LOWER BOUNDS FOR THE NUMBER OF ZEROS OF COSINE POLYNOMIALS IN THE PERIOD: A PROBLEM OF LITTLEWOOD

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Abstract. Littlewood in his 1968 monograph "Some Problems in Real and Complex Analysis" [9, problem 22] poses the following research problem, which appears to still be open: "If the n_m are integral and all different, what is the lower bound on the number of real zeros of $\sum_{m=1}^{N} \cos(n_m\theta)$? Possibly $N-1$, or not much less." Here real zeros are counted in a period. In fact no progress appears to have been made on this in the last half century. In a recent paper [2] we showed that this is false. There exists a cosine polynomial $\sum_{m=1}^{N} \cos(n_m\theta)$ with the n_j integral and all different so that the number of its real zeros in the period is $O(N^{9/10}(\log N)^{1/5})$ (here the frequencies $n_m = n_m(N)$ may vary with N). However, there are reasons to believe that a cosine polynomial $\sum_{m=1}^{N} \cos(n_m\theta)$ always has many zeros on the period. Denote the number of zeros of a trigonometric polynomial T in the period $[-\pi, \pi)$ by $\mathcal{N}(T)$. In this paper we prove the following.

Theorem. *Suppose the set $\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ is finite and the set $\{j \in \mathbb{N} : a_j \neq 0\}$ is infinite. Let*

$$T_n(t) = \sum_{j=0}^{n} a_j \cos(jt)\,.$$

Then $\lim_{n\to\infty} \mathcal{N}(T_n) = \infty\,.$

One of our main tools, not surprisingly, is the resolution of the Littlewood Conjecture [4].

1. Introduction

Let $0 \le n_1 < n_2 < \cdots < n_N$ be integers. A cosine polynomial of the form $T_n(\theta) = \sum_{j=1}^{N} \cos(n_j\theta)$ must have at least one real zero in a period. This is obvious if $n_1 \neq 0$, since then the integral of the sum on a period is 0. The above statement is less obvious if $n_1 = 0$, but for sufficiently large N it follows from Littlewood's Conjecture simply. Here we mean the Littlewood's Conjecture proved by S. Konyagin [5] and independently by McGehee, Pigno, and Smith [11] in 1981. See also [4] for a book proof. It is not difficult to prove the statement in general even in the case $n_1 = 0$. One possible way is to use the identity

$$\sum_{j=1}^{n_N} T_n((2j-1)\pi/n_N) = 0\,.$$

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See [6], for example. Another way is to use Theorem 2 of [12]. So there is certainly no shortage of possible approaches to prove the starting observation of this paper even in the case $n_1 = 0$.

It seems likely that the number of zeros of the above sums in a period must tend to ∞ with N. In a private communication B. Conrey asked how fast the number of zeros of the above sums in a period tend to ∞ as a function N. In [3] the authors observed that for an odd prime p the Fekete polynomial $f_p(z) = \sum_{k=0}^{p-1} \left(\frac{k}{p}\right) z^k$ (the coefficients are Legendre symbols) has $\sim \kappa_0 p$ zeros on the unit circle, where $0.500813 > \kappa_0 > 0.500668$. Conrey's question in general does not appear to be easy.

Littlewood in his 1968 monograph "Some Problems in Real and Complex Analysis" [9, problem 22] poses the following research problem, which appears to still be open: "If the n_m are integral and all different, what is the lower bound on the number of real zeros of $\sum_{m=1}^{N} \cos(n_m \theta)$? Possibly $N-1$, or not much less." Here real zeros are counted in a period. In fact no progress appears to have been made on this in the last half century. In a recent paper [2] we showed that this is false. There exists a cosine polynomials $\sum_{m=1}^{N} \cos(n_m \theta)$ with the n_m integral and all different so that the number of its real zeros in the period is $O(N^{9/10}(\log N)^{1/5})$ (here the frequencies $n_m = n_m(N)$ may vary with N). However, there are reasons to believe that a cosine polynomial $\sum_{m=1}^{N} \cos(n_m \theta)$ always has many zeros in the period. In this paper we prove the following.

2. New Result

Theorem 1. *Suppose the set $\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ is finite and the set $\{j \in \mathbb{N} : a_j \neq 0\}$ is infinite. Let*

$$T_n(t) = \sum_{j=0}^{n} a_j \cos(jt) \,.$$

Then $\lim_{n\to\infty} \mathcal{N}(T_n) = \infty \,.$

3. Lemmas and Proofs

To prove the new result we need a few lemmas. The first two lemmas are straightforward from [4, pages 285-288] which offers an elegant book proof of the the Littlewood Conjecture first shown in [5] and [10]. The book [1] deals with a number of related topics. Littlewood [7,8,9,10] was interested in many closely related problems.

