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THE MAHLER MEASURE OF POLYNOMIALS WITH ODD COEFFICIENTS

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Abstract

The minimum value of the Mahler measure of a nonreciprocal polynomial whose coefficients are all odd integers is proved here to be the golden ratio. The smallest measures of reciprocal polynomials with ±1 coefficients and degree at most 72 are also determined.

1. *Introduction*

The *Mahler measure* of a polynomial

$$f(x) = \sum_{i=0}^{n} a_i x^i = a_n \prod_{i=1}^{n} (x - \alpha_i)$$

is defined by

$$M(f) = |a_n| \prod_{i=1}^{n} \max\{1, |\alpha_i|\}. \tag{1}$$

It is easy to check that the measure of a polynomial is unchanged if its coefficients are reversed: if $f^*(x) = x^n f(1/x)$, then $M(f^*) = M(f)$. The polynomial f^* is called the *reciprocal polynomial* of f, and a polynomial is said to be *reciprocal* if $f = \pm f^*$.

For polynomials with integer coefficients, a well-known result of Kronecker implies that $M(f) = 1$ if and only if $f(x)$ is a product of cyclotomic polynomials and the monomial x. In 1933, D. H. Lehmer [**7**] asked if, for any $\varepsilon > 0$, there exists

$$f(x) \in \mathbb{Z}[x], \quad \text{with } 1 < M(f) < 1 + \varepsilon;$$

this problem remains open. Lehmer noted that

$$\ell(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

has measure $M(\ell) = 1.1762808\ldots$; this remains the smallest known measure greater than 1 of a polynomial with integer coefficients. Smyth [**11**] answered Lehmer's question for the case of nonreciprocal polynomials, proving that if $f(x) \in \mathbb{Z}[x]$ is nonreciprocal and $f(0) \neq 0$, then $M(f) \geqslant M(x^3 - x - 1) = 1.324717\ldots$.

A polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ is said to be a *Littlewood polynomial* if $a_i = \pm 1$ for $0 \leqslant i \leqslant n$. Borwein and Choi [**2**] characterize the Littlewood polynomials of even degree with measure 1, providing a sharper version of Kronecker's theorem for this class of polynomials.

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In this paper, we prove a sharp lower bound for the Mahler measure of a nonreciprocal Littlewood polynomial, improving Smyth's bound for this family. In fact, our main result provides a lower bound on the measure for a larger class of nonreciprocal polynomials. If $f(x) = \sum_{i=0}^{n} a_i x^i$ has integer coefficients and $m \geqslant 2$ is an integer, we write $f \equiv \pm f^* \bmod m$ if either

$$a_i \equiv a_{n-i} \bmod m, \quad \text{for } 0 \leqslant i \leqslant n, \quad \text{or} \quad a_i \equiv -a_{n-i} \bmod m, \quad \text{for each } i. \tag{2}$$

THEOREM 1.1. *Suppose that f is a monic, nonreciprocal polynomial with integer coefficients satisfying $f \equiv \pm f^* \bmod m$ for some integer $m \geqslant 2$. Then*

$$M(f) \geqslant \frac{m + \sqrt{m^2 + 16}}{4}, \tag{3}$$

and this bound is sharp when m is even.

We prove this theorem in Section 2. Taking $m = 2$, we immediately obtain the golden ratio as a sharp lower bound for the measure of a nonreciprocal Littlewood polynomial.

COROLLARY 1.2. *If f is a nonreciprocal polynomial whose coefficients are all odd integers, then*

$$M(f) \geqslant M(x^2 - x - 1) = \frac{1 + \sqrt{5}}{2}.$$

In particular, this bound holds for nonreciprocal Littlewood polynomials.

Recall that a *Pisot number* is a real algebraic integer, greater than 1, whose conjugates lie inside the open unit disk. We remark that Smyth's lower bound is the smallest Pisot number; the golden ratio is the smallest limit point of Pisot numbers (see [**1**, Chapter 6]).

An exhaustive search of Littlewood polynomials up to degree 31 initially led us to suspect the golden ratio as the lower bound for the measure in the nonreciprocal case. Section 3 describes some computations for the reciprocal case. We describe an algorithm for searching for reciprocal Littlewood polynomials with small measure, summarize its results through degree 72, and exhibit a list of fifteen measures of Littlewood polynomials less than 1.6. The smallest measure that we find is 1.496711 . . . , associated with the polynomial

$$\begin{aligned} x^{19} &+ x^{18} + x^{17} + x^{16} - x^{15} + x^{14} - x^{13} + x^{12} - x^{11} \\ &- x^{10} - x^9 - x^8 + x^7 - x^6 + x^5 - x^4 + x^3 + x^2 + x + 1. \end{aligned}$$

2. *Proof of Theorem* 1.1

Our proof follows that of Smyth [**11**]. We require the following inequality regarding coefficients of power series.

