Polyphase sequences with low autocorrelation

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***Abstract*— Low autocorrelation for sequences is usually described in terms of low base energy, i.e., the sum of the sidelobe energies, or the maximum modulus of its autocorrelations, a Barker sequence occurring when this value is ≤ 1. We describe first an algorithm combining stochastic methods and calculus to finding polyphase sequences which are good local minima for the base energy. Starting from these, a second algorithm uses calculus to locate sequences which are local minima for the maximum modulus on autocorrelations. In our tabulation of smallest base energies found at various lengths, statistical evidence suggests we have good candidates for global minima or ground states up to length 45. We extend the list of known polyphase Barker sequences to length 63.**

I. Introduction

The acyclic autocorrelation coefficients or *sidelobes* for a sequence $A = [a_1, a_2, \ldots, a_n]$ are given by

$$c_k = \sum_{j=1}^{n-k} a_j \overline{a}_{j+k}, \quad (k = 1, ..n-1),$$

with $\overline{a}_{j+k}$ denoting the complex conjugate of a_{j+k}. Binary sequences, where $a_j = \pm 1$, with low sidelobe values have been extensively studied. Barker sequences [1], where $|c_k| = 0$ or 1 depending on whether the number of terms in this sum is even or odd, represent ideal cases and are known only for lengths 2,3,4,5,7,11 and 13. The Barker sequence condition was extended to $|c_k| \leq 1$ for sequences where the entries are roots of unity of the form $a_j = e^{2\pi i n_j / K}$ where $i^2 = -1$, drawn from a fixed alphabet of size K (see Golomb and Scholtz [9] for a more complete reference, Golomb and Zhang [12] for examples. For $K = 3, 4$ or 6 this still means $|c_k| = 0$ or 1, but for other values of K, it is possible to have $0 < |c_k| < 1$. Barker sequences for alphabet sizes of 3,4 and 6 are known up to lengths 15. These have been compiled through exhaustive searches. However, Barker sequences for longer lengths require larger alphabets and the possibility for exhaustive searches diminishes.

The more general setting allows for *polyphase* sequences, where the entries of A lie on the unit circle in the complex plane (see Boemer and Antweiler [2], and later work by Friese and Zottmann [7], Friese [6], Brenner [4]). Then, for $1 \leq j \leq n$, we have $a_j = e^{it_j}$ for some unique t_j with $0 \leq t_j < 2\pi$ leading to an equivalent formulation of the sequence as $T = [t_1, t_2, \ldots, t_n]$. The kth acyclic autocorrelation coefficient c_k may then be written

$$c_k = \sum_{j=1}^{n-k} e^{i(t_j - t_{j+k})}.$$

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Since $|c_{n-1}| = 1$ in any case, the Barker condition concerns the maximum value or l_∞-norm on the remaining coefficients $C = [c_1, c_2, \ldots, c_{n-2}]$, requiring

$$l_\infty(C) = \max\{|c_k| : 1 \leq k \leq n-2\} \leq 1.$$

Below, we tabulate the lowest l_∞-norms we have found at lengths up to 63. These values tend to rise as the lengths, but, because of the increasing complexity of the problem, it is difficult to speculate on a first length where a polyphase Barker sequence may no longer exists.

Earlier authors speculated that they may not exist beyond length 30. Friese [6] presented polyphase Barker sequences to length 36 and stated his disbelief that they would exist at lengths significantly greater. Brenner [4] gave sequences up to length 45. We extend this to length at least 63.

Other measures to describe low autocorrelation are the l_p-norms,

$$l_p(C) = \left(\sum_{k=0}^{n-2} |c_k|^p \right)^{1/p}$$

where $p \geq 1$. The most tractable of these, from a computational standpoint, seems to be $l_2(C)$. Indeed, historically, low autocorrelation for sequences has been described in terms of low *base energy*

$$E = \sum_{k=1}^{n-1} |c_k|^2 = 1 + l_2(C)^2,$$

which is the sum of the sidelobe energies, or high *merit factor* (see, for example, [8])

$$F = \frac{n^2}{2E}.$$

For binary sequences, Mertens [11] has tabulated minimum base energies up to length 48 (later extended to length 60), obtained through exhaustive search. Stochastic methods have been used to find longer sequences with low base energy, or equivalently, high merit factors. We suggest that extension of this effort to polyphase sequences may be the more useful approach. In any case, low l_∞ norm will mean low base energy. We describe a method for converting sequences of low base energy to sequences with low l_∞ norm.

