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RATIONAL FUNCTIONS WITH POSITIVE COEFFICIENTS, POLYNOMIALS AND UNIFORM APPROXIMATIONS

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Upper bounds are established for the uniform approximation of continuous functions on [1, 0] by rational functions with positive coefficients. These bounds are obtained by rewriting polynomials with no positive roots as rational functions with positive coefficients.

1. Introduction. The uniform closure in $C[1, 0]$ of the set of polynomials with positive coefficients includes only those functions analytic in the unit disc whose power series expansions have nonnegative coefficients. The uniform closure of the set of rational functions with positive coefficients consists of all continuous functions which are never negative on [0, 1]. This is a consequence of the following interesting factorization theorem.

THEOREM 1. (*E. Meissner* [3].) *Suppose that p is a polynomial with real coefficients and that $p(x) > 0$ for $x > 0$. Then there exists a rational function $r(x)$ with nonnegative coefficients so that $p(x) = r(x)$.*

We will provide an accurate bound for the degree of the above r in terms of the degree of p and some knowledge of the location of the roots of p. We will also derive some estimates concerning how efficiently polynomials can be approximated on [0, 1] by rational functions with positive coefficients. We will exploit these results to prove a number of approximation theorems. For instance: if f is analytic in some neighborhood of [0, 1] and positive on [0, 1], then there exists a sequence of rational functions $\{r_n\}$ where each r_n is of degree n and has nonnegative coefficients so that $\|f-r_n\|_{[0,1]}= 0(\alpha^{-\sqrt{n}})$ for some $\alpha > 1$.

We employ the following notation. Let Π_n denote the polynomials with real coefficients of degree at most n. Let Π_n^+ be the sub class of Π_n whose elements have nonnegative coefficients. Let R_n^{++} denote those rational functions p_n/q_n where $p_n, q_n \in \Pi_n^+$. For $f \in C[a, b]$ define

$$\Pi_n(f: [a, b]) = \inf_{p \in \Pi_n} \|f - p\|_{[a,b]}$$

$$\Pi_n^+(f: [a, b]) = \inf_{p \in \Pi^+} \|f - p\|_{[a,b]}$$

$$R_n^{++}(f: [a, b]) = \inf_{r \in R^{++}} \|f - r\|_{[a,b]}$$

where $\| \quad \|_{[a,b]}$ is the supremum norm on $[a, b]$. We note that all the above infimums are attained.

2. Expressing polynomials as rational functions with non-negative coefficients. The first two results of this section are concerned with expressing quadratic polynomials as rational functions in R_m^{++} where m is as small as possible. The final theorem is an extension of these results to general polynomials.

THEOREM 2. *Suppose that $\alpha, \beta > 0$ and suppose that $x^2 - \alpha x + \beta$ has no positive roots. Then*

(a) *for each $\varepsilon > 0$ there exists a constant A_ε so that*

$$x^2 - \alpha x + \beta = r_m(x)$$

where

$$r_m \in R_m^{++} \text{ and } m \leqq A_\varepsilon \left[\frac{1}{4 - \alpha^2/\beta}\right]^{1/2+\varepsilon} .$$

(b) *for $\varepsilon = 1/14$,*

$$x^2 - \alpha x + \beta = r_m(x)$$

where

$$r_m \in R_m^{++} \text{ and } m \leqq 20\left[\frac{1}{4 - \alpha^2/\beta}\right]^{1/2+1/14} .$$

Proof. The quadratic $x^2 - \alpha x + \beta$ has no positive root if and only if $\alpha^2 < 4\beta$. We set $c = \alpha^2/\beta$ and note that $0 < c < 4$. Consider

(1)
$$\begin{aligned}(x^2 - \alpha x + \beta)(x^2 + \alpha x + \beta) &= x^4 + (2\beta - \alpha^2)x^2 + \beta^2 \\ &= x^4 + \beta(2 - c)x^2 + \beta^2 .\end{aligned}$$

If $c \leqq 2$ we have the desired factorization. In general we proceed as follows:

Define C_n inductively by

(2)
$$C_0 = c^{1/2} \text{ and } C_{n+1} = 2 - C_n^2 .$$

Let

$$p_n(x) = x^{2^{n+1}} + \beta^{2^{n-1}} C_n x^{2^n} + \beta^{2^n}$$

and let

$$\overline{p_n}(x) = x^{2^{n+1}} - \beta^{2^{n-1}} C_n x^{2^n} + \beta^{2^n} .$$

