PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 93, Number 1, January 1985

MARKOV'S INEQUALITY FOR POLYNOMIALS WITH REAL ZEROS

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ABSTRACT. Markov's inequality asserts that $\|p_n'\| \leqslant n^2\|p_n\|$ for any polynomial p_n of degree n. (We denote the supremum norm on $[-1,1]$ by $\|\cdot\|$.) In the case that p_n has all real roots, none of which lie in $[-1,1]$, Erdös has shown that $\|p_n'\| \leqslant en\|p_n\|/2$. We show that if p_n has $n-k$ real roots, none of which lie in $[-1,1]$, then $\|p_n'\| \leqslant cn(k+1)\|p_n\|$, where c is independent of n and k. This extension of Markov's and Erdös' inequalities was conjectured by Szabados.

Introduction. Markov's inequality asserts that

$$\|p_n'\|_{[-1,1]} \leqslant n^2\|p_n\|_{[-1,1]} \tag{1}$$

for any poynomial $p_n \in \pi_n$ **[2** and **3]**. (π_n denotes the algebraic polynomials of degree at most n and $\|\cdot\|_A$ denotes the supremum norm on A.) Erdös **[1]** in 1940 offered the following refinement of Markov's inequality. If $p_n \in \pi_n$ and p_n has all its roots in $\mathbf{R} - (-1,1)$, then

$$\|p_n'\|_{[-1,1]} \leqslant \frac{en}{2}\|p_n\|_{[-1,1]}. \tag{2}$$

Inequality (1) iterates to give bounds for the kth derivative of a polynomial. However, we cannot proceed inductively with inequality (2) since some of the roots of the derivatives may be in $[-1,1]$. With this in mind, Szabados and Varma established a version of (2) for polynomials of degree n with all real roots and at most one root in $[-1,1]$, namely, for such a polynomial p_n,

$$\|p_n'\|_{[-1,1]} \leqslant c_1 n\|p_n\|_{[-1,1]}, \tag{3}$$

where c_1 is independent of n **[5]**. This, of course, yields the following inequality:

$$\|p_n''\|_{[-1,1]} \leqslant c_2 n^2\|p_n\|_{[-1,1]} \tag{4}$$

for any $p_n \in \pi_n$ that has all its roots in $\mathbf{R} - (-1,1)$. In **[6]** Szabados proposed the following

Conjecture. If p_n is a polynomial of degree n and p_n has at least $n-k$ roots in $\mathbf{R} - (-1,1)$, then there is a constant c ($c \leqslant 9$) so that

$$\|p_n'\|_{[-1,1]} \leqslant cn(k+1)\|p_n\|_{[-1,1]}. \tag{5}$$

Received by the editors November 28, 1983.
1980 *Mathematics Subject Classification*. Primary 26C05, 26C10.
[1]Research supported, in part, by N.S.E.R.C. of Canada.

It is our intention to prove this slightly strengthened form of Szabados' conjecture. In its original form the conjecture had the additional assumption that all the roots of p_n be real. Up to the constant this result is best possible; Szabados in [**6**] constructs polynomials p_n of degree n with $n - k$ roots in $\mathbf{R} - (-1, 1)$ so that

$$\|p_n'\|_{[-1,1]} \geqslant \frac{n \cdot k}{2} \|p_n\|_{[-1,1]} \qquad (0 < k \leqslant n).$$

It is apparent from (1) and (2) that the best constant must depend on k. Some related results may be found in [**4**].

Inequalities for the higher derivatives of polynomials with real roots can now be derived straightforwardly from (5). For example,

THEOREM. *If $p_n \in \pi_n$ has at least $n - k$ zeros in $\mathbf{R} - (-1, 1)$, then*

$$\|p_n^{(m)}\|_{[-1,1]} \leqslant c_m \frac{n!(k+m)!}{(n-m)!k!} \|p_n\|_{[-1,1]},$$

where $c_m \leqslant 9^m$ depends only on m.

2. Proof of the Conjecture. Let c_{2k} be the $2k$th Chebychev polynomial shifted to the interval $[0, 2]$ and normalized to have lead coefficient 1. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_k$ be the roots of c_{2k} in $[1, 2]$ and let

$$t_k := \prod_{i=1}^{k} (x - \alpha_i). \tag{6}$$

LEMMA 1. *The polynomial $q_k := (x + 2m/k)^{m+k} t_k(x)$ has the following property. If $\alpha_0 = 0$ and $\alpha_{n+1} = 1$, then, for $i = 1, 2, \ldots, n$,*

$$\left\| \left(x + \frac{2m}{k}\right)^{m+k} t_k \right\|_{[\alpha_{i-1}, \alpha_1]} > \left\| \left(x + \frac{2m}{k}\right)^{m+k} t_k \right\|_{[\alpha_i, \alpha_{i+1}]}, \tag{7}$$

where the maximums on successive intervals occur with alternating sign.

