Can. J. Math., Vol. XXXII, No. 2, 1980, pp. 354-361

INEQUALITIES AND INVERSE THEOREMS IN RESTRICTED RATIONAL APPROXIMATION THEORY

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0. Introduction. The following lemma, part a) due to S. N. Bernstein and part b) due to A. A. Markov, is fundamental to the proofs of many inverse theorems in polynomial approximation theory.

LEMMA 1. [**2**, p. 62 and p. 67] *Let Π_n denote the real polynomials of degree at most n. Let $p \in \Pi_n$, then*

a) $$|p'(x)| \leqq \frac{n}{[(x-a)(b-x)]^{1/2}} ||p||_{[a,b]} \quad \textit{and}$$

b) $$|p'(x)| \leqq \frac{2n^2}{b-a} ||p||_{[a,b]}.$$

From the lemma one can deduce, for example:

THEOREM 1. *If there is a sequence of polynomials $p_n \in \Pi_n$ and a $\delta > 0$ so that $||f - p_n||_{[a,b]} \leqq A/n^{k+\delta}$ then f is k times continuously differentiable on (a, b).*

We shall refer to inequalities, such as those of Lemma 1 that bound the derivative r' of a rational function of degree n in terms of its supremum norm $||r||_{[a,b]}$ and n, as Bernstein-type inequalities.

Bernstein-type inequalities do not exist for arbitrary rational functions and neither do inverse theorems of the above type. Consider $r(x) = -\epsilon^2/(x^2 + \epsilon^2)$, then $||r(x)||_{[-1,1]} \leqq 1$ but $r'(\epsilon) = 1/(2\epsilon)$. [**1**, p. 83].

We shall show that for various restricted classes of rational functions, Bernstein-type inequalities hold. In Section 1 we shall develop Bernstein-type inequalities for the following three classes of rational functions: a) rational functions whose denominators are monotone on an interval, b) rational functions whose denominators have positive coefficients and c) rational functions whose denominators have roots bounded away from the interval of approximation.

In Section 2, we derive the corresponding inverse theorems. We obtain, for example:

THEOREM 2. *If there is a sequence of rational functions p_n/q_n with $p_n, q_n \in \Pi_n$ and q_n monotone non-decreasing on $[a, b]$ and a $\delta > 0$, so that $||f - p_n/q_n||_{[a,b]} \leqq A/n^{2k+\delta}$ then f is k times continuously differentiable on $(a, b]$.*

Received April 6, 1978.

We adopt the following notation. Let R_n denote the real rational functions p_n/q_n where $p_n, q_n \in \Pi_n$. Let R_n^+ denote those rational functions $p_n/q_n \in R_n$ where q_n has non-negative coefficients. Let $R_n^{\uparrow}[a, b]$ denote those rational functions $p_n/q_n \in R_n$ where q_n is non-decreasing on $[a, b]$.

1. Bernstein-type inequalities.

INEQUALITY 1. *Let* $r = p/q \in R_n^{\uparrow}[a, b]$. *Then, if* $0 < \epsilon < b - a$,

a) $$\|r'\|_{[a+\epsilon,b]} \leqq \frac{4n^2}{\epsilon} \|r\|_{[a,b]} \quad \textit{and}$$

b) $$\|r^{(m)}\|_{[a+\epsilon,b]} \leqq [m^n 4^{(m(m+1)/2)}] \frac{n^{2m}}{\epsilon^m} \|r\|_{[a,b]}.$$

Proof. a) Let $0 < \epsilon < b - a$, let ζ be a point where $|r'(\zeta)| = \|r'\|_{[a+\epsilon,b]}$, $a + \epsilon \leqq \zeta \leqq b$, and let t be a point where

$$|p(t)| = \|p\|_{[a,\zeta]}, \quad a \leqq t \leqq \zeta.$$

Then

$$r'(\zeta) = \frac{p'(\zeta)}{q(\zeta)} - \frac{q'(\zeta)}{q(\zeta)} r(\zeta).$$

