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GENERALIZATIONS OF GONÇALVES' INEQUALITY

PETER BORWEIN, MICHAEL J. MOSSINGHOFF, AND JEFFREY D. VAALER

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ABSTRACT. Let $F(z) = \sum_{n=0}^{N} a_n z^n$ be a polynomial with complex coefficients and roots $\alpha_1, \ldots, \alpha_N$, let $\|F\|_p$ denote its L_p norm over the unit circle, and let $\|F\|_p$ denote Mahler's measure of F. Gonçalves' inequality asserts that

$$\|F\|_2 \geq |a_N| \left(\prod_{n=1}^{N} \max\{1, |\alpha_n|^2\} + \prod_{n=1}^{N} \min\{1, |\alpha_n|^2\} \right)^{1/2}$$
$$= \|F\|_0 \left(1 + \frac{|a_0 a_N|^2}{\|F\|^4}\right)^{1/2}.$$

We prove that

$$\|F\|_p \geq B_p |a_N| \left(\prod_{n=1}^{N} \max\{1, |\alpha_n|^p\} + \prod_{n=1}^{N} \min\{1, |\alpha_n|^p\} \right)^{1/p}$$

for $1 \leq p \leq 2$, where B_p is an explicit constant, and that

$$\|F\|_p \geq \|F\|_0 \left(1 + \frac{p^2 |a_0 a_N|^2}{4\|F\|^4}\right)^{1/p}$$

for $p \geq 1$. We also establish additional lower bounds on the L_p norms of a polynomial in terms of its coefficients.

1. Introduction

Let $\Delta \subset \mathbb{C}$ denote the open unit disc, $\overline{\Delta}$ its closure, and let $\mathcal{A}(\Delta)$ denote the algebra of continuous functions $f : \overline{\Delta} \to \mathbb{C}$ that are analytic on Δ. Then $\{\mathcal{A}(\Delta), \|\cdot\|_\infty\}$ is a Banach algebra, where

$$\|f\|_\infty = \sup\left\{|f(z)| : z \in \overline{\Delta}\right\} = \sup\left\{|f(e(t))| : t \in \mathbb{R}/\mathbb{Z}\right\},$$

and $e(t)$ denotes the function $e^{2\pi i t}$. If $f \in \mathcal{A}(\Delta)$ and $0 < p < \infty$, we also define

$$\|f\|_p = \left(\int_0^1 |f(e(t))|^p \, dt \right)^{1/p},$$

and we define

$$\|f\|_0 = \exp\left(\int_0^1 \log |f(e(t))| \, dt \right).$$

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It is known (see [2, Chapter 6]) that for each f in $\mathcal{A}(\Delta)$ the function $p \to \|f\|_p$ is continuous on $[0, \infty]$, and if $0 < p < q < \infty$, then these quantities satisfy the basic inequality

$$\|f\|_0 \le \|f\|_p \le \|f\|_q \le \|f\|_\infty . \tag{1.1}$$

Clearly, equality can occur throughout (1.1) if f is constant. On the other hand, if f is not constant in $\mathcal{A}(\Delta)$, then the function $p \to \|f\|_p$ is strictly increasing on $[0, \infty]$.

Now suppose that F is a polynomial in $\mathbb{C}[z]$ of degree $N \ge 1$, and write

$$F(z) = \sum_{n=0}^{N} a_n z^n = a_N \prod_{n=1}^{N} (z - \alpha_n). \tag{1.2}$$

In this case, the quantity $\|F\|_0$ is *Mahler's measure* of F, and by Jensen's formula one obtains the well-known identity

$$\|F\|_0 = |a_N| \prod_{n=1}^{N} \max\{1, |\alpha_n|\}. \tag{1.3}$$

Thus a special case of (1.1) is the inequality (often called Landau's inequality)

$$\|F\|_2 \ge |a_N| \prod_{n=1}^{N} \max\{1, |\alpha_n|\}.$$

For polynomials of positive degree, the sharper inequality

$$\|F\|_2 \ge |a_N| \left(\prod_{n=1}^{N} \max\{1, |\alpha_n|^2\} + \prod_{n=1}^{N} \min\{1, |\alpha_n|^2\} \right)^{1/2} \tag{1.4}$$

was obtained by Gonçalves [1]. Note that equality occurs in (1.4) for constant multiples of $z^N - 1$. Alternatively, the inequality (1.4) may be written in the less symmetrical form

$$\|F\|_2 \ge \|F\|_0 \left(1 + \frac{|a_0 a_N|^2}{\|F\|_0^4} \right)^{1/2} . \tag{1.5}$$

For a positive real number p, define the real number B_p by

$$B_p = \left(\frac{1}{2} \int_0^1 |1 - e(t)|^p \, dt \right)^{1/p} = \left(\frac{\Gamma(p+1)}{2\Gamma(p/2+1)^2} \right)^{1/p}, \tag{1.6}$$

and note that $B_1 = 2/\pi$ and $B_2 = 1$. In this article we establish the following generalizations of Gonçalves' inequality.