Note that, by (2)

$$
\begin{aligned}
p_n(x)\overline{p_n}(x) &= x^{2^{n+2}} - \beta^{2^n}C_n^2 x^{2^{n+1}} + 2\beta^{2^n}x^{2^{n+1}} + \beta^{2^{n+1}} \\
&= x^{2^{n+2}} + \beta^{2^n}C_{n+1}x^{2^{n+1}} + \beta^{2^{n+1}} \qquad (3)\\
&= p_{n+1}(x) .
\end{aligned}
$$

Consider the smallest n (if it exists) so that C_n is nonnegative. Then, by (1) and (3)

$$(x^2 - \alpha x + \beta)(x^2 + \alpha x + \beta) = p_1$$

and

$$p_1 \cdot \overline{p_1} \cdot \overline{p_2} \cdots \overline{p_{n-1}} = p_n$$

where $\overline{p_1} \cdot \cdots \cdot \overline{p_{n-1}} \in \prod^+_{(2^{n+1}-4)}$ since each $C_k < 0$ for $k < n$ and where $p_n \in \prod^+_{2^{n+1}}$ since $C_n \geqq 0$. Thus, we have

$$(4) \qquad x^2 - \alpha x + \beta = \frac{p_n}{(x^2 + \alpha x + \beta)\overline{p_1} \cdot \overline{p_2} \cdot \cdots \cdot \overline{p_{n-1}}} \in R^{+}_{2^{n+1}} .$$

Since $0 < c^{1/2} < 2$ we deduce that $C_n \to 1$. We wish to find a small n as a function of C, so that

$$(5) \qquad C_n \geqq 0 .$$

Suppose that

$$(6) \qquad C_1, \cdots, C_n < 0 .$$

Then

$$C_n = 2 - (C_{n-1})^2 < 0$$

implies

$$(C_{n-1})^2 > 2 \text{ and } -C_{n-1} > 2^{1/2}$$

implies

$$(C_{n-2})^2 - 2 > 2^{1/2} \text{ and } -C_{n-2} > (2 + 2^{1/2})^{1/2}$$

and by iteration

$$(7) \qquad c > 2 + (2 + \cdots (2 + 2^{1/2})^{1/2})^{1/2} \cdots)^{1/2} = \delta_n$$

where (equivalently) $\delta_1 = 2$ and $\delta_n = 2 + \delta_{n-1}^{1/2}$.

We are reduced to finding an n so that $\delta_n > c = \alpha^2/\beta$ since, for such an n (6) is contradicted and hence, (5) is satisfied.

Consider

$$4-\delta_n = 2-\delta_{n-1}^{1/2} = \frac{4-\delta_{n-1}}{2+\delta_{n-1}^{1/2}} = \frac{4-\delta_{n-2}}{(2+\delta_{n-1}^{1/2})(2+\delta_{n-2}^{1/2})}$$

$$\vdots \tag{8}$$

$$= \frac{2}{(2+\delta_{n-1}^{1/2})(2+\delta_{n-2}^{1/2})\cdots(2+\delta_1^{1/2})} \leqq \frac{2}{(2+2^{1/2})^{n-1}} \leqq \frac{7}{(2+2^{1/2})^n}\,.$$

It is now sufficient to pick n so that

$$\frac{7}{(2+2^{1/2})^n} \leqq 4-\frac{\alpha^2}{\beta}\,. \tag{9}$$

A suitable choice is

$$n = 1 + \text{int. part}\left[\frac{\log_2\left[\dfrac{7}{4-\dfrac{\alpha^2}{\beta}}\right]}{\log_2(2+2^{1/2})}\right] \leqq 1 + \frac{4}{7}\log_2\left[\frac{7}{4-\dfrac{\alpha^2}{\beta}}\right].$$

We deduce from (4) that

$$x^2-\alpha x+\beta \in R_{2^{n+1}}^{++}$$

where

$$2^{n+1} \leqq 4\left[\frac{7}{4-\dfrac{\alpha^2}{\beta}}\right]^{4/7} \leqq 20\left[\frac{1}{4-\dfrac{\alpha^2}{\beta}}\right]^{1/2+1/14}.$$

This completes (b). Part (a) is proved analogously with the observation that in (8), for $k<n$,

$$|4-\delta_n| \leqq \frac{7}{(2+\delta_k^{1/2})^{n-k}}\,.$$

Since $\delta_n \to 4$, we can replace (9) by

$$\frac{F_\varepsilon}{(4-\varepsilon)^n} \leqq 4-\frac{\alpha^2}{\beta}$$

and the result follows as above.

