A VARIANT OF LIOUVILLE'S LAMBDA FUNCTION: SOME SURPRIZING FORMULAE

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Abstract. Let

$$\lambda_3(n) = (-1)^{\omega_3(n)}$$

where $\omega_3(n)$ is the number of distinct prime factors congruent to -1 mod 3 in n (with multiple factors counted multiply). We give explicit closed form evaluations of the following variety.

Theorem 0.1.

$$\lambda_3(1) + \lambda_3(2) + \cdots + \lambda_3(n) = D_n$$

where D_n be the number of 1's in the base three expansion of n.

Note that above sum grows logarithmically, equals k for the first time when $n = 3^0 + 3^1 + 3^2 + \cdots + 3^k$ and is never negative.

More generally let χ_p denote the Legendre character and let

$$\lambda_p(n) := (-1)^{\omega_p(n)}$$

where $\omega_p(n)$ is the number of distinct prime factors q with $\chi_p(q) = -1$ (with multiple factors counted multiply). We give analogous formulae for $\lambda_p(1) + \lambda_p(2) + \ldots + \lambda_p(n)$.

Theorem 0.2. *For $p = 5$*

$$\lambda_5(1) + \lambda_5(2) + \ldots + \lambda_5(n) = D_n$$

where D_n be the number of 1's in the base five expansion of n minus the number of 3's in the base five expansion of n.

While the analysis, as usual, conceals the approach all these results where found experimentally.

1. Introduction

The Riemann zeta function is defined, for $\Re s > 1$, by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Definition 1.1 (Liouville Function)**.** Let

$$\lambda(n) = (-1)^{\omega(n)}$$

where $\omega(n)$ is the number of distinct prime factors in n (with multiple factors counted multiply). The function $\lambda(n)$ is called the Liouville function.

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So $\lambda(1) = \lambda(4) = \lambda(6) = \lambda(9) = \lambda(10) = 1$ and $\lambda(2) = \lambda(5) = \lambda(7) = \lambda(8) = -1$.

Note that ω is completely additive and λ is completely multiplicative.

For any completely multiplicative function $\alpha(n)$, it follows from the Euler product formula that, for $\Re(s) \geq s_0$,

$$\prod_p \left(1 - \alpha(p)p^{-s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} \tag{1.1}$$

where the product is over all the primes p.

It is well known (Hardy and Wright p 255) for any $\Re(s) > 1$,

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \tag{1.2}$$

In fact, this follows easily from (1.1) with $\alpha(n) = \lambda(n)$,

$$\frac{\zeta(2s)}{\zeta(s)} = \prod_p (1 + p^{-s})^{-1} = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} - \frac{1}{5^s} - \cdots = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \tag{1.3}$$

because $\lambda(p) = -1$ for any prime p.

Also there is a Lambert series due to Ramanujan (e.g. p.100 of [1])

$$\sum_{n=1}^{\infty} \frac{\lambda(n)x^n}{1 - x^n} = \sum_{n=1}^{\infty} x^{n^2}. \tag{1.4}$$

So

$$\sum_{n=1}^{\infty} \frac{\lambda(n)x^n}{1 + x + \cdots + x^{n-1}} = (1 - x) \sum_{n=1}^{\infty} x^{n^2}. \tag{1.5}$$

It is a result of Landau that the rate of growth of sums of $\lambda(i)$ is equivalent to the Riemann Hypothesis.

Theorem 1.2. *The Riemann Hypothesis is equivalent to*

$$\lambda(1) + \lambda(2) + \ldots + \lambda(n) \ll n^{1/2+\epsilon}$$

for every positive ϵ.

2. A Character Sum

Let χ be a non-principal Dirichlet character modulo q. Let

$$S_\chi(x) := \sum_{1 \leq i \leq x} \chi(i).$$

The best bound known for the character sums, $S_\chi(x)$, was given independently by G. Pólya and I.M. Vinogradov in 1918 that (e.g. see §23 of [2])

$$S_\chi(x) \ll \sqrt{q} \log q. \tag{2.1}$$

The Pólya-Vinogradov inequality (2.1) is close to best possible, for Schur proved that

$$\max_x |S_\chi(x)| > \frac{1}{2\pi} \sqrt{q}$$

for all primitive χ modulo q and Paley showed that

$$\max_x |S_\chi(x)| > \frac{1}{7}\sqrt{q}\log\log q$$

for $\chi(n) = \left(\frac{n}{q}\right)$, Kronecker symbol, and for infinitely many quadratic discriminiants $q > 0$. Also, Montgomery and Vaughan proved in [4] that assuming the generalized Riemann hypothesis,

$$S_\chi(x) \ll \sqrt{q}\log\log q$$

for all non-principal character χ. There has been no subsequent improvements in the Pólya-Vinogradov inequality other than in the implicit constant in (2.1) until recently Granville and Soundararajan proved in [3] that if $\chi \pmod q$ is a primitive character of odd order g, then

$$S_\chi(x) \ll_g \sqrt{q}(\log q)^{1-\delta_g/2+o(1)} \tag{2.2}$$

where $\delta_g := 1 - (g/\pi)\sin(\pi/g)$.