We tabulate the lowest base energies and l_∞ norm found for sequences up to length 64. We present Barker sequences for lengths 46 to 63 in terms of the smallest alphabets found. These, naturally, are not sequences with the lowest l_∞ norm, but are obtained through a rounding process using some stochastic optimization.

II. An algorithm for finding low base energies

We note that $|c_k|^p$ is a sinusoidal function in each of the variables t_j. Consequently, the number of minima for the functions $l_p(C)$ proliferate as the length increases, making the direct use of calculus impractical for finding the best examples with low autocorrelation. Some researchers have used special types, for example Huffman sequences [6], as starting points and then used stochastic methods for improvement. Others have used stochastic methods alone to find sequences with low l_1 or l_2 norm [4]. The method used for this paper combines methods of calculus with stochastic variation.

Starting from some fixed sequence, we choose a coordinate, j' at random and regard the base energy as a function of the single variable $t_{j'}$. This enable us to rewrite

$$c_k = \sum_{j=1}^{n-k} e^{i(t_j - t_{j+k})}.$$

as

$$c_k = f_{k,1} + f_{k,2}e^{-it_{j'}} + f_{k,3}e^{it_{j'}}$$

where $f_{k,1} = \sum_{j=1\ (j,j+k\neq j')}^{n-k} e^{i(t_j - t_{j+k})}$, $f_{k,2} = e^{it_{j'-k}}$ for $j'-k>0$ and 0 otherwise, and $f_{k,3} = e^{-it_{j'+k}}$ for $j'+k \le n$ and 0 otherwise. This gives

$$|c_k|^2 = g_{k,1} + g_{k,2}e^{it_{j'}} + g_{k,3}e^{2it_{j'}} + \overline{g_{k,2}}e^{-it_{j'}} + \overline{g_{k,3}}e^{-2it_{j'}}$$

where $g_{k,1} = |f_{k,1}|^2 + |f_{k,2}|^2 + |f_{k,3}|^2$, $g_{k,2} = \overline{f_{k,1}}f_{k,3} + f_{k,1}\overline{f_{k,2}}$ and $g_{k,3} = \overline{f_{k,2}}f_{k,3}$, and enables us to write

$$\begin{aligned} l_2(C)^2 &= \sum_{k=1}^{n-2} g_{k,1} + \left(\sum_{k=1}^{n-2} g_{k,2}\right)e^{it_{j'}} + \left(\sum_{k=1}^{n-2} g_{k,3}\right)e^{2it_{j'}} \\ &\quad + \left(\sum_{k=1}^{n-2} \overline{g_{k,2}}\right)e^{-it_{j'}} + \left(\sum_{k=1}^{n-2} \overline{g_{k,3}}\right)e^{-2it_{j'}} \\ &= g_1 + g_2e^{it_{j'}} + g_3e^{2it_{j'}} + \overline{g_2}e^{-it_{j'}} + \overline{g_3}e^{-2it_{j'}} \\ &= g_1 + 2|g_2|\cos(t_{j'} + \arg(g_2)) + 2|g_3|\cos(2t_{j'} + \arg(g_3)). \end{aligned}$$

Over the interval $0 \le t_{j'} < 2\pi$, we find to find the point giving the smallest value of this function and reset $t_{j'}$ to this value. A study of this function shows that it can have up to two local minima in this interval. Using gradients or other methods may not proceed to the better one. Instead, we locate the minimum for $\cos(t_{j'} + \arg(g_2))$ and the closer of the two minima for $\cos(2t_{j'} + \arg(g_3))$ in this interval. We pick some intermediate point and use Newton's method from elementary calculus to find where the derivative is zero. This locates the minimum point we seek.

