Pólya and Szegö [8, p. 144, Problem 206.2 with $\beta := 1$, $\alpha := -1$, $\lambda_1 := 0$, $\lambda_l := 2$ and $n := h+3$]. Thus,

$$w_m^* \geq \min_{|z|=1} |p_m^*(z)\, e^{-2z}+q_m^*(z)\, e^{-z}+r_m^*(z)|.$$

If this were not the case we could find $s, t, u \in \pi_m$ with s having lead coefficient 1 so that, for $|z|=1$

$$|s\, e^{-2z}+t\, e^{-z}+u| < |p_m^*\, e^{-2z}+q_m^*\, e^{-z}+r_m^*|.$$

By Rouché's theorem this would imply that

(4.4) $$(s-p_m^*)\, e^{-2z}+(t-q_m^*)\, e^{-z}+(u-r_m^*)$$

has at least $3m+2$ zeros in D. However, since $s-p_m^*$ has degree at most $m-1$ the sum of the degrees of the coefficients in (4.4) is at most $3m-1$ and we have contradicted the above result from Pólya and Szegö. ■

We note that

$$\|p_m e^{-2z}+q_m e^{-z}+r_m\|_D \sim e\frac{2^{m+1}m!}{(3m+2)!}$$

and so, up to a small constant, the quadratic Hermite-Padé approximant is optimal in the sense of Proposition 5. The trick of shifting the center of the approximation to make the error curve have asymptotically constant norm on D is due to Braess [2] who used it to get the right constant in the Meinardus conjecture.

5. Differential Equations

The coefficient polynomials are linked by the following third-order differential equations.

Proposition 6.

(a) $2mp_{m-1}=p_m'''-3p_m''+2p_m'$.
(b) $2mq_{m-1}=q_m'-q_m'''$.
(c) $2mr_{m-1}=r_m'''+3r_m''+2r_m'$.

Proof. We suppress the variable x in the coefficient polynomials and start with the relation

$$p_m e^{-2x}+q_m e^{-x}+r_m=O(x^{3m+2}). \tag{5.1}$$

Then, on differentiating

$$(p_m'-2p_m)\, e^{-2x}+(q_m'-q_m)\, e^{-x}+r_m'=O(x^{3m+1}). \tag{5.2}$$

Adding twice (5.1) to (5.2) gives

$$p_m' e^{-2x}+(q_m'+q_m)\, e^{-x}+r_m'+2r_m=O(x^{3m+1}) \tag{5.3}$$

and differentiating again gives

$$(p_m''-2p_m')\, e^{-2x}+(q_m''-q_m)\, e^{-x}+r_m''+2r_m'=O(x^{3m}). \tag{5.4}$$

Now adding (5.3) to (5.4) and differentiating yields

$$(p_m'''-3p_m''+2p_m')\, e^{-2x}+(q_m'''-q_m')\, e^{-x}+r_m'''+3r_m''+2r_m'=O(x^{3m-1}). \tag{5.5}$$

Since the coefficients in (5.5) are all of degree $(m-1)$ or less we see that up to a constant multiple of $2m$ (5.5) must equal $p_{m-1}\, e^{-2x}+q_{m-1}\, e^{-x}+r_{m-1}$. (Here we have appealed to uniqueness, as in the proof of Proposition 1.)

6. A Four-Term Recursion

The coefficients sequences $\{p_m\}$, $\{q_m\}$, and $\{r_m\}$ satisfy the following relations.

Proposition 7.

(a) $$\operatorname{Det}\begin{pmatrix} p_m(x) & q_m(x) & r_m(x) \\ p_{m+1}(x) & q_{m+1}(x) & r_{m+1}(x) \\ p_{m+2}(x) & q_{m+2}(x) & r_{m+2}(x) \end{pmatrix} = (-1)^{m+1} 9 \cdot 2^{m+2}(4+3m)x^{3m+2}.$$

(b) $\{p_m\}$, $\{q_m\}$, $\{r_m\}$, and $\{\mathscr{E}_m\}$ all satisfy the four-term recursion.

$$T_{m+3} = \left[\frac{3}{3m+4}\right]\{(-2m-\tfrac{14}{3})x^3 T_m$$
$$+[(3m+5)x^2+(3m+4)(3m+5)(3m+7)]T_{m+1}+xT_{m+2}\}.$$