LEMMA 2.1. *Suppose that $\varphi(z) = \sum_{i \geqslant 0} \gamma_i z^i$, with $\gamma_i \in \mathbb{C}$, is analytic in an open disk containing $|z| \leqslant 1$ and satisfies $|\varphi(z)| \leqslant 1$ on $|z| = 1$. Then $|\gamma_i| \leqslant 1 - |\gamma_0|^2$ for $i \geqslant 1$.*

See [**10**, p. 392] for a proof.

Proof of Theorem 1.1. Suppose that $f(z) = \sum_{i=0}^n a_i z^i = \prod_{i=1}^n (z - \alpha_i)$ satisfies the hypotheses of Theorem 1.1, for a given integer $m \geqslant 2$.

Write $f^*(z) = \sum_{i=0}^n d_i z^i$ (so $d_0 = 1$), and let $\sum_{i \geqslant 0} e_i z^i$ be the power series for $1/f^*(z)$. Because

$$\left(\sum_{i=0}^n d_i z^i\right)\left(\sum_{i\geqslant 0} e_i z^i\right) = 1,$$

certainly $e_0 = 1$, and

$$e_k = -\sum_{j=0}^{k-1} d_{k-j} e_j.$$

Thus each e_k is an integer. Let

$$G(z) = f(z)/f^*(z) = \sum_{i\geqslant 0} q_i z^i,$$

so $q_i \in \mathbb{Z}$ for $i \geqslant 0$. Clearly, $q_0 = a_0$. If $|a_0| > 1$, then in view of (1) and (2),

$$M(f) = M(f^*) \geqslant |a_0| \geqslant m - 1 \geqslant \frac{m + \sqrt{m^2 + 16}}{4}$$

for $m \geqslant 3$ (and similarly, $M(f) \geqslant 3$ for $m = 2$), so we may assume that $|a_0| = 1$. Equating the coefficients of z^j in $f^*(z)G(z) = f(z)$ yields $\sum_{i=0}^j d_i q_{j-i} = a_j$, so for $j \geqslant 1$ we have

$$q_j = (a_j - q_0 d_j) - \sum_{i=1}^{j-1} d_i q_{j-i}.$$

Since $f \equiv \pm f^* \bmod m$, we have $a_j \equiv q_0 d_j \bmod m$, so by induction $m \mid q_j$ for $j \geqslant 1$.

Let $\varepsilon = -1$ if $f(z)$ has a zero of odd multiplicity at $z = 1$; otherwise, let $\varepsilon = 1$. Noting that

$$\prod_{|\alpha_i|=1} \frac{z - \alpha_i}{1 - \overline{\alpha_i} z} = \prod_{|\alpha_i|=1} \frac{z - \alpha_i}{1 - z/\alpha_i} = \prod_{|\alpha_i|=1} (-\alpha_i) = \varepsilon,$$

we let

$$g(z) = \varepsilon \prod_{|\alpha_i|<1} \frac{z - \alpha_i}{1 - \overline{\alpha_i} z} \quad \text{and} \quad h(z) = \prod_{|\alpha_i|>1} \frac{1 - \overline{\alpha_i} z}{z - \alpha_i},$$

so

$$\frac{g(z)}{h(z)} = \frac{\prod_{i=1}^n (z - \alpha_i)}{\prod_{i=1}^n (1 - \overline{\alpha_i} z)} = \frac{\prod_{i=1}^n (z - \alpha_i)}{\prod_{i=1}^n (1 - \alpha_i z)} = \frac{f(z)}{f^*(z)} = G(z).$$

Clearly, all poles of both $g(z)$ and $h(z)$ lie outside the unit disk, so both functions are analytic in a region containing $|z| \leqslant 1$. Further, if $|z| = 1$ and $\beta \in \mathbb{C}$, then

$$\left(\frac{z - \beta}{1 - \overline{\beta} z}\right)\overline{\left(\frac{z - \beta}{1 - \overline{\beta} z}\right)} = \left(\frac{z - \beta}{1 - \overline{\beta} z}\right)\left(\frac{1/z - \overline{\beta}}{1 - \beta/z}\right) = 1,$$

so $|g(z)| = |h(z)| = 1$ on $|z| = 1$. Let

$$g(z) = \sum_{i\geqslant 0} b_i z^i \quad \text{and} \quad h(z) = \sum_{i\geqslant 0} c_i z^i.$$