The bound in Theorem 2 is "essentially" correct.

THEOREM 3. *Let* $\alpha_k = 2$ *and* $\beta_k = 1+1/k^2$. *If*

$$x^2-\alpha_k x+\beta_k = r_m \in R_m^{++}$$

then

$$m \geqq \sqrt{2}\left[\frac{1}{4-\dfrac{\alpha_k^2}{\beta_k}}\right].$$

Proof. We first show that if $p_n \in \prod_n^+$ then p_n has no roots in $T_n = \{z: |\arg(z)| < \pi/n\}$. Suppose $p_n(z) = \sum_{h=0}^n a_n z^n$ where $a_n \geqq 0$. Let $\zeta \in \{0 < \arg(z) < \pi/n\}$. Then $a_h(\zeta)^h \in \{0 < \arg(z) < h\pi/n\}$ and hence, $p_n(\zeta) \in \{\mathrm{im}(z) > 0\}$. Thus, p_n has no roots in T_n.

The quadratic $x^2 - \alpha_k x + \beta_k$ has a root at $1 + i/k \in T_k$ and we deduce that if $x^2 - \alpha_k x + \beta_k = r_m \in R_m^{++}$ then $m > k$. We finish the result by observing that

$$\sqrt{2}\left[\frac{1}{4-\dfrac{\alpha_k^2}{\beta_k}}\right]^{1/2} = \frac{k(1+1/k^2)^{1/2}}{\sqrt{2}} \leqq k.$$

THEOREM 4. *Suppose $p_n \in \prod_n$ has no roots in the region $\Omega(1/h) = \{z: |\arg(z)| < 1/h\}$ and suppose that $p_n(x) > 0$ for $x > 0$. Then,*

(a) *for each $\varepsilon > 0$ there exists a constant B_ε, depending only on ε, so that*

$$p_n = r_m \in R_m^{++} \text{ where } m \leqq B_\varepsilon h^{(1+\varepsilon)} n.$$

(b) *for $\varepsilon = 1/7$,*

$$p_n = r_m \in R_m^{++} \text{ where } m \leqq 10h^{8/7}\cdot n.$$

Proof. Let $x^2 - \alpha x + \gamma$ be a quadratic factor of p_n. We assume $\alpha, \gamma < 0$ since otherwise $x^2 - \alpha x + \gamma$ has either nonnegative coefficients or a nonnegative root. We proceed to replace, using Theorem 2, each such factor by an element of R_k^{++}.

Set $\gamma = 1/4(1/h^2 + 1)\alpha^2 + \delta$ and set $\beta = 1/4(1/h^2 + 1)\alpha^2$. Since $x^2 - \alpha x + \gamma$ has no roots in $\Omega(1/h)$ was see that $|\alpha^2 - 4\gamma|^{1/2} \geqq \alpha/h$ and $4\gamma \geqq (1/h^2 + 1)\alpha^2$ from which we deduce that $\delta \geqq 0$. Consider $x^2 - \alpha x + \beta$. By Theorem 2(b) $x^2 - \alpha x + \beta - r_k \in R_k^{++}$ where

$$k \leqq 20\left[\frac{1}{4-\dfrac{\alpha^2}{\beta}}\right]^{1/2+1/14} = 20\left[\frac{h^2(1/h^2+1)}{4}\right]^{4/7} \leqq 20h^{8/7}.$$

We now replace $x^2 - \alpha x + \gamma$ by $r_k + \delta$. Since there are a maximum of $n/2$ such quadratic terms to replace, we have

$$p_n = r_m \in R_m^{++} \text{ where } m \leqq 20h^{8/7}(n/2) = 10h^{8/7}n.$$

This completes part (b). Part (a) is proved analogously using

Theorem 2(a) instead of 2(b).

