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# RATIONAL INTERPOLATION TO $e^x$, II.*

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**Abstract.** The following estimate is derived for the error in approximating $e^x$ by rational functions. Let $\pi_n$ denote the polynomials of degree at most $n$.

THEOREM. *Let $\gamma_1, \gamma_2, \cdots, \gamma_{2n+1}$ be points (not necessarily distinct) in $[0, \alpha]$, $\alpha < 2$. Choose $P_n$, $Q_n \in \pi_n$ so that*

$$P_n(\gamma_i) - Q_n(\gamma_i)e^{-\gamma_i} = 0 \quad \text{for } i = 1, 2, \cdots, 2n+1.$$

*Then for $x \in [0, \alpha]$*

$$|P_n(x)/Q_n(x) - e^{-x}| \le C_\alpha \frac{n!n!}{(2n)!(2n+1)!} \left| \prod_{i=1}^{2n+1} (x - \gamma_i) \right|$$

*and*

$$|P_n(x)/Q_n(x) - e^{-x}| \ge D_\alpha \frac{n!n!}{(2n)!(2n+1)!} \left| \prod_{i=1}^{2n+1} (x - \gamma_i) \right|,$$

*where $C_\alpha$ and $D_\alpha$ depend only on $\alpha$.*

**1. Introduction.** We derive precise estimates for the error in interpolating $e^{-x}$ on $[0, \alpha]$, $\alpha < 2$, by rational functions whose numerators and denominators have the same degree. These estimates show that, up to a constant, the optimal choice of interpolation points are the zeros of the Chebyshev polynomials shifted to the interval $[0, \alpha]$. The estimates provide another proof of the main diagonal case of the Meinardus conjecture concerning the error in best approximation to $e^x$, at least, up to a constant and on a smaller interval. (See [1], [2], [3, p. 168], [4] and [5].)

Let $\pi_n$ denote the real algebraic polynomials of degree at most $n$.

THEOREM. *Let $\gamma_1, \gamma_1, \cdots, \gamma_{2n+1}$ be points (not necessarily distinct) in $[0, \alpha]$, where $\alpha < 2$. Choose $P_n, Q_n \in \pi_n$ so that*

$$P_n(\gamma_i) - Q_n(\gamma_i)e^{-\gamma_i} = 0 \quad \textit{for } i = 1, 2, \cdots, 2n+1.$$

*Then, for $x \in [0, \alpha]$,*

$$|P_n(x)/Q_n(x) - e^{-x}| \le C_\alpha \frac{n!n!}{(2n)!(2n+1)!} \left| \prod_{i=1}^{2n+1} (x - \gamma_i) \right|$$

*and*

$$|P_n(x)/Q_n(x) - e^{-x}| \ge D_\alpha \frac{n!n!}{(2n)!(2n+1)!} \left| \prod_{i=1}^{2n+1} (x - \gamma_i) \right|,$$

*where*

$$\left( \frac{2-\alpha}{163} \right)^2 \le D_\alpha \le C_\alpha \le \frac{9}{(2-\alpha)^3}.$$

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If we set all the $\gamma_i$ to zero in the above theorem then we get bounds for the error in main diagonal Padé approximation.

The theorem is a refinement of a similar result in [1].

**2. Preliminaries.** We proceed, initially, exactly as in [1, p. 143]. Suppose that $P_n, Q_n \in \pi_n$ and suppose that $P_n(x) - Q_n(x)e^{-x}$ has $2n+1$ zeros on the interval $[0,\alpha]$. If $Q_n(x) = q_0 + q_1 x + \cdots + q_n x^n$ then on taking $n+1$ derivatives

$$(1)\quad (P_n(x) - Q_n(x)e^{-x})^{(n+1)} = (Q_n(x)e^{-x})^{(n+1)} = \sum_{k=0}^{n} \binom{n+1}{k} Q_n^{(k)} e^{-x} (-1)^{(n+1-k)}$$

$$= (-1)^{n+1} e^{-x} \sum_{k=0}^{n} \frac{x^k}{k!} \sum_{j=0}^{n-k} \binom{n+1}{j} (-1)^j (k+j)!\, q_{k+j}.$$