Lemma 3.1. *Let $\lambda_0 < \lambda_1 < \cdots < \lambda_m$ be nonnegative integers and let*

$$S_m(t) = \sum_{j=0}^{m} A_j \cos(\lambda_j t) \,, \qquad A_j \in \mathbb{R} \,, \;\; j = 0, 1, \ldots, m \,.$$

Then

$$\int_{-\pi}^{\pi} |S_m(t)| \, dt \geq \frac{1}{60} \sum_{j=0}^{m} \frac{|A_{m-j}|}{j+1} \,.$$

Lemma 3.2. *Let $\lambda_0 < \lambda_1 < \cdots < \lambda_m$ be nonnegative integers and let*

$$S_m(T) = \sum_{j=0}^{m} A_j \sin(\lambda_j t)\,, \qquad A_j \in \mathbb{R}\,,\;\; j = 0, 1, \ldots, m\,.$$

Then

$$\int_{-\pi}^{\pi} |S_m(t)|\, dt \geq \frac{1}{60} \sum_{j=0}^{m} \frac{|A_{m-j}|}{j+1}\,.$$

Lemma 3.3. *Let $\lambda_0 < \lambda_1 < \cdots < \lambda_m$ be nonnegative integers and let*

$$S_m(t) = \sum_{j=0}^{m} A_j \cos(\lambda_j t)\,, \qquad A_j \in \mathbb{R}\,,\;\; j = 0, 1, \ldots, m\,.$$

Let $A := \max\{|A_j| : j = 0, 1, \ldots, m\}$. Suppose S_m has at most K zeros in the period $[-\pi, \pi)$. Then

$$\int_{-\pi}^{\pi} |S_m(t)|\, dt \leq 2KA \left(\pi + \sum_{j=1}^{m} \frac{1}{\lambda_j} \right) \leq 2KA(5 + \log m)\,.$$

Proof. We may assume that $\lambda_0 = 0$, the case $\lambda_0 > 0$ can be handled similarly. Associated with S_m in the lemma let

$$R_m(t) := A_0 t + \sum_{j=0}^{m} \frac{A_j}{\lambda_j} \sin(\lambda_j t)\,.$$

Clearly

$$\max_{t \in [-\pi, \pi]} |R_m(t)| \leq A \left(\pi + \sum_{j=1}^{m} \frac{1}{\lambda_j} \right)\,.$$

Also, for every $c \in \mathbb{R}$ the function $R_m - c$ has at most K zeros in the period $[-\pi, \pi)$, otherwise Rolle's Theorem implies that $S_m = (R_m - c)'$ has at least $K + 1$ zeros in the period $[-\pi, \pi)$. Hence

$$\begin{aligned} \int_{-\pi}^{\pi} |S_m(t)|\, dt &= V_{-\pi}^{\pi}(R_m) \leq 2K \max_{t \in [-\pi, \pi]} |R_m(t)| \\ &\leq 2KA \left(\pi + \sum_{j=1}^{m} \frac{1}{\lambda_j} \right) \leq 2KA(5 + \log m)\,, \end{aligned}$$

and the lemma is proved. $\square$

Proof of the theorem when $(a_n)_{n=0}^\infty$ is NOT eventually periodic. Suppose the theorem is false. Then there are $k \in \mathbb{N}$, a sequence $(n_\nu)_{\nu=1}^\infty$ of positive integers $n_1 < n_2 < \cdots$, and even trigonometric polynomials $Q_{n_\nu} \in \mathcal{T}_k$ with maximum norm 1 on the period such that

$$T_{n_\nu}(t) Q_{n_\nu}(t) \geq 0\,, \qquad t \in \mathbb{R}\,. \tag{3.1}$$

We can pick a subsequence of $(n_\nu)_{\nu=1}^\infty$ (without loss of generality we may assume that it is the sequence $(n_\nu)_{\nu=1}^\infty$ itself) that converges to a $Q \in \mathcal{T}_k$ uniformly on the period $[-\pi, \pi)$. That is,

$$\lim_{\nu \to \infty} \varepsilon_\nu = 0 \qquad \text{with} \qquad \varepsilon_\nu := \max_{t \in [-\pi,\pi]} |Q(t) - Q_{n_\nu}(t)|\,. \tag{3.2}$$

