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# SCIENTIFIC NOTES

## AN EXPLICIT CUBIC ITERATION FOR $\pi$

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**Abstract.**

Using the theory of the cubic modular equation we have discovered a remarkably simple class of cubically convergent algebraic iterations for $\pi$.

In the course of a study of cubic modular equations the following two remarkably simple cubically convergent iterations for $\pi$ were uncovered. Details of the derivation will appear in [1]. A related but less elegant iteration was discussed in [2]. While derivation of our new algorithm is rather elaborate we begin by describing its genesis. The *complete elliptic integrals* $E$ and $K$ are defined by

$$K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}}; \quad E(k) := \int_0^{\pi/2} \sqrt{(1-k^2\sin^2\theta)}d\theta$$

for $0 \leqq k < 1$. We write $k' := \sqrt{(1-k^2)}$ and $K'(k) := K(k')$. In these terms the *singular value function* is defined by solution of

$$\frac{K'}{K}(k(N)) = \sqrt{N}$$

for positive $N$. This uniquely defines $k$ on $[0, \infty)$ as a decreasing function with $k(0) = \infty$, $k(1) = 1/\sqrt{2}$, $k(\infty) = 0$. It is known that $k$ is algebraic when $N$ is rational [5]. Various related invariants are tabulated below. Moreover, for some $N$ one or more of these invariants become very simple. In [1] a corresponding function $\alpha$ (a *singular value of the second kind*) was studied. It is defined by

$$\alpha(N) = \frac{E'}{K} - \frac{\pi}{4K^2} \qquad k := k(N) \tag{1}$$

and also is algebraic at rational values. It is antitone on $[1, \infty)$ with $\alpha(1) = 1/2$ and $\alpha(\infty) = 1/\pi$. Moreover, it satisfies many recursions which allow one to com-

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pute $\alpha$ at many values both numerically and explicitly. For example,

$$\alpha(4N) = \frac{4\alpha(N) - 2\sqrt{N}\,k^2(N)}{[1+k'(N)]^2}, \quad \text{and} \tag{2}$$

$$k(4N) = \frac{1-k'(N)}{1+k'(N)}. \tag{3}$$

This leads to quadratic iterations for $1/\pi$, [1], [2]. Indeed iteration of (2) and (3) is precisely the Gauss-Brent-Salamin quadratic algorithm for $\pi$ [4].

Similarly one can deduce that

$$\alpha(9N) = m^2(N)\alpha(N) - \tfrac{1}{2}\sqrt{N}(m^2(N) + 2m(N) - 3), \tag{4}$$

where

$$m(N) := \frac{K(k(N))}{K(k(9N))} = \left\{1 + 4\left(\frac{k^3(9N)k'^3(9N)}{k(N)k'(N)}\right)^{1/4}\right\}^{1/2} \tag{5}$$

is the cubic multiplier (usually written somewhat differently). Two additional pieces of notation are helpful. We write

$$G_N := [2k(N)k'(N)]^{-1/12}; \qquad g_N := [2k(N)k'^{-2}(N)]^{-1/12}$$

which are Ramanujan's invariants [3].

In Weber's terms $g_N = 2^{-1/4}f_1(\sqrt{(-N)})$ and $G_N = 2^{-1/4}f(\sqrt{(-N)})$, many values of which are listed in Table VI of [5]. Then

$$m(N) = [1 + 2\sqrt{2}\,G_N^3/G_{9N}^9]^{1/2}. \tag{6}$$

We also introduce

$$s(N) := [1 + 2\sqrt{2}\,G_{9N}^3/G_N^9]^{1/2}, \tag{7}$$

$$r(N) := [1 + 2\sqrt{2}\,g_{9N}^3/g_N^9]^{1/2}. \tag{8}$$

These satisfy

$$m(N)s(N) = 3 \qquad \text{and} \tag{9}$$

$$[s(N) - 1][r(N) - 1] = 4. \tag{10}$$

It is a consequence of an extraordinary cubic theta function identity due to

Ramanujan (in chapter 18 of his second notebook) that

$$s(N)s(9N) = \{[s^2(N)-1]^{1/3}+1\}^2. \tag{11}$$

This leads to a surprisingly simple update for $m(N)$ and so to cubic iterations for $\pi$.