Since $(Q_n(x)e^{-x})^{(n+1)}$ has $n$ zeros on $[0,\alpha]$, we deduce that there exist $\beta_1, \cdots, \beta_n \in [0,\alpha]$ so that

$$\sum_{k=0}^{n} \frac{x^k}{k!} \sum_{j=0}^{n-k} \binom{n+1}{j} (-1)^j (k+j)!\, q_{k+j} = q_n \prod_{i=1}^{n} (x - \beta_i).$$

Thus, if $q_n \prod_{i=1}^{n} (x - \beta_i) = b_0 + b_1 x + \cdots + b_n x^n$, we have

(2)

$$\begin{bmatrix} \binom{n+1}{0} & -\binom{n+1}{1} & +\binom{n+1}{2} & \cdots & (-1)^n \binom{n+1}{n} \\ 0 & \binom{n+1}{0} & -\binom{n+1}{1} & \cdots & (-1)^{n-1} \binom{n+1}{n-1} \\ 0 & 0 & \binom{n+1}{0} & \cdots & (-1)^{n-2} \binom{n+1}{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n+1}{0} \end{bmatrix} \begin{bmatrix} q_0 0! \\ q_1 1! \\ q_2 2! \\ \vdots \\ q_n n! \end{bmatrix} = \begin{bmatrix} b_0 0! \\ b_1 1! \\ b_2 2! \\ \vdots \\ b_n n! \end{bmatrix}.$$

We can invert (2) to obtain

$$(3)\quad \begin{bmatrix} \binom{n}{n} & \binom{n+1}{n} & \binom{n+2}{n} & \cdots & \binom{2n}{n} \\ 0 & \binom{n}{n} & \binom{n+1}{n} & \cdots & \binom{2n-1}{n} \\ 0 & 0 & \binom{n}{n} & \cdots & \binom{2n-2}{n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n}{n} \end{bmatrix} \begin{bmatrix} b_0 0! \\ b_1 1! \\ b_2 2! \\ \vdots \\ b_n n! \end{bmatrix} = \begin{bmatrix} q_0 0! \\ q_1 1! \\ q_2 2! \\ \vdots \\ q_n n! \end{bmatrix}.$$

We observe that (3) can be easily derived from (2) combined with the fact that the $(m,n)$ Padé approximant (the case where $b_0 = b_1 = \cdots = b_{n-1} = 0$) to $e^{-x}$ is given by

$$\sum_{v=0}^{m} \frac{\binom{m}{v}}{\binom{m+n}{v}} \frac{(-x)^v}{v!} \Big/ \sum_{v=0}^{n} \frac{\binom{n}{v}}{\binom{n+m}{v}} \frac{x^v}{v!}.$$

We now consider $e^xP_n(x)-Q_n(x)$ and perform similar calculations to those above. We write $P_n(x)=p_0+\cdots+p_nx^n$ and we deduce the existence of $\alpha_1,\cdots,\alpha_n\in[0,\alpha]$ so that

$$(e^xP_n(x))^{(n+1)}=e^xp_n\prod_{i=1}^{n}(x-\alpha_i),$$

where

$$p_n\prod_{i=0}^{n}(x-\alpha_i)=a_0+\cdots+a_nx^n,$$

$$\text{(4)}\quad \begin{bmatrix} \binom{n+1}{0} & \binom{n+1}{1} & \binom{n+1}{2} & \cdots & \binom{n+1}{n} \\ 0 & \binom{n+1}{0} & \binom{n+1}{1} & \cdots & \binom{n+1}{n-1} \\ 0 & 0 & \binom{n+1}{0} & \cdots & \binom{n+1}{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n+1}{0} \end{bmatrix} \begin{bmatrix} p_0 0! \\ p_1 1! \\ p_2 2! \\ \vdots \\ p_n n! \end{bmatrix} = \begin{bmatrix} a_0 0! \\ a_1 1! \\ a_2 2! \\ \vdots \\ a_n n! \end{bmatrix},$$