3. **Approximating polynomials.** We estimate how efficiently polynomials in the class P. P. C. can be approximated by rationals with positive coefficients. A polynomial is in the class P.P.C. (polynomials with positive coefficients in x and $(1-x)$, see [1]) if it can be written $\Sigma a_{ki}x^k(1-x)^i$ where $a_{ki} \geqq 0$. We use this estimate and Theorem 4 to approximate polynomials with no roots in a region containing $[0, 1]$. We adopt the notation R.P.C. (rationals with positive coefficients in x and $(1-x)$) for those rational functions which are a quotient of two elements of the class P.P.C.

LEMMA 1. *Suppose* $p_n = \sum_{k+i\leq n} a_{ki}x^k(1-x)^i$ *is a* P.P.C. *of degree* n. *Then there exists* $r(x) \in R^{++}_{nm}$ *so that for* $x \in [0, 1)$,

$$|r(x) - p_n(x)| \leqq \frac{nx^m p_n(x)}{(1-x)^n} .$$

Proof. We observe that for $x \in [0, 1)$,

$$\left|(1-x) - \frac{1}{1+x\cdots+x^{m-1}}\right| = \left|(1-x) - \frac{1-x}{1-x^m}\right| = \left|\frac{x^m(1-x)}{1-x^m}\right| \leqq x^m .$$

Since $a^i - b^i = (a-b)(a^{i-1} + a^{i-2}b + \cdots + ab^{i-2} + b^{i-1})$,

$$(1)\qquad \left|(1-x)^i - \frac{1}{(1+x+\cdots+x^{m-1})^i}\right| \leqq ix^m .$$

Let

$$s_m(x) = \frac{1}{1+x+\cdots+x^{m-1}}$$

and consider

$$r(x) = \sum_{k+i\leq n} a_{ki}x^k(s_m)^i .$$

Each term of the above sum can be brought to the common denominator $(1+x+\cdots+x^{m-1})^n$ and hence, $r(x) \in R^{++}_{mn}$. Also, by (1),

$$(2)\qquad \begin{aligned} |r(x) - p_n(x)| &\leqq \sum_{k+i\leq n} a_{ki}x^k i x^m \\ &\leqq nx^m \sum_{k+i\leq n} a_{ki}x^k . \end{aligned}$$

Since

$$\sum_{k+i\leq n} a_{ki}x^k = \sum_{k+i\leq n} a_{ki}x^k \frac{(1-x)^i}{(1-x)^i}$$
$$\leq \frac{1}{(1-x)^n} \sum_{k+i\leq n} a_{ki}x^k(1-x)^i = \frac{p(x)}{(1-x)^n},$$

we have

$$|r(x) - p_n(x)| \leq \frac{nx^m p(x)}{(1-x)^n}.$$

LEMMA 2. *Suppose p and q are both* P.P.C. *of degree n. Then there exists $r \in R_{2nm}^{++}$ so that for any $x \in [0, 1]$, satisfying $(1-x)^n > nx^m$,*

$$|p(x)/q(x) - r(x)| \leq \frac{2nx^m}{(1-x)^n - nx^m} \cdot \frac{p(x)}{q(x)}.$$

Proof. By Lemma 1 we can choose s and $t \in R_{nm}^{++}$ so that for $x \in [0, 1)$,

$$|p(x) - s(x)| \leq \frac{nx^m p(x)}{(1-x)^n}$$

and

$$|q(x) - t(x)| \leq \frac{nx^m q(x)}{(1-x)^n}.$$

Then, for $x \in [0, 1)$,

$$\left|\frac{p(x)}{q(x)} - \frac{s(x)}{t(x)}\right| = \left|\frac{p(x)}{q(x)} - \frac{s(x)}{q(x)} + \frac{s(x)}{q(x)} - \frac{s(x)}{t(x)}\right|$$
$$\leq \left|\frac{p(x) - s(x)}{q(x)}\right| + \left|\frac{s(x)(q(x) - t(x))}{t(x)q(x)}\right|$$
$$\leq \frac{nx^m}{(1-x)^n}\left|\frac{p(x)}{q(x)}\right| + \frac{nx^m}{(1-x)^n}\left|\frac{s(x)}{t(x)}\right|$$
$$\leq \frac{2nx^m}{(1-x)^n}\left|\frac{p(x)}{q(x)}\right| + \frac{nx^m}{(1-x)^n}\left|\frac{p(x)}{q(x)} - \frac{s(x)}{t(x)}\right|.$$

The result follows with $r = s/t$.