This is one iteration in the optimization. Since we minimize over the free free coordinate at each stage, the sequence of values for the base energy obtained by repeated iteration is decreasing. Because all values are positive, it will converge to a local minimum for the base energy. Resetting a number of coordinates at random and repeat this process enables us to obtain a selection of local minima for the base energy.

III. Other norms

The Barker condition, $\max_k |c_k| \le 1$, means that $(l_p(C))^p = \sum_{k=1}^{n-2} |c_k|^p \le n-2$ for all $p \ge 1$, i.e., these p-norm values are small. This suggests that a Barker sequence should lie near a minimum point for each of the p-norms, and, in fact, that these minimum points for different p values should be close together. Starting from a minimum point for the base energy, we use methods of calculus to find minimum points for l_p-norms for a sequence of increasing p-values, $p_1 = 2 < p_2 < p_3 \ldots$. Since

$$\lim_{p\to\infty} l_p(C) = l_\infty(C),$$

this is, effectively, a minimizing process for the l_∞-norm. Furthermore, whenever $l_p(C) \le 1$, we must have $\max_k |c_k| \le 1$, so the Barker condition is satisfied.

The components of the gradient of the l_p norm at a point T may be calculated from the formula

$$\begin{aligned} \frac{\partial l_p(C)}{\partial t_j} &= \frac{\partial}{\partial t_j}\left(\sum_{k=1}^{n-2} |c_k|^p\right)^{1/p} \\ &= \frac{1}{p}\left(\sum_{k=1}^{n-2} |c_k|^p\right)^{\frac{1}{p}-1} \sum_{k=1}^{n-2} \frac{p}{2}\left(|c_k|^2\right)^{\frac{p}{2}-1} \frac{\partial}{\partial t_j}|c_k|^2 \\ &= \sum_{k=1}^{n-2}\left(\frac{|c_k|^{p-2}}{2\left(\sum_{k=1}^{n-2}|c_k|^p\right)^{(p-1)/p}}\right)\frac{\partial}{\partial t_j}|c_k|^2 \\ &= \sum_{k=1}^{n-2}\left(\frac{\left|\frac{c_k}{l_\infty(C)}\right|^{p-2}}{2|l_\infty(C)|\left(\sum_{k=1}^{n-2}\left|\frac{c_k}{l_\infty(C)}\right|^p\right)^{(p-1)/p}}\right)\frac{\partial}{\partial t_j}|c_k|^2 \\ &= \sum_{k=1}^{n-2} \mathrm{wt}(k)\frac{\partial}{\partial t_j}|c_k|^2, \end{aligned}$$

a weighted sum of the components of the gradient of the $|c_k|^2$. Note that $|c_k/l_\infty(C)|^p = 1$ for at least one k, while for others it is ≤ 1. Thus, the sum in the denominators of the weights never approaches 0, in fact, it is always bounded between 1 and $n-2$ even when p is very large, so that, in general, these calculations can be carried to high accuracy. For each k, we obtain $\frac{\partial}{\partial t_j}|c_k|^2$ by differentiating the formula $|c_k|^2 = g_{k,1} + 2|g_{k,2}|\cos(t_j + \arg(g_{k,2})) + 2|g_{k,3}|\cos(2t_j + \arg(g_{k,3}))$, derivable from above.

Using gradients starting from a local minimum for the p_k norm, the algorithm used a line search method to find a local minimum for the p_{k+1} norm. Using a formula such as

$$p_{k+1} = 1.05p_k + 0.5$$

means the p values increase exponentially, though the initial growth is fairly slow, which we found led to better final values. We stopped where p is approximately 300,000, which involved finding minimum points for just over 200 p_k values. At this value the l_p norm is essentially the l_∞ norm on C. Though effective, this algorithm is quite expensive computationally. However, it was used only for the relatively small number of sequences found with sufficiently small l_2 norm, so that the running time of the combined algorithms was not appreciably affected.