and

(5)

$$\begin{bmatrix} \binom{n}{n} & -\binom{n+1}{n} & \binom{n+2}{n} & \cdots & (-1)^n\binom{2n}{n} \\ 0 & \binom{n}{n} & -\binom{n+1}{n} & \cdots & (-1)^{n-1}\binom{2n-1}{n} \\ 0 & 0 & \binom{n}{n} & \cdots & (-1)^{n-2}\binom{2n-2}{n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n}{n} \end{bmatrix} \begin{bmatrix} a_0 0! \\ a_1 1! \\ a_2 2! \\ \vdots \\ a_n n! \end{bmatrix} = \begin{bmatrix} p_0 0! \\ p_1 1! \\ p_2 2! \\ \vdots \\ p_n n! \end{bmatrix}.$$

The information about $P_n$ and $Q_n$ that allows us to analyse the error in interpolating $e^x$ is contained in the following lemma.

LEMMA. *Suppose that* $P_n(x)=p_0+p_1x+\cdots+p_nx^n$ *and suppose that* $Q_n=q_0+q_1x+\cdots+q_nx^n$ *where* $q_0>0$. *Suppose also that* $P_n(x)-Q_n(x)e^{-x}$ *has* $2n+1$ *zeros at* $\gamma_1,\cdots,\gamma_{2n+1}\in[0,\alpha]$.

*Then*:

a) $P_n$ *has alternating coefficients*;

b) $|p_n|\le(n!/(2n)!)|p_0|$;

c) *if* $\alpha\le 2$, *then*

$$|p_n|\ge\left(\frac{4}{27}\right)\frac{n!}{(2n)!}|p_0| \quad \textit{and} \quad |q_n|\ge\left(\frac{4}{27}\right)\frac{n!}{(2n)!}|q_0|;$$

d) *if* $\alpha<2$ *then* $Q_n$ *has positive coefficients*,

$$q_n\le\left(\frac{2}{2-\alpha}\right)\frac{n!}{(2n)!}q_0 \quad \textit{and} \quad |Q_n(\alpha)|\le\frac{q_0 2e^\alpha}{(2-\alpha)},$$

e) *if* $\alpha<2$ *then*

$$\frac{1}{e^{3\alpha/2}}\leq\frac{P_n(0)}{Q_n(0)}\leq\frac{4}{(2-\alpha)^2}.$$

*Proof*. That $P_n$ and $Q_n$ can be found with the desired interpolation properties is a consequence of results in [3, pp. 16 and 165].

Part a) is a direct consequence of (5) and the observation that the $a_i$ in (4) alternate in sign.

Part b) follows from (5) and the above, that is,

$$|p_0|=\sum_{i=0}^{n}|i!a_i|\binom{n+i}{n}\geq\frac{(2n)!}{n!}|a_n|=\frac{(2n)!}{n!}|p_n|.$$

To prove part c) we see that, if $0\leq\alpha_1,\cdots,\alpha_n\leq\alpha<2$ and

$$a_n\prod_{i=1}^{n}(x-\alpha_i)=a_0+\cdots+a_nx^n,$$

then

$$|a_k|\leq\binom{n}{k}\alpha^{n-k}|a_n|.$$

Thus, from (5) (or (3) for the second part),

$$|p_0|\leq\sum_{k=0}^{n}\binom{n+k}{n}k!|a_k|\leq|a_n|\sum_{k=0}^{n}\frac{(n+k)!\alpha^{n-k}}{k!(n-k)!}.$$