From Lemma 1 b) and the monotonicity of q it follows that

$$\frac{|p'(\zeta)|}{|q(\zeta)|} \leqq \frac{2n^2\|p\|_{[a,\zeta]}}{(\zeta - a)|q(\zeta)|} = \frac{2n^2|p(t)|}{(\zeta - a)|q(\zeta)|}$$
$$\leqq \frac{2n^2}{(\zeta - a)} \frac{|p(t)|}{|q(t)|} \leqq \frac{2n^2}{\epsilon} |r(t)|$$

and that

$$\frac{|q'(\zeta)|}{|q(\zeta)|} |r(\zeta)| \leqq \frac{2n^2}{\epsilon} \frac{\|q\|_{[a,\zeta]}}{|q(\zeta)|} |r(\zeta)|$$
$$\leqq \frac{2n^2}{\epsilon} |r(\zeta)|.$$

Thus,

$$\|r'\|_{[a+\epsilon,b]} = |r'(\zeta)| \leqq 4n^2\epsilon^{-1}\|r\|_{[a,b]}.$$

b) Note that $r^{(k)} \in R_{n2^k}$ and that $r^{(k)}$ has a monotone denominator if r does. Let $0 < \epsilon < b - a$ and let $\gamma_k = \epsilon k/m$. Then by a)

$$\|r^{(k)}\|_{[a+\gamma_{k-1}+(\gamma_k-\gamma_{k-1}),b]} \leqq \frac{4n^2 2^{2k-2}}{\gamma_k - \gamma_{k-1}} \|r^{(k-1)}\|_{[a+\gamma_{k-1},b]}$$

or

$$\|r^{(k)}\|_{[a+\gamma_k,b]} \leqq mn^2 4^k \epsilon^{-1}\|r^{(k-1)}\|_{[a+\gamma_{k-1},b]}.$$

Thus, by iteration,

$$\|r^{(m)}\|_{[a+\epsilon,b]} \leqq \left(\prod_{k=1}^{m} mn^2 4^k \epsilon^{-1}\right)\|r\|_{[a,b]}$$

$$= [m^m 4^{(m(m+1)/2)}] \frac{n^{2m}}{\epsilon^m} \|r\|_{[a,b]}.$$

The next inequality is a strengthened version of the last inequality that applies to the more restricted class of rational functions R_n^+.

INEQUALITY 2. *Let* $r = p/q \in R_n^+$. *If* $0 \leqq a < \alpha < \beta < b$ *then*

a) $\|r'\|_{[\alpha,\beta]} \leqq \dfrac{n^{3/2} e^{(b-\beta)/\alpha}}{[(\alpha - a)(b - \beta)]^{1/2}} \|r\|_{[a,b]} + \dfrac{n}{\alpha}\|r\|_{[a,b]}$ *and*

b) $\|r^{(m)}\|_{[\alpha,\beta]} \leqq Cn^{3m/2}\|r\|_{[a,b]}$

where C *depends only on* a, α, β, b *and* m.

Proof. Suppose $0 < x < y$ then, since q has non-negative coefficients,

$q(x) = \sum_{m=0}^{n} |a_m|x^m$ and

$$(1)\quad \frac{x^n}{y^n} q(y) = \sum_{m=0}^{n} (|a_m|y^m)\frac{x^n}{y^n}$$

$$= \sum_{m=0}^{n} \frac{x^{n-m}}{y^{n-m}} |a_m|x^m \leqq q(x).$$

Also, if $x > 0$ then

$$(2)\quad xq'(x) = x\sum_{m=0}^{n} m|a_m|x^{m-1} \leqq nq(x).$$

Let ζ be a point where $|r'(\zeta)| = \|r'\|_{[\alpha,\beta]}$. Then by (2)