IV. Barker sequences with small alphabets

For calculations in the algorithms described above, the entries for the polyphase sequences were rounded to millionth roots of unity, which, practically, gave a fine enough grid of points to adequately approximate continuous values. To test for a Barker sequence with smaller alphabet size K near a Barker sequence found for the alphabet of size one million, the coordinates were first rounded off to the coarser alphabet values. Beyond some value for K this would always yield a Barker sequence, but these are, in general, far from being the minimum alphabet size possible in this neighbourhood. To find if a smaller alphabet was possible, a stochastic algorithm for minimizing the l_∞ norm was used after this rounding. Iteratively, coordinates were chosen first at random, but for the final optimization, in sequence, and a switch of coordinate value made to a neighbouring value in the alphabet if the norm was lowered. To find the better alphabets, many repeated applications starting each time from the rounded values was usually required. In the spirit of submissions by other authors, we generally chose to look for alphabets sizes divisible by 10. More effort would undoubtedly find smaller alphabet sizes at most lengths. This algorithm is fairly expensive as well, but the number of polyphase Barker sequences to consider was not large. Figure 1 summarizes the optimal results found for sequences of lengths up to 64. Note that these entries are somewhat independent, i.e., the sequence with the lowest base energy does not necessarily lead to the sequence with the lowest l_∞-norm, which, in turn, may not lead to a Barker sequence with the smallest alphabet.

n	E	l_∞	K	n	E	l_∞	K
64	37.618	1.003		33	9.248	0.746	25
63	40.788	0.996	2000	32	14.107	0.814	32
62	39.882	0.997	3000	31	11.920	0.855	40
61	40.322	0.996	1930	30	14.016	0.869	50
60	36.919	0.952	210	29	10.972	0.787	30
59	37.271	0.976	340	28	13.730	0.809	35
58	31.457	0.988	500	27	9.840	0.821	29
57	35.289	0.964	240	26	11.294	0.810	25
56	31.309	0.965	190	25	8.199	0.755	30
55	28.656	0.949	150	24	10.810	0.784	30
54	32.632	0.961	200	23	7.500	0.731	25
53	27.688	0.918	100	22	9.856	0.823	26
52	30.389	0.952	185	21	7.738	0.814	30
51	22.570	0.830	50	20	8.554	0.777	26
50	27.530	0.963	150	19	4.773	0.642	15
49	24.113	0.902	90	18	9.742	0.844	24
48	24.616	0.885	70	17	6.116	0.711	15
47	17.595	0.896	80	16	6.491	0.795	14
46	24.826	0.921	90	15	4.702	0.753	4
45	22.491	0.898	78	14	5.430	0.788	6
44	21.977	0.891	75	13	4.507	0.715	2
43	20.362	0.842	60	12	6.718	0.897	6
42	21.460	0.894	60	11	3.258	0.674	2
41	16.044	0.842	60	10	4.493	0.825	6
40	16.992	0.871	55	9	1.059	0.112	3
39	15.561	0.872	60	8	2.743	0.662	6
38	17.113	0.820	40	7	2.142	0.522	2
37	15.102	0.818	43	6	5.000	1.000	6
36	17.588	0.872	55	5	2.000	0.770	2
35	13.838	0.857	46	4	1.500	0.500	2
34	14.782	0.838	45	3	1.000	0.000	2

Fig. 1. Statistics of the three different sequences found at each length n with (1) lowest base energy E, (2) lowest l_∞ norm, (3) smallest alphabet size K

V. Statistical analysis

In collecting data on sequences with base energy less than $0.6N$, we continued to run our programs at lengths up to 45 to the point that we could have good expectation that the best examples were collected. In advance, we do not know the size of the sample space for minimum points with energies in our range of interest. However, at some point, the continual repetition of previous examples suggests that we come close to exhausting the sample space. We use the statistical model described below to give substance to this observation. Part of this may be described as the inverse collector's problem as described in [5] and [10]. However, our problem is not specifically to establish the most probable size of the sample space, but to estimate the likelihood that we have found the example with the lowest base energy.