Since $2^{n-k}/(n-k)!\leq(\frac{2}{3})^{n-k}\cdot\frac{9}{2}$,

$$|p_0|\leq\frac{9|a_n|}{2}\sum_{k=0}^{n}\frac{(n+k)!}{k!}\left(\frac{2}{3}\right)^{n-k}\leq\frac{9|a_n|}{2}\cdot\frac{(2n)!}{n!}\sum_{k=0}^{n}\left(\frac{1}{2}\right)^{n-k}\left(\frac{2}{3}\right)^{n-k}\leq\frac{27}{4}|p_n|\frac{(2n)!}{n!}.$$

The first part of d) follows from an examination of (3) using the facts that, for $i\leq n$,

$$(i-1)!|b_{i-1}|\leq\alpha(i!)|b_i|\quad\text{and}\quad\binom{n+i-1}{n}\leq\frac{1}{2}\binom{n+i}{n}.$$

The second part of d) is proved by noting that

$$q_0\geq n!|b_n|\binom{2n}{n}-(n-1)!|b_{n-1}|\binom{2n-1}{n}\geq\left(1-\frac{\alpha}{2}\right)\frac{(2n)!}{n!}q_n.$$

To see the final part of d), note that

$$i!q_i=\sum_{k=0}^{n-i}\binom{n+k}{n}(k+i)!b_{k+i}$$

and

$$\binom{n+k-1}{n}(k+i-1)!|b_{k+i-1}|\leq\binom{n+k}{n}(k+i)!|b_{k+i}|.$$

Hence, since the $b_k$ alternate in sign,

$$i!q_i \le \binom{2n-i}{n}(n!)|b_n| \le \binom{2n}{n} n!|b_n| = \frac{(2n)!}{n!} q_n$$

or

$$q_i \le \left(\frac{2}{2-\alpha}\right)\frac{q_0}{i!}.$$

Thus

$$Q_n(x) \le \frac{2q_0}{2-\alpha} e^x.$$

Finally, from (5), one can show that

$$|(i+1)!p_{i+1}| \le \frac{1}{2}|i!p_i|.$$

Since,

$$e^{-\gamma_1} = \frac{p_0}{q_0} \frac{\sum_{i=0}^n (p_i/p_0)(\gamma_1)^i}{\sum_{i=0}^n (q_i/q_0)(\gamma_1)^i}.$$

It follows that

$$1-\frac{\gamma_1}{2} \le \sum_{i=0}^n \frac{p_i}{p_0}(\gamma_1)^i \le e^{\gamma_1/2},$$
$$1 \le \sum_{i=0}^n \frac{q_i}{q_0}(\gamma_1)^i \le \frac{2e^{\gamma_1}}{2-\alpha}$$

and

$$\frac{1}{e^{3\gamma_1/2}} \le \frac{p_0}{q_0} \le \frac{4}{(2-\gamma_1)(2-\alpha)}.$$

**3. Proof of the theorem.** Let $P_{n+k}$, $Q_{n+k} \in \pi_{n+k}$ be such that

$$\frac{P_{n+k}(x)}{Q_{n+k}(x)} - e^{-x}, \qquad k=0,1,\cdots$$

has $2k$ zeros at zero and a single zero at each of the $\gamma_i$. Then, for $x \in [0,\alpha]$

(6) $$R_k(x) = \frac{P_{n+k+1}(x)}{Q_{n+k+1}(x)} - \frac{P_{n+k}(x)}{Q_{n+k}(x)} = \frac{\alpha_{k+1}x^{2k}\prod_{i=1}^{2n+1}(x-\gamma_i)}{Q_{n+k+1}(x)Q_{n+k}(x)}.$$

Also, if $P_{n+k}(x) = 1 + \cdots + p_{n+k,k}x^{n+k}$ and $Q_{n+k} = q_{0,k} + \cdots + q_{n+k,k}x^n$, then

$$\alpha_{k+1} = p_{n+k+1,k+1} \cdot q_{n+k,k} - p_{n+k,k} \cdot q_{n+k+1,k+1}$$

and by parts a) and d) of the lemma

$$|\alpha_{k+1}| = |p_{n+k+1,k+1} \cdot q_{n+k,k}| + |p_{n+k,k} \cdot q_{n+k+1,k+1}|.$$