We now restrict our character χ to be a non-principal character modulo p for prime p. We define $f(p) := 1$ and $f(q) := \chi(q)$ if q is a prime other than p and extend f to be completely multiplicative and

$$f(p^l m) = \chi(m) \tag{2.3}$$

for $l \geq 0$ and $p \nmid m$. Define the L-series associated to f by

$$\mathcal{L}_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Then we have

$$\begin{aligned}
\mathcal{L}_f(s) &= \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \\
&= \sum_{l=0}^{\infty} \sum_{\substack{n=1 \\ p\nmid n}}^{\infty} \frac{f(p^l n)}{(p^l n)^s} \\
&= \sum_{l=0}^{\infty} \frac{1}{p^{ls}} \times \sum_{\substack{n=1 \\ p\nmid n}}^{\infty} \frac{\chi(n)}{n^s} \\
&= \left(1 - \frac{1}{p^s}\right)^{-1} \times \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \\
&= L_\chi(s)\left(1 - \frac{1}{p^s}\right)^{-1} \qquad (2.4)
\end{aligned}$$

where $L_\chi(s)$ is the L-Dirichlet series associated to χ

$$L_\chi(s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Formula (2.4) defines the analytic continuation of $\mathcal{L}_p(s)$. Therefore the non-trivial zeros of $\mathcal{L}_p(s)$ are precisely those of $L_p(s)$ and hence is expected to satisfy the generalized Riemann Hypothesis.

Proposition 2.1. *For $\epsilon > 0$ and $x \geq 1$, we have*

$$S_f(x) \ll x^\epsilon$$

where $S_f(x) := \sum_{1 \leq j \leq x} f(j)$.

Proof. Similar to $\lambda(n)$, we have

$$\mathcal{L}_f(s) = \frac{1}{1-p^{-s}} L_\chi(s) = \frac{1}{1-p^{-s}} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = s \int_0^\infty S_f(x) x^{-s-1} dx. \tag{2.5}$$

Since the left hand side of (2.5) is analytic when $\Re(s) > 0$, so

$$S_f(x) \ll x^\epsilon$$

for any $\epsilon > 0$. □

Let $N(n,k)$ be the number of digits k in the base p expansion of n.

Proposition 2.2. *We have*

$$\sum_{j=1}^{n} f(j) = \sum_{k=0}^{p-1} N(n,k) S_\chi(k). \tag{2.6}$$

Proof. We proceed by induction on n. For $n = 1$, it is clear that

$$f(1) = S_\chi(1) = \chi(1).$$

Suppose that

$$\sum_{j=1}^{n} f(j) = \sum_{k=0}^{p-1} N(n,k) S_\chi(k).$$

Write

$$n = a_0 + a_1 p + a_2 p^2 + \ldots + a_k p^k$$

where $0 \leq a_j \leq p-1$.

If $0 \leq a_0 \leq p-2$, then we have

$$N(n+1, a_0) = N(n, a_0) - 1$$

$$N(n+1, a_0+1) = N(n, a_0+1) + 1$$

and

$$N(n+1, k) = N(n,k)$$

for $k \neq a_0, a_0+1$. It follows that

$$\begin{aligned}
\sum_{j=1}^{n+1} f(j) &= \sum_{k=0}^{p-1} N(n,k) S_\chi(k) + f(n+1) \\
&= \sum_{k=0}^{p-1} N(n+1,k) S_\chi(k) + S_\chi(a_0) - S_\chi(a_0+1) + f(a_0+1) \\
&= \sum_{k=0}^{p-1} N(n+1,k) S_\chi(k).
\end{aligned}$$

This proves (2.6) if $0 \leq a_0 \leq p-2$.

Suppose $a_0 = p-1$. Write $n+1 = p^l((a_l+1)+a_{l+1}p+\ldots+a_kp^{k-l})$ for $k \geq l \geq 1$ and $a_l \neq p-1$. Then

$$n = (p-1) + (p-1)p + \ldots + (p-1)p^{l-1} + a_lp^l + a_{l+1}p^{l+1} + \ldots + a_kp^k$$

and

$$n+1 = 0 + 0p + \ldots + 0p^{l-1} + (a_l+1)p^l + a_{l+1}p^{l+1} + \ldots + a_kp^k.$$

Note that $f(n+1) = f(a_l+1) = \chi(a_l+1)$.