Let S be the size of the sample space, i.e., the total number of sequences, unique up to normalization as described in [9], with base energy $\leq 0.6N$ at a particular length N. Let t be the number of trials in terms of examples collected, d the number of these examples with different energy. Where M is the event that we have collected the optimal example, what we seek to evaluate is

$$P(M|t,d),$$

the probability that we have the best example, given the values t and d arising from our data. We make assumptions
(1) The the occurrence of the different examples are equilikely.
(2) There is no specific bias on the actual size of the sample space in the range where this is significant.

Then, we use

$$P(M|t,d) = \sum_{i\geq 0} P(M|S=d+i,d)P(S=d+i|t,d).$$

From our assumptions, we have $P(M|S=d+i,d) = d/(d+i)$. Using Baysian probability theory, we then derive

$$P(S=d+i|t,d) = \frac{P(d|S=d+i,t)}{\sum_{j\geq 0} P(d|S=d+j,t)}.$$

For $S = d+i$, the number of possible outcomes for t trials is $(d+i)^n$. Let $Z_d(t)$ be the number of permutations of d distinct objects taken t at a time, with each object appearing at least once. The the total number of possible outcomes with k different samples occurring is ${}_{d+i}C_d \; Z_d(t)$ and we obtain

$$P(M|t,d) = \sum_{i\geq 0} \frac{d}{d+i} \left(\frac{\binom{d+i}{d} Z_d(t)/(d+i)^t}{\sum_{j\geq 0} \binom{d+j}{d} Z_d(t)/(d+j)^t} \right)$$
$$= \frac{\sum_{i\geq 0} \binom{d+i}{d} d/(d+i)^{t+1}}{\sum_{i\geq 0} \binom{d+i}{d} /(d+i)^t}$$

n	d	t	E(S)	P(M)
40	31	126	31.6	0.98
41	62	292	62.6	0.99
42	33	410	33.0	0.99+
43	57	284	57.4	0.99
44	27	192	27.0	0.99+
45	43	129	45.9	0.94
46	9	19	11.9	0.79

TABLE I

RESULTS OF STATISTICAL ANALYSIS APPLIED AT LENGTH n, WITH d DIFFERENT BASE ENERGIES FOUND, t SAMPLES COLLECTED, $E(S)$ THE EXPECTED SIZE OF SAMPLE SPACE, $P(M)$ THE PROBABILITY GROUND STATE FOUND

In Table I we present results for data collected at lengths 40 through 46. For lengths up to 45 it was possible to run the programs longer so that confidence in these results is strong. At lengths beyond 45, the running times became prohibitive for obtaining much repetition, so our results for lengths 46 – 63 should not be considered to be global optima to within any reasonably high confidence level.

VI. New Barker sequences

Figure 2 lists examples of polyphase Barker sequences found at lengths n not previously reported in the literature. The phases are sequence coordinates in terms of powers of $\exp(2\pi i/K)$, normalized according to equivalences described in [9].

VII. Comments

In the process of optimizing the l_∞-norm in the neighbourhood of a small local minimum for the base energy, the base energy can increase considerably, and the end sequence may not be a Barker sequence. The local minima for the base energy are distinct; they are not related in terms of Golomb's invariance transformations [9]. This, in general, carries forward as well to the Barker sequences found for different minima. The number of these Barker sequences found increases from 1 at length 6 to 50 at length 22. Between lengths 22 and 45 at least 50 Barker sequences were found in each case, while above this the number decreased, so that only one example was found for each of the lengths 57 through 63. However, we suspect that this is more a factor the increasing difficulty in finding low base energy examples rather than their unavailability. At length 42, the algorithm found approximately 10 Barker sequences per day per machine (1.2 Gig). A rough analysis of running times versus data collected at different lengths suggests that at length 64, this might extrapolate to finding approximately 1 Barker sequence every 3 weeks using 50 machines, which is of the order of our effort expended at this length.

The analogy drawn between the coupon collector's problem and the collection of low base energy examples seems natural. The first assumption, namely, that of each different example is equilikely, we cannot expect to be strictly true. This method of analysis needs further study and probably refinement. We compare our experience here with in using a stochastic algorithm to locate low base energy examples for binary sequences, see [3]. Repeated application again leads to repetition of the lower base energy examples. At lengths up to, say, 55, for binary sequences, the examples found may be cross referenced to lists generated by exhaustive search to confirm that we are locating all of the sequences with base energy in this range.