$$(3)\quad \|r'\|_{[\alpha,\beta]} \leqq \frac{|p'(\zeta)|}{|q(\zeta)|} + \frac{|q'(\zeta)|}{|q(\zeta)|}|r(\zeta)|$$

$$\leqq \frac{|p'(\zeta)|}{|q(\zeta)|} + \frac{n}{\alpha}\|r\|_{[a,b]}.$$

Set $\gamma = b - \beta$ then, by Lemma 1 a),

$$(4)\quad \frac{|p'(\zeta)|}{|q(\zeta)|} \leqq \frac{n}{[(\zeta - a)(\zeta + \gamma/n - \zeta)]^{1/2}} \frac{\|p\|_{[a,\zeta+\gamma/n]}}{|q(\zeta)|}$$

$$= \frac{n}{\left[(\zeta - a)\left(\frac{b-\beta}{n}\right)\right]^{1/2}} \frac{\|p\|_{[a,\zeta+\gamma/n]}}{|q(\zeta)|}.$$

From (1) and (4)

$$(5)\quad \frac{|p'(\zeta)|}{|q(\zeta)|} \leqq \frac{n^{3/2}}{[(\alpha-a)(b-\beta)]^{1/2}} \frac{||p||_{[a,\zeta+\gamma/n]}}{|q(\zeta+\gamma/n)|} \left(\frac{\zeta+\gamma/n}{\zeta}\right)^n$$

$$\leqq \frac{n^{3/2}}{[(\alpha-a)(b-\beta)]^{1/2}} ||r||_{[a,\zeta+\gamma/n]} e^{\gamma/\alpha}.$$

Part a) now follows from (3) and (5).

Part b) is deduced from part a) by a similar iteration to that used in the proof of Inequality 1 b).

We now consider rational functions with restricted poles. Let $D(x, \delta)$ be the closed disc in C with (real) center at x and radius δ.

INEQUALITY 3. *Let $r = p/q \in R_n$ and suppose that q has no roots in $D(x, \epsilon + \delta)$ where $\delta \leqq \epsilon/(k+1)$ for some integer k. Then*

$$||r'||_{[x-\delta,x+\delta]} \leqq 2n\delta^{-1} e^{2n/k} ||r||_{[x-\epsilon-\delta,x+\epsilon+\delta]}.$$

We need the following lemma in the proof of Inequality 3.

LEMMA 2. *Suppose $q \in \Pi_n$ and suppose q has no roots in $D(x, (k+1)\epsilon)$ for some integer k. Then supremum$_{z,w\in D(x,\epsilon)}$ $|q(z)|/|q(w)| \leqq e^{2n/k}$.*

Proof. Let z_0 and w_0 be points in $D(x, \epsilon)$ where

$$|q(z_0)| = \max_{z\in D(x,\epsilon)} |q(z)| \quad \text{and} \quad |q(w_0)| = \min_{z\in D(x,\epsilon)} |q(z)|.$$

Suppose $q(z) = a\prod_{m=1}^{n} (x + \alpha_m)$. Then

$$\frac{|q(z_0)|}{|q(w_0)|} = \prod_{m=1}^{n} \frac{|z_0+\alpha_m|}{|w_0+\alpha_m|} = \prod_{m=1}^{n} \left|1 + \frac{z_0 - w_0}{w_0+\alpha_m}\right|.$$

Since $z_0, w_0 \in D(x, \epsilon)$ and $\alpha_m \notin D(x, (k+1)\epsilon)$,

$$\frac{|q(z_0)|}{|q(w_0)|} \leqq \prod_{m=1}^{n} \left(1 + \frac{2\epsilon}{k\epsilon}\right) \leqq e^{2n/k}.$$

Proof of Inequality 3. Let ζ be a point where $|r'(\zeta)| = ||r'||_{[x-\delta,x+\delta]}$. Now

$$|r'(\zeta)| \leqq \frac{|p'(\zeta)|}{|q(\zeta)|} + \frac{|q'(\zeta)|}{|q(\zeta)|} |r(\zeta)|.$$