Theorem 1. *Let* $F(z) \in \mathbb{C}[z]$ *be given by* (1.2). *If* $1 \le p \le 2$, *then*

$$\begin{aligned} \|F\|_p &\ge B_p \, |a_N| \left(\prod_{n=1}^{N} \max\{1, |\alpha_n|^p\} + \prod_{n=1}^{N} \min\{1, |\alpha_n|^p\} \right)^{1/p} \\ &= B_p \, \|F\|_0 \left(1 + \frac{|a_0 a_N|^p}{\|F\|_0^{2p}} \right)^{1/p} , \end{aligned} \tag{1.7}$$

and if $p \geq 1$, *then*

$$\|F\|_p \geq \|F\|_0 \left(1 + \frac{p^2 |a_0 a_N|^2}{4 \|F\|_0^4}\right)^{1/p}. \tag{1.8}$$

Equality occurs in (1.7) for constant multiples of $z^N - 1$. The inequality (1.8) is never sharp for $p \neq 2$, but since $B_p < 1$ for $1 \leq p < 2$ it is clearly stronger than (1.7) in this range when $\|F\|_0^2 / |a_0 a_N|$ is large. For example, one may verify that (1.8) produces a better bound in the case $p = 1$ whenever

$$\frac{\|F\|_0^2}{|a_0 a_N|} > \frac{\pi}{2\left(2 - \sqrt{4 + 2\pi - \pi^2}\right)} = 1.1576382\ldots.$$

Also, for fixed F the right side of (1.8) achieves a maximum at $p = 2c \|F\|_0^2 / |a_0 a_N|$, where $c = 1.9802913\ldots$ is the unique positive number satisfying

$$2c^2 = (1 + c^2) \log(1 + c^2).$$

In view of (1.1), inequality (1.8) is therefore only of interest when $1 \leq p \leq 2c \|F\|_0^2 / |a_0 a_N|$.

To prove Theorem 1, we first establish some lower bounds on the L_p norms of a polynomial in terms of two of its coefficients a_L and a_M, provided $|M - L|$ is sufficiently large. These inequalities have some independent interest, and we record the results in the following theorem.

Theorem 2. *Let* $F(z) \in \mathbb{C}[z]$ *be given by* (1.2), *and let* L *and* M *be integers satisfying* $0 \leq L < M \leq N$ *and* $M - L > \max\{L, N - M\}$. *Then*

$$\|F\|_\infty \geq |a_L| + |a_M|. \tag{1.9}$$

Further, if $1 \leq p \leq 2$, *then*

$$\|F\|_p \geq B_p \left(|a_L|^p + |a_M|^p\right)^{1/p}, \tag{1.10}$$

and if $p \geq 1$ *and* a_L *and* a_M *are not both* 0, *then*

$$\|F\|_p \geq \max\{|a_L|, |a_M|\} \left(1 + \left(\frac{p \min\{|a_L|, |a_M|\}}{2 \max\{|a_L|, |a_M|\}}\right)^2\right)^{1/p}. \tag{1.11}$$

At this point, it is instructive to recall the Hausdorff-Young inequality. If $p = 2$ and $F(z)$ is given by (1.2), then by Parseval's identity we have

$$\|F\|_2 = \left(|a_0|^2 + |a_1|^2 + \cdots + |a_N|^2\right)^{1/2}. \tag{1.12}$$

If $p = 1$, then the inequality

$$\|F\|_1 \geq \max\{|a_0|, |a_1|, \ldots, |a_N|\} \tag{1.13}$$

follows immediately from the identity

$$a_n = \int_0^1 F(e(t)) e(-nt)\, dt.$$

Now suppose that $1 < p < 2$ and let q be the conjugate exponent for p, so $p^{-1} + q^{-1} = 1$. Then the Hausdorff-Young inequality [3, p. 123] asserts that

$$\|F\|_p \geq \left(|a_0|^q + |a_1|^q + \cdots + |a_N|^q\right)^{1/q}, \tag{1.14}$$

and so interpolates between (1.12) and (1.13). If $p = 2$, then (1.10) and (1.11) are equivalent and clearly follow from the identity (1.12). But for $1 < p < 2$, the inequalities (1.10) and (1.11) are not immediate consequences of (1.14). In fact, it is easy to see that the lower bounds in (1.10), (1.11), and (1.14) are not comparable. If $p = 1$, the same remarks apply to (1.10), (1.11), and (1.13).