We introduce the formal trigonometric series

$$\sum_{j=0}^\infty b_j \cos(\beta_j t) := \left(\sum_{j=0}^\infty a_j \cos(jt) \right) Q(t)^3\,, \qquad b_j \neq 0\,,\ \ j = 0, 1, \ldots\,,$$

and

$$\sum_{j=0}^\infty d_j \cos(\delta_j t) := \left(\sum_{j=0}^\infty a_j \cos(jt) \right) Q(t)^4\,, \qquad d_j \neq 0\,,\ \ j = 0, 1, \ldots\,,$$

where $\beta_0 < \beta_1 < \cdots$, and $\delta_0 < \delta_1 < \cdots$ are integers. Since the set $\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ is finite, the sets

$$\{b_j : j \in \mathbb{N}\} \subset \mathbb{R} \qquad \text{and} \qquad \{d_j : j \in \mathbb{N}\} \subset \mathbb{R}$$

are finite as well. Hence there are $\rho, M \in (0, \infty)$ such that

$$|a_j| \leq M\,, \qquad \rho \leq |b_j|\,, |d_j| \leq M\,, \qquad j = 0, 1, \ldots\,. \tag{3.3}$$

Let

$$K_\nu := |\{j \in \mathbb{N} : 0 \leq \beta_j \leq n_\nu\}|$$

and

$$L_\nu := |\{j \in \mathbb{N} : 0 \leq \delta_j \leq n_\nu\}|\,.$$

Since the sequence $(a_n)_{n=0}^\infty$ is not eventually periodic, we have

$$\lim_{\nu \to \infty} K_\nu = \infty \qquad \text{and} \qquad \lim_{\nu \to \infty} L_\nu = \infty\,. \tag{3.4}$$

We claim that

$$K_\nu \leq c_1 L_\nu \tag{3.5}$$

with some $c_1 > 0$ independent of $\nu \in \mathbb{N}$. Indeed, using Parseval's formula and (3.2) – (3.4), we deduce

$$\frac{1}{\pi}\int_{-\pi}^{\pi} T_{n_\nu}(t)^2 Q(t)^4 Q_{n_\nu}(t)^2\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} (T_{n_\nu}(t)Q(t)^2 Q_{n_\nu}(t))^2\,dt \geq (K_\nu - 3k)\frac{1}{2}\rho^2 \geq \frac{1}{4}\rho^2 K_\nu \tag{3.6}$$

for every sufficiently large $\nu \in \mathbb{N}$. Also, (3.1) – (3.4) imply

$$\begin{aligned}
&\frac{1}{\pi}\int_{-\pi}^{\pi} T_{n_\nu}(t)^2 Q(t)^4 Q_{n_\nu}(t)^2\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} (T_{n_\nu}(t)Q_{n_\nu}(t))(T_{n_\nu}(t)Q(t)^4)Q_{n_\nu}(t)\,dt \\
&\leq \frac{1}{\pi}\left(\int_{-\pi}^{\pi} T_{n_\nu}(t)Q_{n_\nu}(t)\,dt\right)\left(\max_{t\in[-\pi,\pi]} |T_{n_\nu}(t)Q(t)^4|\right)\left(\max_{t\in[-\pi,\pi]} |Q_{n_\nu}(t)|\right) \\
&\leq \frac{1}{\pi}\left(\int_{-\pi}^{\pi} T_{n_\nu}(t)Q_{n_\nu}(t)\,dt\right)(L_\nu M + 4kM)\left(\max_{t\in[-\pi,\pi]} |Q_{n_\nu}(t)|\right) \\
&\leq c_2 L_\nu
\end{aligned} \tag{3.7}$$

with a constant $c_2 > 0$ independent of ν for every sufficiently large $\nu \in \mathbb{N}$. Now (3.5) follows from (3.6) and (3.7). From Lemma 3.1 we deduce

$$\int_{-\pi}^{\pi} |T_{n_\nu}(t)Q(t)^4|\,dt \geq c_3 \rho \log L_\nu - c_4 \tag{3.8}$$

with some constants $c_3 > 0$ and $c_4 > 0$ independent of $\nu \in \mathbb{N}$. On the other hand, using (3.1), Lemma 3.3, (3.2), (3.3), (3.5), and (3.4), we obtain