Our algorithm, given $N$, computes $m_n := m(N3^{2n})$ and $\alpha_n := \alpha(N3^{2n})$ for $n$ in $\mathbb{N}$. The method is initialized by $\alpha(N)$ and $s(N)$ which may be computed from (6) and (9), from (7) or from (8) and (10).

**Algorithms for $\pi$.**

Fix $N > 0$. Let $s_0 := s(N)$ and $\alpha_0 := \alpha(N)$ (various initial values are given in Table 1). For $n$ in $\mathbb{N}$ compute

$$m_n := 3/s_n, \tag{12}$$

$$s_{n+1} := [(s_n^2-1)^{1/3}+1]^2/s_n, \tag{13}$$

$$\alpha_{n+1} := m_n^2\alpha_n - \sqrt{N}\,3^n(m_n^2+2m_n-3)/2. \tag{14}$$

Then, for $7N3^{2n} \geqq 1$,

$$0 < \alpha_n - \pi^{-1} \leqq 16\sqrt{N}\,3^n \exp(-\sqrt{N}\,3^n\pi). \tag{15}$$ ■

With $N := 1$, the algorithm gives 0, 2, 10, 34, 107 and 327 correct digits of $\pi$ (for $n = 0, 1, 2, 3, 4, 5$). Similarly, with $N := 7$, the method gives 1, 8, 30, 93, 288 and 873 digits. This agrees closely with the upper bound in (15) which is very near to the actual error. It requires only twenty steps of the algorithm to compute over a billion digits of $\pi$ and this, in turn, requires approximately 100 full precision multiplications and only 20 divisions and 20 cube root extractions.

The quantity $g_N$ (and hence $r(n)$) is usually simpler than $G_n$ for $N$ even, and $G_N$ (and hence $s(n)$) is simpler for $N$ odd. Thus we give the following information.

*Table* 1

| $N$ | $s(N)$ | $\alpha(N)$ |
|---|---|---|
| 1/3 | $\sqrt{3}$ | $(\sqrt{3}+1)/6$ |
| 1 | $\sqrt{(3+2\sqrt{3})}$ | 1/2 |
| 3 | $(1+2^{1/3})^2/\sqrt{3}$ | $(\sqrt{3}-1)/2$ |
| 5 | $\sqrt{(1+2\sqrt{3}+2\sqrt{5})}$ | $(\sqrt{5}-\sqrt{(2\sqrt{5}-2)})/2$ |
| 7 | $[6+\sqrt{21}+\sqrt{(27+6\sqrt{21})/2}]^{1/2}$ | $(\sqrt{7}-2)/2$ |
| 9 | $(\sqrt{3}-1)[(2+2\sqrt{3})^{1/3}+1]^2/(\sqrt{2}\,3^{1/4})$ | $[3-3^{3/4}(\sqrt{6}-\sqrt{2})]/2$ |

| $N$ | $r(N)$ | $\alpha(N)$ |
|---|---|---|
| 2/3 | $\sqrt{6}+\sqrt{3}$ | $[5\sqrt{6}-6\sqrt{3}-8\sqrt{2}+11]/3$ |
| 2 | $\sqrt{2}+\sqrt{3}$ | $\sqrt{2}-1$ |
| 6 | $(\sqrt{2}-1)[\sqrt{2}(\sqrt{2}+1)^{2/3}+1]^2/\sqrt{3}$ | $5\sqrt{6}+6\sqrt{3}-8\sqrt{2}-11$ |
| 18 | $(\sqrt{3}-\sqrt{2})[(4+2\sqrt{6})^{1/3}+1]^2$ | $3(\sqrt{3}+\sqrt{2})^4(\sqrt{6}-1)^2(7\sqrt{2}-5-2\sqrt{6})$ |

Many more initial values of $r(N)$, $m(N)$ or $s(N)$ are easily computed. Moreover, since (10) holds we compute $r(\frac{1}{3}) = 3+2\sqrt{3}$, $s(2) = \sqrt{6}+\sqrt{2}-1$, and so on. From (4) with $N := 3$ we discover that $\alpha(27) = 3((\sqrt{3}+1)/2-2^{1/3})$.

Further, one easily verifies from (7), (8) and (10) that

$$r(4N) = \sqrt{(r(N)+s(N)+3)}. \tag{16}$$

Hence $r(\frac{4}{3}) = \sqrt{(6+3\sqrt{3})}$. Also $\alpha(\frac{4}{3}) = 2[54\sqrt{6}+77\sqrt{3}-94\sqrt{2}-132]/3$.

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