In section 2 we develop some preliminary results concerning lower bounds on L_p norms of binomials, and we use these facts to establish Theorems 1 and 2 in section 3.

2. Norms of binomials

For $0 < r < 1$ and real t, recall that the Poisson kernel is defined by

$$P(r,t) = \sum_{n=-\infty}^{\infty} r^{|n|} e(nt) = \Re\left(\frac{1+re(t)}{1-re(t)}\right) = \frac{1-r^2}{|1-re(t)|^2}.$$

This is a positive summability kernel that satisfies

$$\int_0^1 P(r,t)\,dt = 1$$

and

$$\lim_{r\to 1-} \int_{\epsilon}^{1-\epsilon} P(r,t)\,dt = 0$$

for $0 < \epsilon < 1/2$.

Lemma 3. *If $p > 0$, then*

$$\lim_{r\to 1-} \int_0^1 |1-re(t)|^p\, P(r,t)\,dt = 0.$$

Proof. Let $0 < \epsilon < 1/2$ so that

$$\begin{aligned}\int_0^1 |1-re(t)|^p\, P(r,t)\,dt &= \int_{-\epsilon}^{\epsilon} |1-re(t)|^p\, P(r,t)\,dt + \int_{\epsilon}^{1-\epsilon} |1-re(t)|^p\, P(r,t)\,dt \\ &\le |1-re(\epsilon)|^p \int_{-\epsilon}^{\epsilon} P(r,t)\,dt + 2^p \int_{\epsilon}^{1-\epsilon} P(r,t)\,dt \\ &\le |1-re(\epsilon)|^p + 2^p \int_{\epsilon}^{1-\epsilon} P(r,t)\,dt.\end{aligned}$$

We conclude that

$$\limsup_{r\to 1-} \int_0^1 |1-re(t)|^p\, P(r,t)\,dt \le |1-e(\epsilon)|^p \le (2\pi\epsilon)^p,$$

and the statement follows. □

For positive numbers p and r, we define

$$\mathcal{I}_p(r) = \int_0^1 \left|1-r^{1/p}e(t)\right|^p\,dt. \tag{2.1}$$

It follows easily that $r \to \mathcal{I}_p(r)$ is a continuous, positive, real-valued function that satisfies the functional equation

$$\mathcal{I}_p(r) = r\mathcal{I}_p(1/r) \tag{2.2}$$

for all positive r. Also, if $0 < r < 1$, then $\left(1 - r^{1/p}e(t)\right)^{p/2}$ has the absolutely convergent Fourier expansion

$$\left(1 - r^{1/p}e(t)\right)^{p/2} = \sum_{m \geq 0} \binom{p/2}{m} (-1)^m r^{m/p} e(mt),$$

so by Parseval's identity, we have

$$\mathcal{I}_p(r) = \sum_{m \geq 0} \binom{p/2}{m}^2 r^{2m/p}. \tag{2.3}$$

The following lemmas record some further information about this function.

Lemma 4. *For any positive number p, the function $r \to \mathcal{I}_p(r)$ has a continuous derivative at each point of $(0, \infty)$ and satisfies the identity $\mathcal{I}_p'(1) = \mathcal{I}_p(1)/2$.*

Proof. Suppose first that $0 < r < 1$. Then (2.3) shows that $r \to \mathcal{I}_p(r)$ is represented on $(0, 1)$ by a convergent power series in $r^{1/p}$ and therefore has infinitely many continuous derivatives on this interval. Next, we observe that

$$\frac{\partial}{\partial r} \left|1 - r^{1/p}e(t)\right|^p = \frac{\left|1 - r^{1/p}e(t)\right|^p}{2r} \left(1 - P(r^{1/p}, t)\right).$$