$$\begin{aligned}
&\int_{-\pi}^{\pi} |T_{n_\nu}(t)Q(t)^4|\,dt \\
&\leq \int_{-\pi}^{\pi} T_{n_\nu}(t)Q_{n_\nu}(t)|Q(t)|^3\,dt + \int_{-\pi}^{\pi} |T_{n_\nu}(t)Q(t)^3|\,|Q(t) - Q_{n_\nu}(t)|\,dt \\
&\leq \left(\int_{-\pi}^{\pi} T_{n_\nu}(t)Q_{n_\nu}(t)\,dt\right)\left(\max_{t\in[-\pi,\pi]} |Q(t)|^3\right) \\
&\qquad + \left(\int_{-\pi}^{\pi} |T_{n_\nu}(t)Q(t)^3|\,dt\right)\left(\max_{t\in[-\pi,\pi]} |Q(t) - Q_{n_\nu}(t)|\right) \\
&\leq c_5 + c_6(\log K_\nu)\varepsilon_\nu \leq c_5 + c_6(\log(c_1 L_\nu))\varepsilon_\nu \\
&\leq c_7 + c_6(\log L_\nu)\varepsilon_\nu = o(\log L_\nu)\,,
\end{aligned} \tag{3.9}$$

where c_1, c_5, c_6, and c_7 are constants independent of $\nu \in \mathbb{N}$. Since (3.9) contradicts (3.8), the proof of the theorem is finished in the case when the sequence $(a_n)_{n=0}^\infty$ is not eventually periodic. □

Proof of the theorem when $(a_n)_{n=0}^\infty$ is eventually periodic. The theorem now follows from Lemmas 3.4 below. □

To prove the theorem in the case when $(a_n)_{n=0}^\infty$ is eventually periodic we need one more lemma.

Lemma 3.4. *Let $(a_j)_{j=0}^{\infty}$ be a an eventually periodic sequence of real numbers. Suppose the set $\{j \in \mathbb{N} : a_j \neq 0\}$ is infinite. Then the trigonometric polynomials*

$$T_n(t) := \sum_{j=0}^{n} a_j \cos(jt)$$

have at least $c_8 \log n$ zeros in the period $[-\pi, \pi)$ with a constant $c_8 > 0$ depending only on $(a_j)_{j=0}^{\infty}$.

Proof. It is a well known classical result that for the trigonometric polynomials

$$Q_n(t) := \sum_{j=1}^{n} \frac{\sin(jt)}{j}$$

we have

$$|Q_n(t)| \leq 1 + \pi\,, \qquad t \in \mathbb{R}, \quad n = 1, 2, \ldots\,.$$

Using the standard way to show this, it can be easily shown that if $(a_j)_{j=0}^{\infty}$ is an eventually periodic sequence of real numbers, then for the functions

$$S_n(t) := a_0 t + \sum_{j=1}^{n} \frac{a_j \sin(jt)}{j}$$

we have

$$|S_n(t)| \leq M\,, \qquad t \in [-\pi, \pi), \quad n = 1, 2, \ldots\,, \tag{3.10}$$

with a constant $M > 0$ depending only on $(a_j)_{j=0}^{\infty}$. Observe that $S_n'(t) = T_n(t)$, so Lemma 3.1 (a consequence of the resolution of the Littlewood Conjecture) implies that

$$V_{-\pi}^{\pi}(S_n) = \int_{-\pi}^{\pi} |S_n'(t)|\, dt = \int_{-\pi}^{\pi} |T_n(t)|\, dt \geq \eta \log n \tag{3.11}$$

with a constant $\eta > 0$ depending only on $(a_j)_{j=0}^{\infty}$. Combining (3.10) and (3.11) we can easily deduce that there is a $c \in [-M, M]$ such that $S_n - c$ has at least $(2M)^{-1}(\eta \log n)$ distinct zeros in the period $[-\pi, \pi)$. Hence by Rolle's Theorem $T_n = (S_n - c)'$ has at least $(2M)^{-1}(\eta \log n) - 1$ distinct zeros in the period $[-\pi, \pi)$. $\square$

In fact, in the case when $(a_n)_{n=0}^{\infty}$ is eventually periodic and it does not contain 0 infinitely many times, then we can prove a better lower bound in the theorem. This is the content of Theorem 3.6. To prove Theorem 3.6 we need the observation below.