Let k be the smallest positive integer for which $q_k \neq 0$, so $|q_k| \geqslant m$. Since $g(z) = h(z)G(z)$, we obtain $b_i = c_i q_0$ for $0 \leqslant i < k$ and $b_k = c_0 q_k + c_k q_0$. Thus

$$|c_0 m| \leqslant |c_0 q_k| = |b_k - c_k q_0| \leqslant 2 \max\{|b_k|, |c_k|\}. \tag{4}$$

Assume, without loss of generality, that $|c_k| \geqslant |b_k|$. By Lemma 2.1, we have $|c_k| \leqslant 1 - c_0^2$, and combining this with (4) and the observation that

$$|c_0| = |h(0)| = \prod_{|\alpha_i|>1} 1/|\alpha_i| = 1/M(f)$$

yields

$$M(f)m \leqslant 2(M(f)^2 - 1).$$

The inequality (3) follows, and this bound is achieved when m is even by $f(z) = z^2 \pm mz/2 - 1$. □

3. *Reciprocal Littlewood polynomials with small measure*

We describe an algorithm for searching for reciprocal Littlewood polynomials with small Mahler measure, provide some details on its implementation, and report on its results.

3.1. *Algorithm*

Given a positive integer d, we wish to determine all reciprocal Littlewood polynomials $f(x) = \sum_{i=0}^{d} a_i x^i$ having $1 < M(f) < M$, where M is a fixed constant. If f is reciprocal of even degree d, then necessarily $f = f^*$, since the middle coefficient of f is nonzero. Further, $f(-x)$ also has this property, and clearly $M(f(-x)) = M(f(x))$, so we may assume that $a_0 = a_1 = 1$. If d is odd and $f = -f^*$, set $g(x) = f(-x)$ so that $g = g^*$. Thus we may assume that $a_0 = 1$ and $f = f^*$ for odd d.

Following [**3**, **4**, **8**], we use the Graeffe root-squaring algorithm to screen out most polynomials f having $M(f) > M$, and all polynomials with $M(f) = 1$, in an efficient way. Recall that the Graeffe operator G applied to a polynomial $f(x)$ written as

$$f(x) = g(x^2) + xh(x^2)$$

yields the polynomial

$$Gf(x) = g(x)^2 - xh(x)^2.$$

The roots of Gf are precisely the squares of the roots of f, and $M(Gf) = M(f)^2$. Let $a_{k,m}$ denote the coefficient of x^k in $G^m f(x)$. Boyd [**3**] shows that if $M(f) \leqslant M$, then

$$|a_{k,m}| \leqslant \binom{d}{k} + \binom{d-2}{k-1}\left(M^{2^m} + M^{-2^m} - 2\right) \tag{5}$$

for all m, and if in addition $a_{1,m} \geqslant d - 4$ and $m \geqslant 1$, then

$$\begin{aligned} |a_{k,m}| \leqslant \binom{d}{k} &+ \binom{d-4}{k-2}\left(M^{2^m} + M^{-2^m} - 2\right) \\ &+ 2\left(M^{2^{m-1}} + M^{-2^{m-1}} - 2\right)\left(\binom{d-4}{k-3} + \binom{d-4}{k-1}\right). \end{aligned} \tag{6}$$

We apply the Graeffe operator to each polynomial at most m_0 times, where m_0 is another fixed parameter of the algorithm. A polynomial f is rejected at stage m if the appropriate inequality (5) or (6) is not satisfied for some k, or if $G^m f = G^{m-1} f$. In the latter case, Kronecker's theorem implies that f is a product of cyclotomic polynomials. Let Φ_n denote the nth cyclotomic polynomial. If $n = 2^r s$ with s odd, then $G^m \Phi_{2^r s} = \Phi_s^{2^{r-1}}$ when $m \geqslant r$, so the Graeffe method is guaranteed to detect a product of cyclotomic polynomials with total degree d if $m \geqslant 1 + \log_2 d$.

3.2. *Implementation*

We use $M = 5/3$ and $m_0 = 10$ in our C++ implementation. All root-squaring is performed using exact integer arithmetic. For each f, we store the coefficients of the polynomial $G^m f$ using native 32-bit integers for as many m as possible for efficiency, and then switch to a representation in software for larger m. We use the highly optimized package GMP [**6**] for arithmetic with big integers; tests with our application showed a 30% improvement in speed over the package LIP used in [**8**]. We use Maple to compute the measure of each polynomial that survives the Graeffe iteration.

We also allow parallel processing by partitioning the set of polynomials of a particular degree into 2^n parts of equal size, fixing n particular coefficients to some combination of 1 and -1 in each part. Fixing n leading coefficients yields rather uneven search times, since fewer iterations of root-squaring are required on average for polynomials with certain prefixes. Instead, we fix the n coefficients nearest the middle term for quite uniform times across all 2^n parts. We use a Gray code to iterate over the possible values of the free coefficients.