We now prove an analogue of Theorem 4 for rationals in the class R.P.C. Define a diamond-shaped region in the complex plane $G(\alpha)$ by

$$G(\alpha) = \{z: |\arg(z)| \leq \alpha\} \cap \{z: |\arg(1-z)| \leq \alpha\}.$$

LEMMA 3. *Let $\varepsilon > 0$. Suppose $p_n \in \prod_n$ has no roots in the region $G(1/h)$ and $p_n(x) > 0$ for $x \in [0, 1]$. Then $p_n(x) = r_m(x)$ where*

$r_m(x)$ is a R.P.C. of degree m, $m \leqq B_\varepsilon h^{(1+\varepsilon)} \cdot n$ and B_ε is the same constant as appears in Theorem 4.

Proof. We write $p_n(x) = s_k(x)t_{n-k}(x)$ where $s_k \in \prod_k$ has no roots in $\{z: |\arg(z)| \leqq 1/h\}$ and $t_{n-k} \in \prod_{n-k}$ has no roots in $\{z: |\arg(1-z)| \leqq 1/h\}$. By Theorem 4,

$$s_k(x) = U_j(x) \in R_j^{++} \text{ where } j \leqq B_\varepsilon h^{(1+\varepsilon)}k$$

and since $t_{m-k}(x) = q_{m-k}(1-x)$ where $q_{m-k}(1-x)$ has no roots in $\{z: \arg(z) \leqq 1/h\}$,

$$t_{n-k}(x) = V_i(1-x) \text{ where } V_i(x) \in R_i^{++} \text{ and } i \leqq B_\varepsilon h^{(1+\varepsilon)}(n-k)\ .$$

We set $r_m(x) = U_j(x)V_i(1-x)$ to complete the result.

LEMMA 4. *Let $\varepsilon > 0$. If $p_n \in \prod_n$ has no roots in the region $G(1/h)$ and $p_n(x) > 0$ for $x \in [0, 1]$, then there exists $r \in R^{++}_{2cmn}$ where $c = B_\varepsilon h^{(1+\varepsilon)}$ so that for $x \in [0, 1)$,*

$$|p(x) - r(x)| \leqq \frac{2nx^m\,|p(x)|}{(1-x)^{cn} - cnx^m}$$

provided $(1-x)^{cn} \geqq cnx^m$.

Proof. By Lemma 3, there exists s an R.P.C. of degree at most $cn = B_\varepsilon h^{(1+\varepsilon)}n$ so that $p = s$. By Lemma 2, there exists $r \in R^{++}_{2cnm}$ so that

$$|p(x) - r(x)| = |s(x) - r(x)| \leqq \frac{2nx^m\,|p(x)|}{(1-x)^{cn} - cnx^m}\ .$$

4. Approximating analytic functions. Let $\rho > 1$ and let E_ρ be the closed ellipse in the complex plane with foci at 0 and 1 and with semiaxes $1/4(\rho + \rho^{-1})$ and $1/4\,|\rho - \rho^{-1}|$. S.N. Bernstein proved:

THEOREM 5. ([2] p. 76.) *If f is analytic on E_ρ then there exist polynomials $p_n \in \prod_n$ so that*

$$\|f - p_n\|_{[0,1]} = 0(1/\rho^n)$$

and $p_n \to f$ uniformly on E_ρ.

We show that positive analytic functions can be approximated almost as efficiently by rational functions from the class R.P.C.

THEOREM 6. *If f is analytic and never zero on E_ρ and $f(x) > 0$ for $x \in [0, 1]$, then there exists a sequence of $r_n \in$ R.P.C., r_n of degree*

n, so that for each $\varepsilon > 0$,

$$\|f - r_n\|_{[0,1]} = O(1/\rho^{n/c_\varepsilon})$$

where $c_\varepsilon = B_\varepsilon[(\tan^{-1}(\rho + \rho^{-1})/2)]^{-(1+\varepsilon)}$ and B_ε is the same constant as in Theorem 4.

