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Note

On Monochromatic Triangles

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Let A and B be two disjoint finite sets in R^2. Simple conditions that guarantee the existence of a triangle with vertices in one of the sets and with no points from the other set in its interior are given. The analogous problem for d-simplices in R^d is treated. Conditions are derived that guarantee the existence of a triangle with vertices in one of the sets and with no points from either set on its boundary.

INTRODUCTION

Let A and B be two disjoint nonempty finite sets in R^2. Under the assumption that $A \cup B$ spans R^2 Motzkin [2] proves the existence of a monochromatic line (a line through at least 2 points of one of the sets that misses the other set). A special case of this result is the following lemma.

LEMMA 1 (J. B. Kelly [3, p. 298]). *Let A and B be two finite sets in R^n. Suppose that every open segment joining two points of A contains a point of B, and vice versa. Then the sets A and B lie on a line.*

J. B. Kelly's proof of Lemma 1 is a *minimum-altitude proof* based on L. M. Kelly's proof of Sylvester's theorem. We offer the following particularly simple proof of Lemma 1.

Proof. Let $T_1 = \{p_1, p_2, p_3\}$ be a nondegenerate triangle of smallest area with all vertices either in A or all vertices in B. We show that no such triangle exists. We assume that $\{p_1, p_2, p_3\} \subset A$. By assumption there exists $\{b_1, b_2, b_3\} \subset B$ so that each b_i lies on a different edge of T_1. The triangle $T_2 = \{b_1, b_2, b_3\}$ now has smaller area that T_1 and contradicts the initial assumption. ∎

This result motivated Baston and Bostock [1] to examine various generalizations. It is our intention to do the same.

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Two Results Concerning Monochromatic Triangles

The first theorem concerns triangles with monochromatic interiors.

Theorem 1. *Let A and B be two finite disjoint sets of points in R^2. Suppose A contains five points in general position. Then there exists a triangle with vertices in one of the sets and with no point from the other set in its interior.*

Theorems 1 and 3 are similar in flavour to results in [1]. Figure 1 shows that the assumption of five points in general position in Theorem 1 is necessary.

Proof. Suppose that $\text{card}(A) = n \geqslant 5$. Let Γ denote the convex hull of A, let $A' = A \cap \text{int}(\Gamma)$ and let $B' = B \cap \text{int}(\Gamma)$. Let $k = \text{card}(A')$. Suppose that A and B satisfy the conditions of the theorem but contradict the conlcusion.

We observe that there is a triangulation of Γ consisting of $n + k - 2$ triangles having A as its vertex set. To see this, first, partition Γ into $n - k - 2$ triangles using the points of A on the boundary of Γ as vertices, then add the interior vertices one at a time. Each additional interior vertex increases the number of triangles by two.

Since each triangle in such a triangulation of Γ contains a point of B', it follows that card $B' \geqslant n + k - 2$. Any line 1 has at least three noncollinear points of A on or to one side of it, hence, there is a point of B' not on 1. Thus, the points of B' are not all collinear, so the convex hull of B' has a triangulation consisting of at least $(n + k - 2) - 2$ triangles. This implies that $\text{card}(A') \geqslant n + k - 4 > k = \text{card}(A')$ which is impossible. ∎

Theorem 2. *Suppose A and B are disjoint finite sets in R^2. Suppose that A contains 5 points $p_1, \ldots, p_5$ that are the vertices of a strictly convex pentagon. Let Π denote the convex hull of $\{p_1, \ldots, p_5\}$ and suppose that $\Pi \cap A$ has no three points collinear. Then there exists a triangle T with*

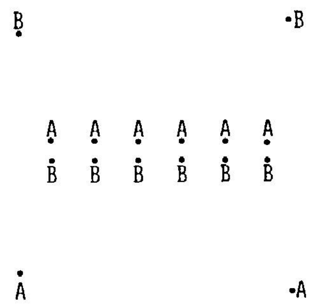

Figure 1

vertices only in A or only in B so that no other point from either A or B lies on any of the edges of T.

Proof. Assume we have sets A and B which satisfy the hypothesis but not the conclusion of the theorem. Let Γ be the smallest (in area) strictly convex pentagon with vertices in A and with the property that $\Gamma \cap A$ has no three points collinear. Let $A' = A \cap \Gamma$ and $B' = B \cap \Gamma$. Let q_1, q_2, q_3, q_4, q_5 be the vertices of Γ in order. Notice that no other set of five points of A' spans a convex pentagon.