3.3. *Results and analysis*

We ran our program at HPC@SFU, the high-performance computing centre at Simon Fraser University, on the Bugaboos, a Beowulf cluster with 96 nodes, each with two AMD Athlon 1.2 GHz processors. In two weeks we searched through degree 72, using as many as 64 processors at once and totalling 426 days of CPU time. Our program found 1643 Littlewood polynomials with degree at most 72 that survive ten iterations of root-squaring; of these, 1487 have measure less than $M = 5/3$. Only 127 distinct measures less than 5/3 appear, since most measures occur several times. For example, if f is a Littlewood polynomial of degree d, and k is a positive integer, then clearly $(x^{d+1} + 1)f(x)$ and $f(x^k)(x^k - 1)/(x - 1)$ are Littlewood polynomials with the same measure as f. The complete list of polynomials found is available at [**9**].

Table 1 lists the fifteen measures less than 1.6 found in our search. For each measure, the table lists the minimal degree d of a Littlewood polynomial with this measure, the degree d_0 of its noncyclotomic part, and the first half of the sequence of coefficients of a Littlewood polynomial realizing the measure, abbreviating to '+' for 1 and '−' for -1. Two of the polynomials in our table merit closer attention.

The seventh polynomial in Table 1 is the only one listed whose noncyclotomic part is reducible. Its noncyclotomic factors are $x^{10} - x^7 - x^5 - x^3 + 1$, with measure $1.230391\ldots$, and $x^{10} - x^9 + x^5 - x + 1$, with measure $1.283582\ldots$. Both appear in lists of known polynomials with particularly small measure [**3**, **9**]; the first one has the 33rd smallest known measure greater than 1. In fact, our search finds only one other example of a Littlewood polynomial whose noncyclotomic part is reducible. Its measure is $1.651512\ldots$, and its factors are Lehmer's polynomial $\ell(-x)$ and the polynomial

$$x^{20} + 2x^{19} + x^{18} - x^{17} - x^{16} - x^{13} - x^{12} + x^{10} - x^8 - x^7 - x^4 - x^3 + x^2 + 2x + 1.$$

Table 1. *Known measures of Littlewood polynomials less than* 1.6.

	Measure	d	d_0	Coefficients a_i, for $0 \leqslant i \leqslant d/2$.
1.	1.49671107561	19	12	++++-+-+--
2.	1.50613567955	11	6	+-+---
3.	1.50646000575	35	12	++------+++--+---+
4.	1.53691794778	23	14	++++----+-+-
5.	1.55107223951	23	12	+++++++--++-
6.	1.55603019132	6	6	++--
7.	1.57930874185	65	20	+--++--++--+--++--++--+--++--++--
8.	1.58234718368	7	6	++-+
9.	1.58501169305	35	24	++++++-------+-+-+
10.	1.59185616779	71	16	+-+--+-+--+-++-++++-+--+------++-+++
11.	1.59287323067	65	44	+-+-+-+-+-+-+-+-+-+-+-+++++++++++
12.	1.59341317381	19	12	++++-++++-
13.	1.59504631133	53	36	+-+-+-+-+-+-+-+-+----------
14.	1.59700500917	17	10	++-++-++-
15.	1.59918220880	41	26	+-+-+-+-+-+-+-+++++++

The tenth entry in Table 1 is the product of a noncyclotomic polynomial of degree 16 and a number of cyclotomic factors with total degree 55. The degree of 71 is best possible: no Littlewood polynomial of smaller degree has the same measure.

There is certainly an infinite number of polynomials having $\{-1, 0, 1\}$ coefficients with smaller measure. For example, the measure of

$$x^{2n+2} + x^{2n+1} + x^{n+2} + x^{n+1} + x^n + x + 1$$

approaches $1.255433\ldots$ as $n \to \infty$. This is the smallest known limit point of measures of integer polynomials. There are, in fact, an infinite number of limit points of measures of polynomials with $\{-1, 0, 1\}$ coefficients less than 1.382. The structure of known limit points of measures, and the 48 smallest known limit points, are described in [**5**]. There are also infinitely many integer polynomials with reducible noncyclotomic part having measure less than 1.4967, since two noncyclotomic polynomials are known with measure less than $1.4967/1.2554 \approx 1.1922$.

It therefore seems quite possible that Littlewood polynomials with Mahler measure smaller than $1.496711\ldots$ exist. It appears likely, however, that additional techniques would be required in further searches.

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