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n	K	Phase
63	2000	$0, 0, 88, 200, 250, 89, 1832, 1668, 1792, 145, 308, 290, 528, 819, 1357, 1558, 1407, 1165, 930, 869, 274, 97, 10, 1857, 731, 789, 1736, 150, 1332, 1229, 390, 944, 1522, 1913, 648, 239, 1114, 1708, 200, 666, 1870, 1124, 1464, 265, 845, 1751, 1039, 53, 737, 1760, 798, 1880, 851, 1838, 1103, 419, 1711, 1155, 546, 1985, 1325, 754, 44$
62	3000	$0, 0, 459, 324, 361, 2987, 152, 432, 2963, 2907, 112, 598, 1276, 1489, 2216, 1814, 1505, 2536, 2949, 197, 1039, 1241, 2809, 2780, 1388, 590, 2233, 1352, 2458, 2284, 962, 172, 1453, 2245, 799, 558, 2461, 1258, 34, 1666, 2834, 1364, 2755, 1369, 2284, 796, 724, 2118, 198, 1327, 2858, 2962, 2021, 1774, 1604, 698, 1059, 100, 2995, 1923, 2278, 884$
61	1930	$0, 0, 58, 1761, 1762, 1703, 1724, 193, 721, 241, 247, 1855, 187, 416, 1379, 1421, 1385, 922, 362, 784, 1401, 1383, 584, 1709, 284, 807, 285, 373, 1404, 1739, 1173, 179, 750, 1, 1239, 1215, 1691, 1092, 490, 17, 160, 1047, 704, 536, 1515, 820, 1892, 1138, 1630, 139, 288, 1065, 1780, 733, 613, 1309, 1452, 550, 1673, 1049, 143$
60	210	$0, 0, 16, 208, 180, 153, 126, 161, 135, 78, 83, 98, 143, 127, 162, 153, 183, 141, 72, 207, 149, 167, 15, 13, 146, 58, 23, 109, 169, 208, 75, 143, 173, 199, 51, 50, 31, 142, 152, 84, 74, 6, 147, 205, 151, 66, 51, 151, 27, 101, 170, 75, 172, 91, 20, 131, 1, 78, 166, 68$
59	340	$0, 0, 5, 321, 293, 253, 251, 285, 268, 262, 286, 14, 96, 65, 33, 43, 152, 220, 235, 142, 71, 49, 262, 176, 285, 31, 181, 150, 305, 337, 108, 138, 13, 209, 274, 165, 24, 100, 320, 169, 221, 4, 48, 209, 339, 109, 192, 33, 222, 301, 128, 46, 228, 100, 299, 188, 45, 288, 124$
58	500	$0, 0, 1, 47, 209, 191, 154, 364, 437, 363, 420, 51, 437, 413, 277, 382, 78, 4, 428, 267, 308, 352, 238, 115, 205, 179, 474, 425, 234, 52, 443, 311, 382, 491, 400, 234, 297, 495, 492, 169, 397, 464, 75, 259, 476, 121, 437, 183, 34, 263, 0, 64, 242, 496, 292, 68, 318, 127$
57	240	$0, 0, 18, 51, 31, 37, 6, 39, 43, 64, 128, 167, 187, 19, 22, 226, 163, 103, 97, 238, 200, 172, 111, 201, 72, 95, 75, 172, 2, 91, 49, 220, 209, 57, 212, 168, 116, 206, 110, 102, 25, 131, 2, 30, 143, 182, 42, 107, 216, 89, 10, 161, 29, 170, 106, 205, 86$
56	190	$0, 0, 13, 37, 43, 95, 83, 115, 109, 145, 111, 12, 117, 86, 127, 116, 184, 109, 65, 121, 126, 116, 36, 92, 79, 85, 12, 1, 72, 183, 156, 135, 62, 139, 95, 16, 67, 134, 17, 138, 59, 92, 161, 46, 79, 176, 10, 127, 114, 48, 23, 148, 162, 88, 117, 35$
55	150	$0, 0, 8, 18, 18, 19, 22, 105, 100, 127, 119, 128, 117, 118, 53, 33, 112, 147, 132, 46, 30, 1, 133, 48, 117, 83, 31, 35, 38, 64, 144, 129, 100, 56, 39, 92, 104, 32, 140, 49, 110, 88, 14, 91, 134, 38, 84, 3, 111, 33, 95, 140, 43, 101, 19$