It follows that if $0 < \epsilon \leq 1/4$ and $\epsilon \leq r \leq 1 - \epsilon$, then there exists a positive constant $C(\epsilon, p)$ such that

$$\left| \frac{\partial}{\partial r} \left|1 - r^{1/p}e(t)\right|^p \right| \leq C(\epsilon, p).$$

From the mean value theorem and the dominated convergence theorem, we find that

$$\mathcal{I}_p'(r) = \frac{1}{2r} \int_0^1 \left|1 - r^{1/p}e(t)\right|^p \left(1 - P(r^{1/p}, t)\right) dt,$$

and therefore

$$\mathcal{I}_p(r) - 2r\mathcal{I}_p'(r) = \int_0^1 \left|1 - r^{1/p}e(t)\right|^p P(r^{1/p}, t)\, dt.$$

Using the continuity of $r \to \mathcal{I}_p(r)$ and Lemma 3, we conclude that

$$\lim_{r \to 1-} \mathcal{I}_p'(r) = \mathcal{I}_p(1)/2. \tag{2.4}$$

Again using the mean value theorem, it follows that $r \to \mathcal{I}_p(r)$ has a left-hand derivative at 1 with the value $\mathcal{I}_p(1)/2$.

Suppose then that $r > 1$. From (2.2) and (2.3) we find that

$$\mathcal{I}_p(r) = r \left(\sum_{m \geq 0} \binom{p/2}{m}^2 r^{-2m/p} \right). \tag{2.5}$$

Thus $r \to \mathcal{I}_p(r)$ is represented by r times a convergent power series in $r^{-1/p}$, and so has infinitely many continuous derivatives on the interval $(1, \infty)$. We differentiate both sides of (2.2) to obtain the identity

$$\mathcal{I}_p'(r) = \mathcal{I}_p(1/r) - \frac{\mathcal{I}_p'(1/r)}{r}$$

for $r > 1$, and using the continuity of $r \to \mathcal{I}_p(r)$ and (2.4), we conclude that

$$\lim_{r \to 1+} \mathcal{I}_p'(r) = \mathcal{I}_p(1) - \lim_{s \to 1-} \mathcal{I}_p'(s) = \mathcal{I}_p(1)/2.$$

It follows that $r \to \mathcal{I}_p(r)$ has a right-hand derivative at 1 with value $\mathcal{I}_p(1)/2$.

We conclude then that $r \to \mathcal{I}_p(r)$ is continuously differentiable on $(0,\infty)$ and $\mathcal{I}_p'(1) = \mathcal{I}_p(1)/2$. □

The following lower bound is obtained by establishing the convexity of the function $r \to \mathcal{I}_p(r)$ for each fixed p in $(0,2]$.

Lemma 5. *If $0 < p \le 2$, then the function $r \to \mathcal{I}_p(r)$ satisfies the inequality*

$$\mathcal{I}_p(r) \ge \frac{\mathcal{I}_p(1)(1+r)}{2} \tag{2.6}$$

for $r > 0$.

Proof. If $p = 2$, then $\mathcal{I}_2(r) = 1 + r$ and the result is trivial. Suppose then that $0 < p < 2$. If $r < 1$, then we may differentiate the power series (2.3) termwise to obtain

$$\mathcal{I}_p'(r) = \sum_{m \ge 0} \binom{p/2}{m}^2 \frac{2m}{p} r^{(2m/p)-1}.$$

As $0 < p < 2$, it follows that $r \to \mathcal{I}_p'(r)$ is strictly increasing on $(0,1)$, so $r \to \mathcal{I}_p(r)$ is strictly convex on this interval. Thus, if r and s are in $(0,1)$, then

$$\mathcal{I}_p(r) \ge \mathcal{I}_p(s) + (r-s)\mathcal{I}_p'(s). \tag{2.7}$$

Letting $s \to 1-$ and using Lemma 4, we obtain

$$\mathcal{I}_p(r) \ge \mathcal{I}_p(1) + \mathcal{I}_p'(1)(r-1) = \frac{\mathcal{I}_p(1)(1+r)}{2} \tag{2.8}$$

for $0 < r < 1$.

In a similar manner, if $r > 1$, we differentiate (2.5) termwise to obtain

$$\mathcal{I}_p'(r) = \sum_{m \ge 0} \binom{p/2}{m}^2 \left(1 - \frac{2m}{p}\right) r^{-2m/p},$$

and again $r \to \mathcal{I}_p'(r)$ is strictly increasing on $(1,\infty)$, so $r \to \mathcal{I}_p(r)$ is strictly convex on this interval. Thus (2.7) holds as well for $r > 1$ and $s > 1$, and letting $s \to 1+$ we obtain (2.8) for $r > 1$.