PROOF. Let $0 < \beta_1 < \cdots < \beta_k$ be the roots of c_{2k} in $[0, 1]$. We observe that $x/(x - \beta_i)$ is positive and decreasing on $(\beta_k, \infty]$ and that c_{2k} equioscillates on the intervals in question (i.e. c_{2k} satisfies (7) with equality). We now note that

$$x^k t_k = \left(\prod_{i=1}^{k} \frac{x}{(x - \beta_i)} \right) c_{2k}$$

satisfies the conclusion of Lemma 1. To finish the proof we need only observe that $(x + 2m/k)^{m+k}/x^k$ is decreasing on $[0, 2]$. □

Let $n = 2k + m$ and let

$$s_n(x) := \frac{1}{(1 + m/k)^n} q_k\left(\left(1 + \frac{m}{k}\right)x + \left(1 - \frac{m}{k}\right)\right).$$

(We have shifted from $[-2m/k, 2]$ to $[-1, 1]$.) This polynomial will act as a kind of near extremal polynomial for the Conjecture. Let $\gamma_1 < \gamma_2 < \cdots < \gamma_k$ be the roots of s_n in $(-1, 1)$. We collect the properties of s_n that we require in the next lemma.

LEMMA 2. *For* $n = m + 2k$ *and* s_n *as above*:
(a) $s_n(x) = (x+1)^{m+k}\prod_{i=1}^{k}(x - \gamma_i)$,
(b) $\sum_{i=1}^{k}(1/(1-\gamma_i)) \leqslant 4k(n-k)$, *and*
(c) *for* $i = 1, \dots, k$, $\gamma_0 = -1$ *and* $\gamma_{k+1} = 1$

$$\|s_n\|_{[\gamma_{i-1},\gamma_i]} \geqslant \|s_n\|_{[\gamma_i,\gamma_{i+1}]}.$$

PROOF. Parts (a) and (c) are immediate from the construction of s_n. Part (b) follows from the observation that, for α_i as in (6).

$$\sum_{i=1}^{k} \frac{1}{2-\alpha_i} \leqslant \frac{c_{2k}'(2)}{c_{2k}(2)} = 4k^2$$

and the observation that

$$1 - \gamma_i = \frac{1}{1 + m/k}(2 - \alpha_i). \quad \square$$

Let $p_n^* \in \pi_n$ maximize

(8) $$|p_n'(1)|/\|p_n\|_{[-1,1]},$$

where the maximum is taken over all polynomials in π_n that have all but at most k roots in $\mathbf{R} - (-1,1)$. The information we need about p_n^* is contained in the next lemma.

LEMMA 3. *Let* p_n^* *be as above. Then*
(a) p_n^* *has* k *simple roots* $\delta_1 < \cdots < \delta_k$ *in* $(-1,1)$, p_n^* *has* $n-k$ *roots at* ± 1, *and* p_n^* *achieves its maximum modulus on each of the intervals* $[-1,\delta_1], [\delta_1,\delta_2], \dots, [\delta_k, 1]$.
(b) *Either* p_n^* *has no roots in* $[-1,\infty)$ *or* p_n^* *has exactly one root at* 1.

PROOF. The proof of (a) is a simple and standard perturbation argument (if p_n^* did not satisfy (a) then it would be possible to perturb p_n^* to reduce its norm on $[-1,1]$ without decreasing the derivative at 1). We will prove only that p_n has no roots in $(1,\infty)$, the other parts are similar. First suppose that p_n^* has two roots at $\alpha > 1$ and $\beta > 1$. Consider

$$v_n(x) := \frac{p_n^*(x)(x-1)^2}{(x-\alpha)(x-\beta)}.$$

Then for sufficiently small $\varepsilon > 0$:
(i) $\|p_n^* - \varepsilon v_n\|_{[-1,1]} < \|p_n^*\|_{[-1,1]}$,
(ii) $|(p_n^* - \varepsilon v_n)'(1)| = |p_n^{*\prime}(1)|$,
(iii) $p_n^* - \varepsilon v_n$ *has all but at most* k *roots in* $\mathbf{R} - (1,1)$.

Part (iii) follows since $(x-\alpha)(x-\beta) - \varepsilon(x-1)^2$ has two roots in $[1,\infty]$ for sufficiently small ε. (Note α may equal β.) However, this contradicts the maximality of p_n^*.

Next we suppose that p_n^* has exactly one (nonrepeat) root at $\alpha > 1$. Now we argue as before by considering

$$v_n(x) := p_n^*(x)\frac{(x-1)^2}{(x-1)(x-\alpha)}.$$

If $p_n^*(1) \neq 0$ we must observe that in this case sign$(p_n^*(1)) =$ sign$(p_n^{*\prime}(1))$ and, hence, that

$$v_n'(1) = \frac{(1-\alpha)p_n^*(1)}{(1-\alpha)^2}$$

has the opposite sign to $p_n^{*\prime}(1)$. The last observation requires noticing that if $p_n^{*\prime}(1)$ has opposite sign to $p_n^*(1)$, then $p_n^{*\prime}$ has all its zeros in $(-\infty, 1]$ and, hence, $p_n''(1) \neq 0$. Thus, $|p_n^{*\prime}|$ is increasing on $[1, \infty)$, $|p_n|$ is decreasing on $[1, \alpha)$ and $p_n^*(x + (\alpha - 1))$ violates the maximality assumptions on $[-1, 1]$.