By Lemma 1 a)

$$\frac{|p'(\zeta)|}{|q(\zeta)|} \leqq \frac{n}{\delta} \frac{||p||_{[\zeta-\delta,\zeta+\delta]}}{|q(\zeta)|}$$

$$\leqq \frac{n}{\delta} ||r||_{[\zeta-\delta,\zeta+\delta]} \left(\sup_{z,w\in D(\zeta,\delta)} \frac{|q(z)|}{|q(w)|}\right).$$

Since $\delta \leqq \epsilon/(k+1)$ and $\zeta \in [x-\delta, x+\delta]$, an application of Lemma 2 to the above inequality yields

$$\frac{|p'(\zeta)|}{|q(\zeta)|} \leqq \frac{n}{\delta} \|r\|_{[\zeta-\delta,\zeta+\delta]} e^{2n/k}.$$

Similarly, by Lemma 1 a) and Lemma 2,

$$\frac{|q'(\zeta)|}{|q(\zeta)|} |r(\zeta)| \leqq \frac{n}{\delta} \frac{\|q\|_{[\zeta-\delta,\zeta+\delta]}}{|q(\zeta)|} |r(\zeta)|$$

$$\leqq \frac{n}{\delta} e^{2n/k} |r(\zeta)|.$$

Thus

$$\|r'\|_{[x-\delta,x+\delta]} \leqq 2n\delta^{-1} e^{2n/k} \|r\|_{[x-\epsilon-\delta,x+\epsilon+\delta]}.$$

Inequality 4 is an analogue to the following lemma due to S. N. Bernstein. Let $E_\rho(x, \epsilon)$, $\rho \geqq 1$, be the closed ellipse with foci $x - \epsilon$, $x + \epsilon$ and semi-axes $\frac{1}{2}\epsilon(\rho + \rho^{-1})$, $\frac{1}{2}\epsilon(\rho - \rho^{-1})$.

LEMMA 3. [**1**, p. 42] *If* $p \in \Pi_n$ *then* $\|p(z)\|_{E_\rho(x,\epsilon)} \leqq \rho^n \|p\|_{[x-\epsilon,x+\epsilon]}$.

INEQUALITY 4. *Let* $r = p/q \in R_n$ *and suppose that* q *has no roots in* $D(x, (k + 1)\epsilon)$. *Then, if* $\delta \leqq \epsilon$,

$$\|r(z)\|_{D(x,\epsilon) \cap E_\rho(x,\delta)} \leqq \rho^n \|r\|_{[x-\delta,x+\delta]} \cdot e^{2n/k}.$$

Proof.

$$\|p\|_{[x-\delta,x+\delta]} \leqq \|r\|_{[x-\delta,x+\delta]} \|q\|_{[x-\delta,x+\delta]}$$

$$\leqq \|r\|_{[x-\delta,x+\delta]} \|q\|_{D(x,\epsilon)}.$$

Thus, by Lemma 3,

$$\|p\|_{E_\rho(x,\delta)} \leqq \rho^n \|r\|_{[x-\delta,x+\delta]} \|q\|_{D(x,\epsilon)}.$$

By Lemma 2, since q has no roots in $D(x, (k + 1)\epsilon)$,

$$\min_{z \in D(x,\epsilon)} |q(z)| e^{2n/k} \geqq \|q\|_{D(x,\epsilon)}.$$

Thus,

$$\|r\|_{D(x,\epsilon) \cap E_\rho(x,\rho)} \leqq \rho^n \|r\|_{[x-\delta,x+\delta]} \cdot e^{2n/k}.$$

2. Inverse theorems. If a function f is k times continuously differentiable on $[a, b]$ then there is a sequence of polynomials $p_n \in \Pi_n$ so that $\|f - p_n\|_{[a,b]} \leqq A/n^k$ for each n. [**1**, p. 66]. Thus, in light of Theorem 1, $1/n^k$ is, in some senses, the right speed of polynomial approximation for $C^k[a, b]$. The following inverse theorems show that for $C^k[a, b]$ restricted rational approximation cannot be dramatically more efficient. Note that we only deduce that f is k times differentiable on the half open interval $(a, b]$.