We have therefore verified (2.6) at each point r in $(0,1) \cup (1,\infty)$, and it is trivial at $r = 1$. □

Next, we use these facts about the function $r \to \mathcal{I}_p(r)$ to establish some lower bounds on $\|F\|_p$ when F has just two terms.

Proposition 6. *Let $0 \le L < M$ be integers and let α and β be complex numbers. If $p > 0$ and α and β are not both zero, then*

$$\left\|\alpha z^L + \beta z^M\right\|_p \ge \max\{|\alpha|, |\beta|\} \left(1 + \left(\frac{p\min\{|\alpha|, |\beta|\}}{2\max\{|\alpha|, |\beta|\}}\right)^2\right)^{1/p}, \tag{2.9}$$

with equality precisely when $\alpha\beta = 0$ or $p = 2$. Also, if $0 < p \le 2$, then

$$\left\|\alpha z^L + \beta z^M\right\|_p \ge B_p \left(|\alpha|^p + |\beta|^p\right)^{1/p}. \tag{2.10}$$

Proof. The results are trivial if either α or β is zero, so we assume that this is not the case. We may then assume by homogeneity that $\alpha = 1$, and it is clear from the definition of $\|f\|_p$ that we may assume that $L = 0$, $M = 1$, and that β is real and negative. Suppose $p > 0$. If $|\beta| < 1$, then taking $r = |\beta|^p$ in (2.3) and keeping just the first two terms of the sum, we obtain

$$\|1 + \beta z\|_p^p \geq 1 + \frac{p^2 |\beta|^2}{4}.$$

If $|\beta| > 1$, then

$$\|1 + \beta z\|_p^p = |\beta|^p \left\|\beta^{-1} + z\right\|_p^p = |\beta|^p \|1 + z/\beta\|_p^p.$$

So taking $r = |\beta|^{-p}$, we obtain in the same way

$$\|1 + \beta z\|_p^p \geq |\beta|^p \left(1 + \frac{p^2}{4|\beta|^2}\right).$$

The case $\beta = -1$ follows by continuity, and this establishes (2.9). For the case of equality, notice that the sum (2.3) has precisely two nonzero terms only when $p = 2$.

Last, using Lemma 5 we find

$$\begin{aligned}\|1 + \beta z\|_p^p &= \mathcal{I}_p(|\beta|^p)\\ &\geq \frac{\mathcal{I}_p(1)\,(1 + |\beta|^p)}{2}\\ &= B_p^p\,(1 + |\beta|^p),\end{aligned}$$

establishing (2.10). □

3. Proofs of the theorems

The proof of Theorem 2 employs an averaging argument and makes use of the triangle inequality for L_p norms. We therefore require the restriction $p \geq 1$ in the statement of the theorem.

Proof of Theorem 2. Suppose that $F(z) = \sum_{n=0}^N a_n z^n$ is a polynomial with complex coefficients, and L and M are as in the statement of the theorem. Set $K = M - L$, and let ζ_K denote a primitive Kth root of unity in $\mathbb{C}$. Then

$$\begin{aligned}\frac{1}{K}\sum_{k=1}^K \zeta_K^{-kL} F\left(\zeta_K^k z\right) &= \frac{1}{K}\sum_{n=0}^N \left(\sum_{k=1}^K \zeta_K^{k(n-L)}\right) a_n z^n\\ &= \sum_{\substack{0\leq n\leq N\\ n\equiv L \pmod K}} a_n z^n\\ &= a_L z^L + a_M z^M.\end{aligned}$$

Using the triangle inequality and the fact that the polynomials $\zeta_K^{-kL} F(\zeta_K^k z)$ all have the same L_p norm, we find that

$$\|F\|_p \geq \left\|\frac{1}{K}\sum_{k=1}^K \zeta_K^{-kL} F\left(\zeta_K^k z\right)\right\|_p = \left\|a_L z^L + a_M z^M\right\|_p \tag{3.1}$$

for $1 \le p \le \infty$. Inequalities (1.10) and (1.11) are then established by combining (3.1) with (2.10) and (2.9), respectively. When $p = \infty$, inequality (1.9) follows by selecting a complex number z of unit modulus so that $a_L z^L$ and $a_M z^M$ have the same argument. □

The proof of Theorem 1 proceeds by applying Theorem 2 to a polynomial having the same values over the unit circle as the given polynomial F. Ostrowski [6] and Mignotte [5] (see also [4, p. 80]) employ a similar construction in their proofs of Gonçalves' inequality (1.5) in the case $p = 2$.