Lemma 3.5. *Suppose $k > 2m \geq 0$, k is even. Let*

$$z_j := \exp\left(\frac{2\pi j i}{k}\right), \qquad j = 0, 1, \ldots, k-1\,,$$

be the kth roots of unity. Suppose

$$0 \notin \{b_0, b_1, \dots, b_{k-1}\}$$

and

$$Q(z) := z^m \sum_{j=0}^{k-1} b_j z^j .$$

Then there is a value of $j \in \{0, 1, \dots k-1\}$ *for which* $\mathrm{Im}(Q(z_j)) \neq 0$.

Proof. If the statement of the lemma were false, then

$$z^{m+k-1}(Q(z) - Q(1/z)) = (z^k - 1) \sum_{\nu=0}^{2m+k-2} \alpha_\nu z^\nu .$$

Obviously

$$z^{m+k-1}(Q(z) - Q(1/z)) = -b_{k-1} - b_{k-2}z - b_{k-3}z^2 - \cdots - b_0 z^{k-1} + \\ + b_0 z^{2m+k-1} + b_1 z^{2m+k} + b_2 z^{2m+k+1} + \cdots + b_{k-1} z^{2m+2k-2} .$$

Hence

$$\alpha_\nu = -b_{k-1-\nu} , \qquad \nu = 0, 1, \dots, k-1 ,$$

and

$$\alpha_{2m+k-2-\nu} = b_{k-1-\nu} , \qquad \nu = 0, 1, \dots, k-1 .$$

Then for $\nu := m + (k/2) - 1 < k - 1$ we have

$$-b_{k-1-\nu} = b_{k-1-\nu} , \qquad \text{that is} \quad b_{k-1-\nu} = 0 ,$$

a contradiction. □

Theorem 3.6. *Let* $0 \notin \{b_0, b_1, \dots, b_{k-1}\}$ *and*

$$a_{m+lk+j} = b_j , \qquad l = 0, 1, \dots , \quad j = 0, 1, \dots, k-1 .$$

Suppose $k > 2m \geq 0$, k *is even. Let* $n = m + lk + u$ *with integers* $m \geq 0$, $l \geq 0$, $k \geq 1$, *and* $0 \leq u \leq k-1$. *Then for every sufficiently large* n

$$T_n(t) := \mathrm{Re} \left(\sum_{j=0}^{n} a_j e^{ijt} \right)$$

has at least $c_9 n$ *zeros in* $[-\pi, \pi)$, *where* $c_9 > 0$ *is independent of* n.

Proof of Theorem 3.6. Note that

$$\sum_{j=0}^{n} a_j z^j = \sum_{j=0}^{m-1} a_j z^j + z^m \left(\sum_{j=0}^{k-1} b_j z^j \right) \frac{z^{(l+1)k} - 1}{z^k - 1} + z^{m+lk} \sum_{j=0}^{u} b_j z^j = P_1(z) + P_2(z) ,$$

where

$$P_1(z) := \sum_{j=0}^{m-1} a_j z^j + z^{m+lk} \sum_{j=0}^{u} b_j z^j$$

and

$$P_2(z) := z^m \sum_{j=0}^{k-1} b_j z^j \frac{z^{(l+1)k} - 1}{z^k - 1} = Q(z) \frac{z^{(l+1)k} - 1}{z^k - 1}\,,$$

with

$$Q(z) := z^m \sum_{j=0}^{k-1} b_j z^j\,.$$

By Lemma 3.5 there is a kth root of unity $\xi = e^{i\tau}$ such that $\mathrm{Im}(Q(\xi)) \neq 0$. Then for every $K > 0$ there is a $\delta \in (0, 2\pi/k)$ such that $\mathrm{Re}(P_2(e^{it}))$ oscillates between $-K$ and K at least $c_{10}(l+1)k\delta$ times, where $c_{10} > 0$ is a constant independent of n. Now we choose $\delta \in (0, 2\pi/k)$ for

$$K := 1 + \sum_{j=0}^{m-1} |a_j| + \sum_{j=0}^{k-1} |b_j|\,.$$

Then

$$T_n(t) := \mathrm{Re}\left(\sum_{j=0}^{n} a_j e^{ijt}\right) = \mathrm{Re}(P_1(e^{it})) + \mathrm{Re}(P_2(e^{it}))$$

has at least one zero on each interval on which $\mathrm{Re}(P_2(e^{it}))$ oscillates between $-K$ and K, and hence it has at least $c_{10}(l+1)k\delta > c_9 n$ zeros on $[-\pi, \pi)$, where $c_9 > 0$ is a constant independent of n. $\square$

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