Proof. By Theorem 5 there exists a sequence of polynomials p_n so that

$$(1) \qquad \|f - p_n\|_{[0,1]} = O(1/\rho^n)$$

and each p_n has no zeros on E_ρ. We note that the region

$$G\left(\tan^{-1}\left(\frac{\rho + \rho^{-1}}{2}\right)\right) \subset E_\rho$$

and hence, by Lemma 3,

$$p_n = r_m \in \text{R.P.C. where } m \leq B_\varepsilon\left(\tan^{-1}\left(\frac{\rho + \rho^{-1}}{2}\right)\right)^{-(1+\varepsilon)} \cdot n\,.$$

The result is finished by substituting r_m into (1).

We have the following two theorems for approximating analytic functions by rational functions with positive coefficients.

THEOREM 7. *Let $0 < \rho < 1$. If f is analytic and never zero on E_ρ and $f(x) > 0$ for $x \in [0, 1]$, then there exists a constant γ so that*

$$R_n^{++}(f\colon [0, \rho]) = O(1/\gamma^{\sqrt{n}})$$

where γ depends only on ρ and δ.

Under stronger assumptions on f we recover exponential rates of convergence.

THEOREM 8. *Let $0 < \delta < 1$. Suppose that $f(z) = \Sigma a_k z^k$, a_k real, is analytic in a region containing $\{z\colon |z| \leq 1\}$ and suppose that*

$$f(x) > 0 \text{ for } x \in [0, 1]\,.$$

Then there exists $\eta > 1$ so that

$$R_n^{++}(f\colon [0, \delta]) = 0(1/\eta^n)$$

where η is independent of n.

Proof of Theorem 7. By Theorem 4, there exists a sequence of polynomials $p_n \in \prod_n$ so that

$$\|f - p_n\|_{[0,1]} = O(1/\rho^n) \tag{1}$$

where each p_n has no roots in E_ρ. Since $G(\tan^{-1}[(\rho + \rho^{-1})/2] \subset E_\rho$ we deduce, from Lemma 4 with $h = 1/\tan^{-1}[(\rho + \rho^{-1})/2]$ and $m = in$, that there exists $r_{k_n} \in R^{++}_{2icn^2}$ so that

$$|p_n - r_{k_n}| \leq \frac{2nx^{in} \|p\|_{[0,1]}}{(1-x)^{cn} - cnx^{in}} . \tag{2}$$

From (1) and (2) we have, for fixed i sufficiently large,

$$\|f - r_{k_n}\|_{[0,\delta]} = 0\left[\frac{1}{\rho^n} + \frac{n\delta^{in}}{(1-\delta)^{cn}}\right].$$

Since $k_n \leqq 2icn^2$, the result follows.

We need the next lemma in the proof of Theorem 8. Let D_α be the open disc of radius α centered at the origin.

LEMMA 5. *Let $\beta > \alpha$. Suppose $f(z) = \sum_{k=0}^\infty a_k z^k$ is analytic on D_β. Then, for $z \in D_\alpha$,*

$$f(z) = \frac{\sum_{k=0}^\infty (s_k(f:\alpha)/\alpha^k) z^k}{\sum_{k=0}^\infty z^k/\alpha^k}$$

where $s_k(f:\alpha)$ is the kth Taylor polynomial of f evaluated at α.

Proof. Let

$$g(z) = \frac{1}{1 - z/\alpha} = \sum_{k=0}^\infty z^k/\alpha^k .$$

Then,

$$\begin{aligned} f(z)g(z) &= \sum_{k=0}^\infty \left(\sum_{m=0}^k \frac{a_m}{\alpha^{k-m}}\right) z^k \\ &= \sum_{k=0}^\infty \frac{1}{\alpha^k}\left(\sum_{m=0}^k a_m \alpha^m\right) z^k \\ &= \sum_{k=0}^\infty \left[\frac{s_k(f:\alpha)}{\alpha^k}\right] z^k . \end{aligned}$$