Part (b) follows since if p_n^* has two or more zeros at 1, then $p_n'(1)$ would also equal zero. □

Lemma 4. *If $p_n \in \pi_n$ has at least $(n-k)$ roots in $\mathbf{R} - (-1, 1)$, then*

$$|p_n'(1)| \leqslant \tfrac{9}{2}(k+1)n\|p_n\|_{[-1,1]}.$$

Proof. If $2k \geqslant n$, then the lemma follows from Markov's inequality, so we may suppose $2k < n$. Suppose there exists p_n, as above, so that

$$|p_n'(1)| > \tfrac{9}{2}(k+1)(n-k)\|p_n\|_{[-1,1]},$$

and let q_n be the maximal such p_n. By Lemma 3, this q_n equioscillates $k+1$ times on $[-1, 1]$.

We shall first consider the case where q_n has no root at 1.

The key to the proof is to observe that the roots of q_n lie to the left of the roots of s_n (as defined in Lemma 3). We may write

$$q_n(x) = (x+1)^{n-k} \prod_{i=1}^{k} (x - \rho_i),$$

where $-1 < \rho_1 < \cdots < \rho_k < 1$. Also,

$$s_n(x) = (x+1)^{n-k} \prod_{i=1}^{k} (x - \gamma_i).$$

The claim is that $\gamma_i \geqslant \rho_i$ for each i. This is seen as follows.Choose the largest i for which $\rho_i > \gamma_i$. Then pick η so that $\|\eta q_n\|_{[\gamma_i,1]} = \|s_n\|_{[\gamma_i,1]}$. (We will specify the sign of η later.)We can deduce from the equioscillation of q_n that $\eta q_n - s_n$ has at least $k - i$ roots on $[\beta, 1]$, where β is the first point greater than ρ_i where ηq_n achieves its maximum modulus. From Lemma 2(c) we deduce that $\eta q_n - s_n$ has at least $i - 1$ roots on $(-1, \alpha)$, where α is the largest point less than γ_i where s_n achieves its maximum modulus. We need only observe that if we choose the sign of η so that

$$\operatorname{sign} \eta q_n(\beta) = -\operatorname{sign} s_n(\alpha),$$

then $\eta q_n - s_n$ must have 2 roots in (α, β). Thus, $\eta q_n - s_n$ has $n+1$ roots which is a contradiction and we conclude that $\rho_i \leqslant \gamma_i$.

We now observe that, since $\rho_i \leqslant \gamma_i < 1$,

$$\frac{|q_n'(1)|}{\|q_n\|_{[-1,1]}} = \frac{q_n'(1)}{q_n(1)} = \sum_{i=1}^{k} \frac{1}{1-\rho_i} + \sum_{i=1}^{n-k} \frac{1}{1-(-1)}$$

$$\leqslant \sum_{i=1}^{k} \frac{1}{1-\gamma_i} + \frac{n-k}{2} \leqslant 4k(n-k) + \frac{n-k}{2},$$

where the later inequality follows from Lemma 2. This is a contradiction.

In the case where q_n has exactly one root at 1 we proceed as follows. Let $d > 1$ be the unique point in $(1, \infty)$, where $|q_n(d)| = \|q_n\|_{[-1,1]}$. We can now consider q_n on $[-1, d]$. We note that

$$\frac{|q_n'(d)|}{\|q_n\|_{[-1,1]}} \geqslant \frac{|q_n'(1)|}{\|q_n\|_{[-1,1]}}$$

since $|q_n'|$ is increasing on $[1, \infty)$. We can repeat verbatim the argument of the first part applied to

$$\tilde{q}(x) = q_n\left(x\left(\frac{d+1}{2}\right) + \left(\frac{d-1}{2}\right)\right)$$

with k replaced by $k+1$. This allows us to deduce the contradiction that

$$\frac{|q'(1)|}{\|q_n\|_{[-1,1]}} \leqslant \frac{|\tilde{q}_n(1)|}{\|\tilde{q}_n\|_{[-1,1]}} \leqslant 4(k+1)(n-k) + \frac{n-k}{2}. \quad \square$$

The proof of the Conjecture is now straightforward.

Proof of Conjecture. Let p_n be a polynomial of degree n with $n-k$ roots in $\mathbf{R} - (-1,1)$. Let x_0 be a point in $[-1,1]$, where p_n' achieves its maximum modulus. We suppose $x_0 \leqslant 0$ ($x_0 > 0$ follows analogously). Let $ax + b$ map $[x_0, 1]$ one-to-one onto $[-1,1]$ in such a way that $x_0 \to 1$.

Note that $|a| < 2$. Thus, if $v_n(ax+b) = p_n(x)$, then

$$\frac{|p_n'(x_0)|}{\|p_n\|_{[-1,1]}} \leqslant \frac{2|v_n'(1)|}{\|v_n\|_{[-1,1]}} = 9n(k+1),$$

where the last inequality follows from Lemma 4 and the observation that v_n has at least as many roots as p_n in $\mathbf{R} - [-1,1]$. $\square$

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