THEOREM 3. *Suppose there is a sequence* $r_n \in R_n^{\uparrow}[a, b]$ *such that*

$$\sum_{n=1}^{\infty} ||f - r_n||_{[a,b]} n^{2k-1} < \infty,$$

then $f \in C^k(a, b]$.

Proof. We may assume that $||f - r_n||_{[a,b]}$ is monotone non-increasing. Consider the expansion:

(1) $f(x) = r_1(x) + \sum_{n=0}^{\infty} (r_{2^{n+1}}(x) - r_{2^n}(x))$

and observe that

$$\begin{aligned} ||r_{2^{n+1}} - r_{2^n}||_{[a,b]} &\leqq ||f - r_{2^{n+1}}||_{[a,b]} + ||f - r_{2^n}||_{[a,b]} \\ &\leqq 2||f - r_{2^n}||_{[a,b]}. \end{aligned}$$

Formal differentiation of the right side of (1) yields

(2) $r_1^{(k)}(x) + \sum_{n=0}^{\infty} (r_{2^{n+1}}^{(k)}(x) - r_{2^n}^{(k)}(x))$

where, by Inequality 1 b), for fixed ϵ there is a C independent of n such that

$$||r_{2^{n+1}}^{(k)} - r_{2^n}^{(k)}||_{[a+\epsilon,b]} \leqq C[2^{n+2}]^{2k} ||r_{2^{n+1}} - r_{2^n}||_{[a,b]}.$$

Thus,

$$\begin{aligned} \sum_{k=0}^{\infty} ||r_{2^{n+1}}^{(k)} - r_{2^n}^{(k)}||_{[a+\epsilon,b]} &\leqq \sum_{n=0}^{\infty} C[2^{n+2}]^{2k} 2||f - r_{2^n}||_{[a,b]} \\ &\leqq 2C4^{2k} \sum_{n=0}^{\infty} [2^n]^{2k} ||f - r_{2^n}||_{[a,b]} \end{aligned}$$

which converges because $||f - r_n||_{[a,b]}$ is monotone and

$$\sum_{n=1}^{\infty} ||f - r_n||_{[a,b]} n^{(2k-1)}$$

converges. Thus, (2) converges uniformly on $[a + \epsilon, b]$ and $f \in C^k[a + \epsilon, b]$.

Theorem 2, stated in the introduction, is an immediate corollary to the above result. The next theorem is deduced from Inequality 2 in the same way that Theorem 3 is deduced from Inequality 1.

THEOREM 4. *Suppose* $0 \leqq a$ *and suppose there is a sequence* $r_n \in R_n^+$ *such that*

$$\sum_{n=1}^{\infty} ||f - r_n||_{[a,b]} n^{\frac{3}{2}k-1} < \infty,$$

then $f \in C^k(a, b)$.

The following is an example of the type of inverse theorem attainable from Inequality 3.

THEOREM 5. *Let* $c > 0$ *and suppose there is a sequence* $r_n = p_n/q_n \in R_n$ *so* *that no root (complex or real) of q_n lies within distance c/n of $[a, b]$. If*

$$\sum_{n=1}^{\infty} \|f - r_n\|_{[a,b]} n^2 < \infty,$$

then $f \in C^1(a, b)$.

Proof. r_n satisfies the conditions of Inequality 3 with $\epsilon = c/(n+1)$ and $\delta = c/(n+1)^2$. Thus, for each n,

$$\|r_n'\|_{[a+\epsilon, b-\epsilon]} \leqq 2n(n+1)^2 c^{-1} e^2 \|r_n\|_{[a,b]}.$$

The result now follows analogously to Theorem 4.