Proof of Theorem 8. By assumption, f is analytic in some disc D_β where $\beta > 1$. Setting $\alpha = 1$ in Lemma 5 yields, for $z \in D_1$,

$$f(z) = \frac{\sum_{k=0}^\infty s_k(f:1) z^k}{\sum_{k=0}^\infty z^k} .$$

Since $f(x) > 0$ for $x \in [0, 1]$, there exist N so that for $n \geqq N$, $s_n(f\colon 1) > 0$ and so that $\sum_{k=0}^{N} s_k(f\colon 1)x^k$ is strictly positive on $[0, \infty]$. For $m \geqq N$ set

$$(1) \qquad r_m(z) = \frac{\sum_{k=0}^{N} s_k(f\colon 1)z^k}{\sum_{k=0}^{m} z^k} + \frac{\sum_{k=N+1}^{m} s_k(f\colon 1)z^k}{\sum_{k=0}^{m} z^k} .$$

The second term of the right side of (1) is an element of R_m^{++}. The first term has a fixed numerator which is positive on $[0, \infty]$ and by Theorem 4, there exists a constant A, independent of m, so that this term is an element of R_{Am}^{++}. Thus, there exists A so that for each $m \geqq N$

$$r_m \in R_{Am}^{++} .$$

We finish the proof by observing that

$$\begin{aligned} \|f - r_m\|_{[0,\delta]} &= \left\| \frac{\sum_{k=0}^{\infty} s_k(f\colon 1)z^k}{\sum_{k=0}^{\infty} z^k} - \frac{\sum_{k=0}^{m} s_k(f\colon 1)z^k}{\sum_{k=0}^{m} z^k} \right\|_{[0,\delta]} \\ &\leqq \left| \sum_{k=m+1}^{\infty} s_k(f\colon 1)\delta^k \right| + \|f\|_{[0,\delta]} \cdot \left| \sum_{k=m+1}^{\infty} \delta^k \right| \\ &= O(\delta^m) . \end{aligned}$$

5. **Approximating continuous functions.** We prove the following three theorems:

THEOREM 9. *If $f \in C[0, 1/2]$ and $f \geqq 0$ on $[0, 1/2]$ then*

$$R_{mn}^{++}(f\colon [0, 1/2]) \leqq \|f\|_{[0,1/2]} n 2^{n-m} + 2\omega(f, 1/\sqrt{n}) .$$

THEOREM 10. *If $f \in C[0, 1/2]$, $f \geqq 0$ on $[0, 1/2]$ then for each $\delta > 0$ there exists A_δ depending only on δ so that*

$$R_n^{++}(f\colon [0, 1/2]) \leqq A_\delta \omega(f, 1/n^{1/(4+\delta)}) .$$

THEOREM 11. *If $f \in C^k[0, 1/2]$, $f > 0$ on $[0, 1/2]$ and $f^{(k)} \in \operatorname{lip} \alpha$, $0 < \alpha < 1$, then for each $\delta > 0$ there exists A_δ so that*

$$R_n^{++}(f\colon [0, 1/2]) \leqq A_\delta \left[\frac{1}{n^{1/(4+\delta)}} \right]^{k+\alpha} ,$$

where A_δ is independent of n.

We have use the notation $\omega(f, \cdot)$ for the modulus of continuity of f.

We now collect the results we need to prove the above theorems. For $f \in C[0, 1]$ we define the nth Bernstein polynomial by

$$B_n(x) = B_n(f: x) = \sum_{k=0}^{n} f(k/n)\binom{n}{k} x^k(1-x)^{n-k} .$$

THEOREM 12. ([5] p. 15.) *If* $f \in C[0, 1]$ *then*

$$\|f(x) - B_n(f: x)\|_{[0,1]} \leqq 2\omega(f, 1/\sqrt{n}) .$$

THEOREM 13 (Lorentz [1].) *If* $f \in C^k[0, 1]$, $f > 0$ *on* $[0, 1]$ *and* $f^{(k)} \in \text{lip}\,\alpha$, $0 < \alpha \leqq 1$, *then there exists* p_n *a* P.P.C. *of degree* n *so that*

$$\|f(x) - p_n(x)\|_{[0,1]} \leqq C\left(\frac{1}{\sqrt{n}}\right)^{k+\alpha}$$

where C *is independent of* n.

Proof of Theorem 9. We extend f to a continuous function on $[0, 1]$ by setting, for $x \in [0, 1/2]$

$$f\left(x + \frac{1}{2}\right) = f\left(x - \frac{1}{2}\right) .$$

Then the modulus of continuity of f on $[0, 1]$ is the same as the modulus of continuity of f on $[0, 1/2]$.