Parts b), c) and d) of the lemma yield the following bounds for $a_{k+1}$:

$$|\alpha_{k+1}| \leq \left(\frac{2}{2-\alpha}\right) \frac{(n+k+1)!(n+k)!}{(2n+2k+2)!(2n+2k)!}\left(|q_{0,k}\cdot p_{0,k+1}| + |q_{0,k+1}\cdot p_{0,k}|\right)$$

and

$$|\alpha_{k+1}| \geq \left(\frac{4}{27}\right)^2 \frac{(n+k+1)!(n+k)!}{(2n+2k+2)!(2n+2k)!}\left(|q_{0,k}\cdot p_{0,k+1}| + |q_{0,k+1}\cdot p_{0,k}|\right).$$

From part d) of the lemma

$$q_{0,k} \leq Q_{n+k+1}(x) \leq (q_{0,k})\frac{2e^\alpha}{2-\alpha}.$$

For $k \geq 0$ we note that $q_{0,k+1} = p_{0,k} = 1$. Thus, for $k \geq 1$

$$|R_k(x)| \leq \frac{4}{2-\alpha}\left(x^{2k}\prod_{i=1}^{2n+1}|x-\gamma_i|\right)\frac{(n+k+1)!(n+k)!}{(2n+2k+2)!(2n+2k)!}$$

and

$$|R_k(x)| \geq 2\left(\frac{4-2\alpha}{27e^\alpha}\right)^2\left(x^{2k}\prod_{i=1}^{2n+1}|x-\gamma_i|\right)\frac{(n+k+1)!(n+k)!}{(2n+2k+2)!(2n+2k)!}.$$

For $k=0$

$$|R_0(x)| \leq \left(\frac{2}{2-\alpha}\right)\left(\prod_{i=1}^{2n+1}|x-\gamma_i|\right)\frac{(n+1)!n!}{(2n+2)!(2n)!}\left(1+\frac{P_n(0)}{Q_n(0)}\right)$$

and

$$|R_0(x)| \geq \left(\frac{4-2\alpha}{27e^\alpha}\right)^2\left(\prod_{i=1}^{2n+1}|x-\gamma_i|\right)\frac{(n+1)!n!}{(2n+2)!(2n)!}\left(1+\frac{P_n(0)}{Q_n(0)}\right).$$

Note that

$$\frac{(n+k+2)!(n+k+1)!}{(2n+2k+4)!(2n+2k+2)!}\Big/\frac{(n+k+1)!(n+k)!}{(2n+2k+2)!(2n+k)!} \leq \frac{1}{16(n+k)^2}.$$

Thus,

$$\left|e^{-x}-\frac{P_n(x)}{Q_n(x)}\right| = \sum_{k=0}^{\infty}|R_k(x)|$$

$$\leq \left(\frac{2}{2-\alpha}\right)\left(\prod_{i=1}^{2n+1}|x-\gamma_i|\right)\frac{(n+1)!n!}{(2n+2)!2n!}\left[1+\frac{4}{(2-\alpha)^2}+2\sum_{k=1}^{\infty}\frac{1}{16(n+k)^2}\right]$$

and

$$\left|e^{-x}-\frac{P_n(x)}{Q_n(x)}\right|$$

$$\geq \left(\frac{4-2\alpha}{27e^\alpha}\right)^2\left(\prod_{i=1}^{2n+1}|x-\gamma_i|\right)\frac{(n+1)!n!}{(2n+2)!2n!}\left[1+\frac{1}{e^3}-2\sum_{n=1}^{\infty}\frac{1}{16(n+k)^2}\right]. \quad \square$$

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