If $a_l = p-2$, then

$$N(n+1,k) - N(n,k) = \begin{cases} l & \text{if } k = 0; \\ -l+1 & \text{if } k = p-1; \\ -1 & \text{if } k = p-2; \\ 0 & \text{if } k \neq 0, p-2, p-1 \end{cases}$$

and hence

$$\begin{aligned} \sum_{j=1}^{n+1} f(j) &= \sum_{k=0}^{p-1} N(n,k)S_\chi(k) + f(n+1) \\ &= \sum_{k=0}^{p-1} N(n+1,k)S_\chi(k) - lS_\chi(0) - (-l+1)S_\chi(p-1) - (-1)S_\chi(p-2) + f(a_l+1) \\ &= \sum_{k=0}^{p-1} N(n+1,k)S_\chi(k) - f(p-1) + f(p-1) \end{aligned}$$

because $S_\chi(0) = S_\chi(p-1) = 0$.

If $a_l = 0$, then

$$N(n+1,k) - N(n,k) = \begin{cases} l-1 & \text{if } k = 0; \\ 1 & \text{if } k = 1; \\ -l & \text{if } k = p-1; \\ 0 & \text{if } k \neq 0, 1, p-1 \end{cases}$$

and hence

$$\begin{aligned} \sum_{j=1}^{n+1} f(j) &= \sum_{k=0}^{p-1} N(n,k)S_\chi(k) + f(n+1) \\ &= \sum_{k=0}^{p-1} N(n+1,k)S_\chi(k) - (l-1)S_\chi(0) - S_\chi(1) - (-l)S_\chi(p-1) + f(1) \\ &= \sum_{k=0}^{p-1} N(n+1,k)S_\chi(k). \end{aligned}$$

If $a_l \neq 0, p-2, p-1$, then

$$N(n+1,k) - N(n,k) = \begin{cases} -l & \text{if } k = p-1; \\ l & \text{if } k = 0; \\ -1 & \text{if } k = a_l; \\ 1 & \text{if } k = a_l + 1; \\ 0 & \text{otherwise} \end{cases}$$

and hence

$$\begin{aligned} \sum_{j=1}^{n+1} f(j) &= \sum_{k=0}^{p-1} N(n,k) S_\chi(k) + f(n+1) \\ &= \sum_{k=0}^{p-1} N(n+1,k) S_\chi(k) - l S_\chi(0) - (-l) S_\chi(p-1) - (-1) S_\chi(a_l) - S_\chi(a_l+1) + f(a_l+1) \\ &= \sum_{k=0}^{p-1} N(n+1,k) S_\chi(k). \end{aligned}$$

This completes the proof. □

3. The Modified Liouville Function, λ_p

In this section, we further concentrate the character on the Legendre symbol. Let $\chi_p(n)$ be the Legendre symbol modulo p and $L_p(s)$ be the Dirichlet L-function associated with χ_p,

$$L_p(s) := \sum_{n=1}^{\infty} \frac{\chi_p(n)}{n^s}$$

We define the modified Liouville function $\lambda_p(n)$ by

$$\lambda_p(n) := (-1)^{\omega_p(n)}$$

where $\omega_p(n)$ is the number of distinct prime factors q with $\chi_p(q) = -1$. This is the Liouville function for the quadratic residue modulo p.

Corollary 3.1. *Let*

$$n = a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k$$

be the base p expansion of n, where $a_j \in \{0, 1, 2, \cdots, p-1\}$. Then we have

$$\sum_{l=1}^{n} \lambda_p(l) = \sum_{l=1}^{a_0} \lambda_p(l) + \sum_{l=1}^{a_1} \lambda_p(l) + \cdots + \sum_{l=1}^{a_k} \lambda_p(l). \tag{3.1}$$

Proof. This follows readily from Proposition 2.2 with $\chi = \chi_p$. □

Theorem 3.2. *For p prime.*

$$S_p(n) := \lambda_p(1) + \lambda_p(2) + \ldots + \lambda_p(n) \geq 0$$

for all n exactly for those odd primes p for which

$$\chi_p(1) + \chi_p(2) + \ldots + \chi_p(k) \geq 0$$

for all $1 \le k \le p$. *Such* n *are*

$$\{3, 7, 11, 23, 31, 47, 59, 71, 79, 83, 103, 131, 151, 167, 191, 199, 239, 251\ldots\}$$