It is interesting to note the close relationship between Theorem 4 and Theorem 5.

Remark. If $p \in \Pi_n$ has non-negative coefficients then p has no roots in the region $\{z \mid |\arg z| < \pi/n\}$.

Proof. Suppose $p(z) = \sum_{m=0}^{\infty} a_m z^m$, $a_n > 0$, $a_m \geqq 0$, and let ζ be any point where $0 < \arg \zeta < \pi/n$. Then, for each $m \leqq n$, $0 < \arg \zeta^m < \pi$ and hence, $\sum_{m=0}^{n} a_m \zeta^m$ lies in the region $\{z \mid \operatorname{im} z > 0\}$. In particular,

$$\sum_{m=0}^{n} a_m \zeta^m \neq 0.$$

Since p can have no positive real roots, we are finished.

The final result of this section is of a different character. It shows that under sufficiently strong restrictions on the behaviour of q_n, exponential rates of approximation guarantee analyticity.

THEOREM 6. *Suppose $c > 0$ and suppose there is a sequence $r_n = p_n/q_n \in R_n$ so that no root (real or complex) of q_n lies within distance c of $[a, b]$. If $\|f - r_n\|_{[a,b]} \leqq \rho^{-n}$ for some $\rho > 1$, then f is the restriction to $[a, b]$ of some function F analytic in a region containing $[a, b]$.*

Proof. Choose $K > 1$ so that $\rho^{-1/2} e^{4/K} = \beta < 1$. Let $\delta = c/2(K+1)$ and let $\epsilon = 2\delta$. Then applying Inequality 4 to $r_n - r_{n-1} \in R_{2n}$ yields

$$\|r_n - r_{n-1}\|_{D(x,2\delta) \cap E_{\rho^{1/4}}(x,\delta)} \leqq \|r_n - r_{n-1}\|_{[x-\delta, x+\delta]} \cdot \rho^{n/2} e^{4n/K}$$

for each $x \in [a + \delta, b - \delta]$. Set

$$T = \bigcup_{x \in [a+\delta, b-\delta]} \{D(x, 2\delta) \cap E_{\rho^{1/4}}(x, \delta)\}.$$

Then, since $\|r_n - r_{n-1}\|_{[a,b]} \leqq 2\rho^{-(n-1)}$, it follows that

$$|r_n(z) - r_{n-1}(z)| \leqq 2\rho \cdot \rho^{-n} \rho^{n/2} e^{4n/K} \leqq 2\rho \cdot \beta^n$$

for each $z \in T$. Hence, r_n converges uniformly on T, the interior of T contains $[a, b]$ and the result follows.

The next example shows that no speed of approximation from R_n^+ or $R_n^\dagger$

guarantees analyticity and hence there is no analogue of Theorem 6 for these classes.

Example 1. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{|a_n|}{x^n + \left(1 - \frac{1}{n}\right)^n}.$$

Then f is not analytic in any neighbourhood of 1. However, for suitably small a_n, the speed of approximation from R_n^+ on [0, 2] can be made as rapid as desired.

Also, in contrast to polynomial approximation, no speed of approximation from R_n^+ or $R_n^\uparrow$ guarantees differentiability at the endpoints of the interval of approximation.

Example 2. [**1**, p. 91] For suitably chosen $\delta_k \searrow 0$ and $\phi_k \searrow 0$ the function

$$f(x) = \sum_{k=0}^{\infty} \frac{\phi_k \delta_k}{x + \delta_k}$$

fails to be differentiable at 0 while the speed of approximation from R_n^+ can be arbitrarily fast.

It is natural to enquire about the exactness of the constants in the preceding inequalities and theorems. For instance: is the 2, in n^{2k-1}, the "correct" constant in Theorem 3? Should it be 1 as in the comparable theorem for polynomial approximation or perhaps some intermediated value?

Acknowledgment. The author would like to thank Dr. D. W. Boyd for his assistance and encouragement.

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