Consider B_n the nth Bernstein polynomial for f. Since f is nonnegative on $[0, 1]$, B_n is a P.P.C. of degree n and $\|B_n\|_{[0,1/2]} \leqq \|f\|_{[0,1]}$. Thus, by Lemma 1 with $x \leqq 1/2$ and Theorem 12,

$$\begin{aligned} R_{mn}^{++}\left(f: \left[0, \frac{1}{2}\right]\right) &\leqq R_{mn}^{++}\left(B_n: \left[0, \frac{1}{2}\right]\right) + \|B_n - f\|_{[0,1/2]} \\ &\leqq \|f\|_{[0,1]} n 2^{n-m} + 2\omega(f, 1/\sqrt{n}) . \end{aligned}$$

Theorem 10 is a corollary to Theorem 9. We observe that it suffices to prove Theorem 10 under the assumption that f has a zero on $[0, 1/2]$ and that under this assumption $2\omega(f, 1/\sqrt{n}) \geqq (1/n)\|f\|_{[0,1]}$. The result is now completed by choosing $m = n^\delta$ for small δ, and applying Theorem 9.

Theorem 11 is proved analogously to Theorems 9 and 10. We first extend f to $[0, 1]$ in such a way that $f > 0$ on $[0, 1]$ and so that $f \in C^k[0, 1]$ with $f^{(k)} \in \text{lip}\,\alpha$. We now approximate this extended f by a P.P.C. as guaranteed by Theorem 13 and proceed as in the proofs of Theorem 9 and Theorem 10.

6. Remarks.

(1) D. J. Newman and A. R. Reddy [4] show that the best approximant to x^{n+1} from R_n^{++} on [0, 1] is a monomial αx^n and that

$$R_n^{++}(x^{n+1}: [0, 1]) = \Pi_n^+(x^{n+1}: [0, 1]) \sim c/n .$$

This should be compared to the fact ([2] p. 31) that

$$\Pi_n(x^{n+1}: [0, 1]) = \frac{1}{2^{2n+1}} .$$

(2) The restriction that f be strictly positive is essential in Theorems 7 and 11.

LEMMA 6. *Let* $0 < \alpha < \beta$. *If* $f(\alpha) = 0$

$$R_n^{++}(f: [\alpha, \beta]) \geqq \frac{f(\beta)}{(1 + \beta^n/\alpha^n)} .$$

Proof. Let p_n/q_n be a best approximant to f from R_n^{++} on $[\alpha, \beta]$. Then we can write

$$p_n(x) = \sum_{k=0}^{n} a_k x^k \text{ where } a_k \geqq 0 .$$

We have

$$p_n(\beta) = \sum_{k=0}^{n} a_k \beta^k = \sum_{k=0}^{n} \frac{\beta^k}{\alpha^k} a_k \alpha^k \leqq \frac{\beta^n}{\alpha^n} p_n(\alpha)$$

and hence,

$$\begin{aligned} R_n^{++}(f: [\alpha, \beta]) &\geqq f(\beta) - \frac{p_n(\beta)}{q_n(\beta)} \\ &\geqq f(\beta) - \frac{\beta^n}{\alpha^n} \frac{p_n(\alpha)}{q_n(\alpha)} . \end{aligned}$$

Since

$$\frac{p_n(\alpha)}{q_n(\alpha)} \leqq f(\alpha) + R_n^{++}(f: [\alpha, \beta])$$

we have

$$R_n^{++}(f: [\alpha, \beta]) \geqq f(\beta) - \frac{\beta^n}{\alpha^n} R_n^{++}(f: [\alpha, \beta]) .$$

Suppose that f is continuous on [0, 1] and $f(1/2) = 0$. If we set $\alpha = 1/2$ and $\beta = 1/2 + 1/2n$ in Lemma 6 then

$$R_n^{++}(f: [0, 1]) \geqq R_n^{++}\left(f: \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{2}n\right]\right) \geqq \frac{f(1/2 + 1/2n)}{(1 + e)} .$$

In particular

$$R_n^{++}\left(\left(x - \frac{1}{2}\right)^2 : [0, 1]\right) \geqq \frac{1}{4n^2} \frac{1}{(1 + e)} .$$

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