Proof of Theorem 1. Suppose that $F(z) = \sum_{n=0}^{N} a_n z^n = a_N \prod_{n=1}^{N}(z - \alpha_n)$ is a polynomial with complex coefficients. If $F(z)$ has a root at $z = 0$, then (1.7) and (1.8) follow immediately from (1.1), so we assume that $a_0 \neq 0$. Let $\mathcal{E}$ denote the collection of all subsets of $\{1, 2, \ldots, N\}$, and for each E in $\mathcal{E}$, let E' denote the complement of E in $\{1, 2, \ldots, N\}$. For each set E in $\mathcal{E}$, we define the finite Blaschke product $B_E(z)$ by

$$B_E(z) = \prod_{n \in E} \frac{1 - \overline{\alpha_n} z}{z - \alpha_n}$$

and the polynomial $G_E(z)$ by

$$G_E(z) = B_E(z) F(z) = \sum_{n=0}^{N} b_n(E) z^n.$$

Clearly,

$$b_0(E) = a_N \prod_{m \in E'} (-\alpha_m)$$

and

$$b_N(E) = a_N \prod_{n \in E} (-\overline{\alpha_n}).$$

If $|z| = 1$, then the Blaschke product satisfies $|B_E(z)| = 1$, so $\|G_E(z)\|_p = \|F\|_p$ for $0 \le p \le \infty$ and every E in $\mathcal{E}$. Now select $L = 0$ and $M = N$ for the polynomial $G_E(z)$ in Theorem 2. Then from (1.10) we obtain

$$\|F\|_p \ge B_p\, |a_N| \left(\prod_{m \in E'} |\alpha_m|^p + \prod_{n \in E} |\alpha_n|^p \right)^{1/p} \tag{3.2}$$

for $1 \le p \le 2$. Also, since $|b_0(E) b_N(E)| = |a_0 a_N|$, we find from (1.11) that

$$\|F\|_p \ge \max\{|b_0(E)|, |b_N(E)|\} \left(1 + \frac{p^2 |a_0 a_N|^2}{4(\max\{|b_0(E)|, |b_N(E)|\})^4} \right)^{1/p} \tag{3.3}$$

for $p \ge 1$. Inequalities (1.7) and (1.8) then follow from (3.2) and (3.3) by choosing $E = \{n : |\alpha_n| \le 1\}$. □

We remark that the choice of E in the preceding proof produces the best possible inequality in (3.2). To establish this, suppose that $E \in \mathcal{E}$ has $|b_0(E)| = \|F\|_0 / r$ for some real number r, so $|b_N(E)| = r\,|a_0 a_N| / \|F\|_0$. Then certainly $1 \le r \le \|F\|_0^2 / |a_0 a_N|$, and it is easy to check that

$$\frac{\|F\|_0}{r} + \frac{r\,|a_0 a_N|}{\|F\|_0} \le \|F\|_0 + \frac{|a_0 a_N|}{\|F\|_0}$$

in this range, with equality occurring only at the endpoints.

It is possible, however, that a different choice for E in (3.3) could produce a bound better than (1.8) for a particular polynomial. Specifically, if $E \in \mathcal{E}$ has $|b_0(E)| = \|F\|_0 / r$, again with $1 \le r \le \|F\|_0^2 / |a_0 a_N|$, then we obtain an improved bound whenever

$$\left(4\|F\|_0^4 + p^2 |a_0 a_N|^2\right) r^p < 4\|F\|_0^4 + p^2 |a_0 a_N|^2 r^4,$$

and this may occur when p is small. For example, the polynomial $F(z) = 18z^2 - 101z + 90$ has roots $\alpha_1 = 9/2$ and $\alpha_2 = 10/9$; choosing $E = \{\}$ with $p = 1$ yields $\|F\|_1 \ge 90.9$, but selecting $E = \{1\}$ (so $r = 9/2$) produces a lower bound slightly larger than 102.

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Department of Mathematics and Statistics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6

E-mail address: pborwein@cecm.sfu.ca

Department of Mathematics, Davidson College, Davidson, North Carolina 28035

E-mail address: mjm@member.ams.org

Department of Mathematics, University of Texas, Austin, Texas 78712

E-mail address: vaaler@math.utexas.edu