Proof. We first observe from (2.3) that if $0 \le r < p$, then

$$\sum_{l=1}^{r} \lambda_p(l) = \sum_{l=1}^{r} \chi_p(l).$$

From Corollary 3.1,

$$\begin{aligned}\sum_{l=1}^{n} \lambda_p(l) &= \sum_{l=1}^{a_0} \lambda_p(l) + \sum_{l=1}^{a_1} \lambda_p(l) + \cdots + \sum_{l=1}^{a_k} \lambda_p(l) \\ &= \sum_{l=1}^{a_0} \chi_p(l) + \sum_{l=1}^{a_1} \chi_p(l) + \cdots + \sum_{l=1}^{a_k} \chi_p(l)\end{aligned}$$

because all a_j are between 0 and $p-1$. On the other hand,

$$\sum_{l=1}^{n} \chi_p(l) = \sum_{l=1}^{a_0} \chi_p(l) = \sum_{l=1}^{a_0} \lambda_p(l)$$

The result then follows. □

In particular, for $p = 3$, we have

Corollary 3.3.

$$S_3(n) := \lambda_3(1) + \lambda_3(2) + \ldots + \lambda_3(n) = D_3(n)$$

where $D_3(n)$ *be the number of 1's in the base 3 expansion of* n.

Note that $S_n = k$ for the first time when $n = 3^0 + 3^1 + 3^2 + \ldots + 3^k$ and is never negative.

From Proposition 2.1, we have

$$S_3(n) \ll n^{\epsilon}$$

for any $\epsilon > 0$. However, from Corollary 3.3, we obtain better estimate.

Corollary 3.4. *For* $n \ge 1$, *we have*

$$0 \le S_3(n) \le [\log_3 n] + 1.$$

Proof. This follows from Corollary 3.3 and the fact that the number of 1's in the base three expansion of n is $\le [\log_3 n] + 1$. □

For the case of $p = 5$, we have

Corollary 3.5. *We have*

$$S_5(n) := \lambda_5(1) + \lambda_5(2) + \ldots + \lambda_5(n) = D_5(n)$$

where $D_5(n)$ *be the number of 1's in the base 5 expansion of* n *minus the number of 3's in the base 5 expansion of* n. *Also for* $n \ge 1$, *we have*

$$|S_5(n)| \le [\log_5 n] + 1$$

Note that λ_3 is always nonnegative but λ_5 isn't. Also $S_5(n) = k$ for the first time when $n = 5^0 + 5^1 + 5^2 + \ldots + 5^k$ and $S_5(n) = -k$ for the first time when $n = 3 \cdot 5^0 + 3 \cdot 5^1 + 3 \cdot 5^2 + \ldots + 3 \cdot 5^k$.

4. The Modified Mobiüs Function, μ_p

Let

$$\mu_p(n) := (-1)^{\omega_p^*(n)}$$

where $\omega_p^*(n)$ is the number of distinct prime factors q with $\chi_p(q) = -1$ and where q^2 does not divide n. The function μ_p is similar to λ_p but defined on the complementary "residues". Note that if $(n, p) = 1$ then

$$\mu_p(n)\lambda_p(n) = \mu(n)\lambda(n)$$

otherwise

$$\mu_p(n)\lambda_p(n) = -\mu(n)\lambda(n)$$

In particular, if $p \neq q$, then $\mu_p(q^l) = \mu(q^l)^2\chi_p(q^l)$ and $\mu_p(p^l) = -\mu(p^l)^2$. Define

$$\mathcal{L}_p^*(s) := \sum_{n=1}^{\infty} \frac{\mu_p(n)}{n^s}.$$

Since

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\mu_p(n)}{n^s} &= \prod_q \left\{1 + \frac{\mu_p(q)}{q^s}\right\} \\
&= \left(1 - \frac{1}{p^s}\right) \prod_{q \neq p} \left\{1 + \frac{\chi_p(q)}{q^s}\right\} \\
&= \left(1 - \frac{1}{p^s}\right) \prod_q \left\{1 + \frac{\chi_p(q)}{q^s}\right\} \\
&= \left(1 - \frac{1}{p^s}\right) \prod_q \left\{1 - \frac{\chi_p(q)^2}{q^{2s}}\right\} \left\{1 - \frac{\chi_p(q)}{q^s}\right\}^{-1} \\
&= \frac{1 - \frac{1}{p^s}}{1 - \frac{1}{p^{2s}}} \prod_q \left(1 - \frac{1}{q^{2s}}\right) \left(1 - \frac{\chi_p(q)}{q^s}\right)^{-1} \\
&= \frac{L_p(s)}{\zeta(2s)(1 + 1/p^s)}
\end{aligned}$$

this is analytic in the region $0 < \Re(s) < 1$ and hence for any $\epsilon > 0$ we have

$$\sum_{n \leq x} \mu_p(n) \ll x^{\epsilon}.$$

However, similar to λ_p, computational data suggests that

$$\sum_{n \leq x} \mu_p(n) = O(\log(x)).$$

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