Proof. For part (a) observe that

$$\begin{pmatrix} p_m & q_m & r_m \\ p_{m+1} & q_{m+1} & r_{m+1} \\ p_{m+2} & q_{m+1} & r_{m+2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ e^{-x} & 1 & 0 \\ e^{-2x} & 0 & 1 \end{pmatrix} = \begin{pmatrix} O(x^{3m+2}) & q_m & r_m \\ O(x^{3m+5}) & q_{m+1} & r_{m+1} \\ O(x^{3m+8}) & q_{m+2} & r_{m+2} \end{pmatrix}.$$

Thus the determinant in (a), which is a polynomial of degree at most $3m+3$, has a zero of order $3m+2$. It follows that we need only consider the two highest coefficients of the entries. From Section 2,

$$\operatorname{Det}\begin{pmatrix} p_m & q_m & r_m \\ p_{m+1} & q_{m+1} & r_{m+1} \\ p_{m+2} & q_{m+2} & r_{m+2} \end{pmatrix}$$
$$=(-2)^{m+1}\operatorname{Det}\begin{pmatrix} x^m+\frac{3}{2}m(m+1)x^{m-1} & x^m & x^m-\frac{3}{2}m(m+1)x^{m-1} \\ x^{m+1}+\frac{3}{2}(m+1)(m+2)x^m & -2x^{m+1} & x^{m+1}-\frac{3}{2}(m+1)(m+2)x^m \\ x^{m+2}+\frac{3}{2}(m+2)(m+3)^{m+1} & 4x^{m+2} & x^{m+2}-\frac{3}{2}(m+2)(m+3)x^{m+} \end{pmatrix}$$
$$=(-1)^{m+1}9\cdot 2^{m+2}(4+3m)x^{3m+2},$$

where the final calculation is performed by expanding the determinant around the middle column.

In a similar fashion we calculate

$$\operatorname{Det}\begin{pmatrix} p_m & q_m & r_m \\ p_{m+1} & q_{m+1} & r_{m+1} \\ p_{m+3} & q_{m+3} & r_{m+3} \end{pmatrix} = (-1)^{m+1}27\cdot 2^{m+2}x^{3m+3}, \tag{6.1}$$

and

$$\operatorname{Det}\begin{pmatrix} p_m & q_m & r_m \\ p_{m+2} & q_{m+2} & r_{m+2} \\ p_{m+3} & q_{m+3} & r_{m+3} \end{pmatrix} = (-1)^m 27\cdot 2^{m+2}(3m+5)[x^2+(3m+4)(3m+7)]x^{3m+2}. \tag{6.2}$$

Part (b) is now straightforward. Observe from part (a) that

$$\operatorname{Det}\begin{pmatrix} p_m & q_m & r_m & -(4+3m) \\ p_{m+1} & q_{m+1} & r_{m+1} & 0 \\ p_{m+2} & q_{m+2} & r_{m+2} & 0 \\ p_{m+3} & q_{m+3} & r_{m+3} & 2(4+3(m+1))x^3 \end{pmatrix} = 0,$$

which allows us to solve for the coefficient of T_m in the recursion of part (b). The coefficients of T_{m+1} and T_{m+2} are derived in a similar fashion using (6.1) and (6.2). ■

7. Some Numbers

The following tables give the coefficients of first six p_m, q_m, and r_m.

p_m	x^0	x^1	x^2	x^3	x^4	x^5	x^6
$m=1$	3	1					
2	24	9	1				
3	315	123	18	1			
4	5760	2295	375	30	1		
5	135 135	54 495	9450	885	45	1	
6	3 870 720	1 573 425	283 185	28 980	1785	63	1

Recall that $r_m(x) = (-1)^m p_m(-x)$.

q_m	x^0	x^1	x^2	x^3	x^4	x^5	x^6
$m=1$	0	4					
2	−48	0	−8				
3	0	384	0	16			
4	−11 520	0	−1920	0	−32		
5	0	161 280	0	7680	0	64	
6	−7 741 440	0	−1 290 240	0	−26 880	0	−128

Note, for $m=5$ and $x=1$ we have

$$p := p_5(1) = 200011,$$

$$q := q_5(1) = 169024,$$

$$r := r_5(1) = -89249,$$

and

$$\frac{1}{e} - \alpha_5(1) = \frac{1}{e} - \frac{-q+\sqrt{q^2-4pr}}{2p}$$

$$= (0.25\ldots) \times 10^{-16}.$$

In fact, α_5 is giving at least fifteen-digit accuracy over the whole unit disk.

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P. B. Borwein
Department of Mathematics
Statistics and Computing Science
Dalhousie University
Halifax
Nova Scotia
Canada B3H 4H8