Now any three points of A' span a triangle which has no other points of A' on its edges, so one edge must contain a point of B'. Suppose C is a noncollinear subset of B'. The convex hull Δ of C has a triangulation whose vertex set is $B \cap \Delta$. (See the proof of Theorem 1.) Any triangle of this triangulation must contain a point of A', so $A' \cap \Delta \neq \emptyset$.

The triangles $q_1q_2q_3$ and $q_3q_4q_5$ each contain a point of B' and these two points are distinct. Let l be the line joining them. Since l does not contain more than two points of A', we can find either three points of A' on one side of l or two points of A' on one side of l and one on l. In either case we obtain a point of B' not on l. Thus, the convex hull of B' is two dimensional and contains a point r of A' which is necessarily in int Γ and hence, is distinct from q_1, q_2, q_3, q_4, q_5.

The convex hull of r and three consecutive vertices of Γ is a quadrilateral, since if r were within the triangle spanned by q_1, q_2, q_3, say, then r, q_1, q_3, q_4, q_5 would span a smaller convex pentagon, which is impossible.

Consider the five radial segments rq_i. Suppose two of these that are not adjacent each contain a point of B', say rq_1 and rq_3 both meet B'. Then a third point of B' can be found on the triangle $q_1q_2q_3$. These three points of B' cannot be collinear and hence, there is a point s of A' within the triangle $q_1q_2q_3$. Since s is interior to the quadrilateral $rq_1q_2q_3$, it is a new point.

If the configuration just analyzed does not exist, then there are three consecutive radial segments, say rq_1, rq_2, rq_3, none of which contains a point of B'. Then each of the segments q_1q_2, q_2q_3, q_3q_1 must contain a point of B'. These three points of B' span a triangle which must contain a point s of A'. Since s is interior to the triangle $q_1q_2q_3$, it is a new point.

Thus, in either case there is a seventh point s in A'. At least three of the q's lie on one side of the line joining r and s, and these three together with r and s span a convex pentagon. This, contradiction finishes the proof. ∎

The example in Fig. 2 shows that we cannot weaken the assumptions of Theorem 2. Since any set of nine points in general position contain the vertices of a strictly convex pentagon [4, Prob. 31] we have

COROLLARY 1. *Suppose that A and B are disjoint finite sets in R^2 and suppose that there exists a convex set Γ in R^2 so that $\Gamma \cap A$ contains no*

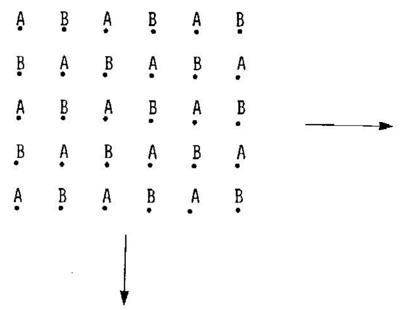

FIGURE 2

three collinear points. Then, if card($\Gamma \cap A$) $\geqslant 9$ *there exists a triangle T with vertices only in A or only in B so that no other point from $A \cup B$ lies on any edge of T.*

We note that both Theorem 1 and Corollary 1 are valid in R^m, $m \geqslant 2$. The proofs in higher dimensions follow, under careful projection, from the two dimensional cases.

The following higher dimensional analogue of Theorem 1 is valid.

THEOREM 3. *Suppose that A and B are two finite disjoint sets in R^d, $d \geqslant 2$. Suppose that A contains $2d + 1$ points in general position. Then there exists a d-simplex with vertices in one of the sets and with no points of either set in its interior.*

This follows, from Lemma 1. The arguments are analogous to those used in the proof of Theorem 1.

LEMMA 1. *Suppose S is a finite set in R^d that spans R^d. Let Γ be its convex hull. Then Γ has a triangulation with vertices in S and with at least $n - d + k(d - 1)$ d-simplices, where $n =$ card(S) and $k =$ card($S \cap \operatorname{int} \Gamma$).*

There are many obvious related questions: What happens if we consider quadrilaterals (pentagons, etc.) in Theorem 1? What conditions yield a result like Theorem 2 with the conclusion that there exists a triangle with both monochromatic edges and monochromatic interior? What is a *correct* analogue to Theorem 3 is we consider three sets instead of just two?

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