NIKOLSKII-TYPE INEQUALITIES FOR SHIFT INVARIANT FUNCTION SPACES

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Abstract. In this note we prove the following.

Theorem. *Let V_n be a shift invariant vectorspace of complex-valued functions defined on $\mathbb{R}$ of dimension $n+1$ over $\mathbb{C}$. Let $p \in (0,2]$. Then*

$$\|f\|_{[a+\delta,b-\delta]} \leq 2^{2/p^2} \left(\frac{n+1}{\delta}\right)^{1/p} \|f\|_{L_p[a,b]}$$

for every $f \in V_n$ and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$.

Let V_n be a vectorspace of complex-valued functions defined on $\mathbb{R}$ of dimension $n+1$ over $\mathbb{C}$. We say that V_n is shift invariant (on $\mathbb{R}$) if $f \in V_n$ implies that $f_a \in V_n$ for every $a \in \mathbb{R}$, where $f_a(x) := f(x-a)$ on $\mathbb{R}$. Let $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ be a set of distinct **complex** numbers. The collection of all linear combinations of of $e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}$ over $\mathbb{C}$ will be denoted by

$$E(\Lambda_n) := \text{span}\{e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}\}.$$

Elements of $E(\Lambda_n)$ are called exponential sums of $n+1$ terms. Examples of shift invariant spaces of dimension $n+1$ include $E(\Lambda_n)$. In this note we prove the following.

Theorem. *Let V_n be a shift invariant vectorspace of complex-valued functions defined on $\mathbb{R}$ of dimension $n+1$ over $\mathbb{C}$. Let $p \in (0,2]$. Then*

$$\|f\|_{[a+\delta,b-\delta]} \leq 2^{2/p^2} \left(\frac{n+1}{\delta}\right)^{1/p} \|f\|_{L_p[a,b]}$$

for every $f \in V_n$ and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$.

Remark It is proved in [2] that if $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ is a set of distinct **real** numbers, then the inequality

$$\|f\|_{[a+\delta,b-\delta]} \leq e8^{1/p} \left(\frac{n+1}{\delta}\right)^{1/p} \|f\|_{L_p[a,b]}$$

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holds for every $f \in E(\Lambda_n)$, $p > 0$, and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$.

Problem Is it possible to extend a version of the theorem for **all** $p > 0$?

Proof. Since V_n is shift invariant, it is sufficient to prove only that

$$|f(0)| \le 2^{2/p^2 - 1/p}(n+1)\|f\|_{L_p[-2,2]}$$

for every $f \in V_n$. Take an orthonormal basis $(L_k)_{k=0}^n$ on $[-\frac{1}{2}, \frac{1}{2}]$ so that

(1) $$L_k \in V_n\,, \qquad k = 0, 1, \ldots, n\,,$$

and

(2) $$\int_{-1/2}^{1/2} L_j(x)\overline{L_k(x)}\,dx = \delta_{j,k}\,, \qquad 0 \le j \le k \le n\,,$$

where $\delta_{j.k}$ is the Kronecker symbol. On writing $f \in V_n$ as a linear combination of $L_0, L_1, \ldots L_n$, and using the Cauchy-Schwarz inequality and the orthonormality of $(L_k)_{k=0}^n$ on $[-\frac{1}{2}, \frac{1}{2}]$, we obtain in a standard fashion that

$$\max_{0 \ne V_n} \frac{|f(t_0)|}{\|f\|_{L_2[-1/2,1/2]}} = \left(\sum_{k=0}^n |L_k(t_0)|^2\right)^{1/2}, \qquad t_0 \in \mathbb{R}\,.$$

Since

$$\int_{-1/2}^{1/2} \sum_{k=0}^n |L_k(x)|^2\,dx = n+1\,,$$

there exists a $t_0 \in [-\frac{1}{2}, \frac{1}{2}]$ such that

$$\max_{0 \ne V_n} \frac{|f(t_0)|}{\|f\|_{L_2[-1/2,1/2]}} = \left(\sum_{k=0}^n |L_k(t_0)|^2\right)^{1/2} \le \sqrt{n+1}\,.$$

Observe that if $f \in V_n$, then g defined by $g(t) := f(t - t_0)$ is also in V_n, so

(3) $$\max_{0 \ne V_n} \frac{|f(0)|}{\|f\|_{L_2[-1,1]}} \le \sqrt{n+1}\,.$$

We introduce

$$\widetilde{V}_n := \{g : g(t) = f(\lambda t)\,, \quad f \in V_n\,,\ \lambda \in [-2,2]\}\,.$$

It follows from (3) that

$$\max_{0 \ne \widetilde{V}_n} \frac{|f(0)|}{\|f\|_{L_2[-1,1]}} \le \sqrt{n+1}\,.$$

Let

$$C := \max_{0 \neq \widetilde{V}_n} \frac{|f(0)|}{\|f\|_{L_p[-2,2]}}\,.$$

Let $0 \neq f \in \widetilde{V}_n$. We define $g \in \widetilde{V}_n$ by $g(t) = f(t/2 + y)$. Then

$$\frac{|f(y)|}{\|f\|_{L_p[-2,2]}} \leq \frac{|f(y)|}{\|f\|_{L_p[y-1,y+1]}} \leq \frac{|g(0)|}{\|g\|_{L_p[-2,2]}} 2^{1/p} \leq 2^{1/p} C\,, \qquad y \in [-1,1]\,.$$

Hence

$$\max_{0 \neq \widetilde{V}_n} \frac{|f(y)|}{\|f\|_{L_p[-2,2]}} \leq 2^{1/p} C\,, \qquad y \in [-1,1]\,.$$

Therefore, for every $f \in \widetilde{V}_n$,

$$\begin{aligned}
|f(0)| &\leq \sqrt{n+1}\|f\|_{L_2[-1,1]} \\
&\leq \sqrt{n+1}\left(\|f\|^p_{L_p[-1,1]}\|f\|^{2-p}_{[-1,1]}\right)^{1/2} \\
&\leq \sqrt{n+1}\left(\|f\|^p_{L_p[-1,1]}\left(2^{1/p}C\right)^{2-p}\|f\|^{2-p}_{L_p[-2,2]}\right)^{1/2} \\
&\leq \sqrt{n+1}\left(2^{1/p}C\right)^{1-p/2}\|f\|_{L_p[-2,2]} \\
&\leq 2^{1/p-1/2}\sqrt{n+1}C^{1-p/2}\|f\|_{L_p[-2,2]}\,.
\end{aligned}$$

Hence

$$C = \max_{0 \neq \widetilde{V}_n} \frac{|f(0)|}{\|f\|_{L_p[-2,2]}} \leq 2^{1/p-1/2}\sqrt{n+1}C^{1-p/2}$$

and we conclude that

$$C \leq 2^{2/p^2-1/p}(n+1)^{1/p}\,.$$

So

$$|f(0)| \leq 2^{2/p^2-1/p}(n+1)^{1/p}\|f\|_{L_p[-2,2]}$$

for every $f \in \widetilde{V}_n$, and the result follows. □

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