n	K	Phase
54	200	$0, 0, 23, 43, 16, 9, 40, 51, 20, 7, 67, 126, 178, 180, 71, 120, 144, 151, 61, 25, 45, 100, 86, 9, 172, 161, 142, 22, 85, 8, 96, 128, 81, 1, 18, 137, 0, 95, 132, 59, 44, 155, 16, 129, 157, 98, 47, 174, 73, 18, 145, 65, 170, 100$
53	100	$0, 0, 5, 3, 4, 5, 9, 13, 23, 58, 79, 99, 42, 63, 66, 99, 2, 41, 68, 29, 41, 76, 22, 25, 94, 98, 74, 59, 16, 58, 35, 62, 22, 93, 85, 19, 54, 17, 56, 94, 64, 92, 43, 26, 13, 70, 47, 95, 57, 21, 13, 86, 51$
52	185	$0, 0, 20, 11, 30, 26, 15, 27, 57, 26, 133, 97, 177, 149, 123, 45, 11, 140, 76, 85, 105, 3, 133, 31, 28, 58, 150, 103, 143, 39, 32, 137, 170, 100, 122, 58, 42, 86, 2, 172, 50, 128, 163, 49, 136, 76, 122, 17, 57, 20, 108, 171$
51	50	$0, 0, 4, 4, 18, 20, 27, 25, 25, 26, 24, 15, 15, 14, 9, 32, 36, 2, 21, 17, 9, 27, 46, 49, 19, 29, 9, 32, 7, 45, 21, 46, 22, 47, 18, 35, 0, 22, 9, 31, 44, 5, 29, 21, 4, 49, 33, 24, 9, 49, 29$
50	150	$0, 0, 16, 20, 44, 48, 72, 66, 103, 40, 142, 59, 4, 92, 129, 96, 112, 82, 58, 71, 94, 67, 1, 52, 58, 112, 92, 37, 14, 59, 107, 3, 68, 146, 71, 102, 40, 58, 0, 124, 62, 67, 129, 41, 51, 138, 136, 76, 66, 13$
49	90	$0, 0, 5, 12, 7, 1, 0, 88, 6, 25, 43, 68, 72, 51, 29, 13, 55, 62, 10, 21, 73, 79, 28, 23, 63, 50, 81, 57, 37, 5, 9, 23, 84, 61, 47, 54, 24, 75, 23, 88, 51, 7, 43, 78, 35, 65, 15, 51, 7$
48	70	$0, 0, 1, 5, 14, 23, 35, 36, 26, 22, 17, 5, 68, 16, 16, 51, 53, 0, 21, 13, 63, 50, 59, 43, 21, 1, 52, 27, 53, 62, 28, 28, 0, 55, 24, 51, 5, 22, 51, 15, 50, 8, 44, 21, 64, 24, 52, 12$
47	80	$0, 0, 10, 13, 15, 11, 9, 15, 31, 41, 66, 74, 5, 77, 46, 35, 65, 53, 32, 15, 77, 59, 37, 30, 42, 4, 8, 39, 74, 71, 25, 57, 60, 24, 54, 23, 31, 75, 19, 58, 13, 55, 11, 61, 33, 65, 28$
46	90	$0, 0, 3, 14, 21, 34, 50, 70, 75, 79, 57, 61, 47, 61, 79, 22, 55, 71, 71, 25, 44, 85, 9, 67, 5, 56, 81, 59, 26, 64, 11, 58, 25, 14, 83, 85, 62, 42, 4, 56, 23, 81, 50, 24, 11, 71$

Fig. 2. Barker sequences at lengths n with alphabet sizes K, the smallest found.