THE AVERAGE NORM OF POLYNOMIALS OF FIXED HEIGHT

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Abstract. Let $n \geq 0$ be any integer and

$$\mathfrak{F}_n := \left\{ \sum_{i=0}^{n} a_i z^i : a_i = 0, \pm 1 \right\}$$

be the set of all polynomials of height 1 and degree n. Let

$$\beta_n(m) := \frac{1}{3^{n+1}} \sum_{P \in \mathfrak{F}_n} \|P\|_m^m.$$

Here $\|P\|_m^m$ is the mth power of the L_m norm on the boundary of the unit disc. So $\beta_n(m)$ is the average of the mth power of the L_m norm over $\mathfrak{F}_n$.

In this paper we give exact formulae for $\beta_n(m)$ for various values of m. We also give a variety of related results for different classes of polynomials including: polynomials of fixed height H, polynomials with coefficients ± 1 and reciprocal polynomials. The results are surprisingly precise. Typical of the results we get is the following.

Theorem 0.1. *For $n \geq 0$, we have*

$$\beta_n(2) = \frac{2}{3}(n+1),$$

$$\beta_n(4) = \frac{8}{9}n^2 + \frac{14}{9}n + \frac{2}{3}$$

and

$$\beta_n(6) = \frac{16}{9}n^3 + 4n^2 + \frac{26}{9}n + \frac{2}{3}.$$

1. Introduction

We explore a variety of questions concerning the average norm of certain classes of polynomials on the unit disk. The Littlewood polynomials $\mathfrak{L}_n$ are defined as follows:

$$\mathfrak{L}_n := \left\{ \sum_{i=0}^{n} a_i z^i : a_i = \pm 1 \right\}.$$

Now let

$$\mu_n(m) := \frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \|P\|_m^m$$

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be the average of the mth power of the L_m norms over $\mathfrak{L}_n$. Here and throughout

$$\|P\|_m := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(z)|^m d\theta \right\}^{\frac{1}{m}}, \qquad (z = e^{i\theta})$$

is the L_m norm of P over the unit circle.

There are many limiting results concerning expected norms. For example the expected norms of random Littlewood polynomials P of degree n, satisfy

$$\lim_{n \longrightarrow \infty} \frac{1}{n^{m/2}} \left\{ \frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \|P\|_m^m \right\} = \Gamma(1 + m/2)$$

and for their derivatives

$$\lim_{n \longrightarrow \infty} \frac{1}{n^{m/(2r+1)2}} \left\{ \frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \|P^{(r)}\|_m^m \right\} = \frac{\Gamma(1 + m/2)}{(2r+1)^{m/2}}.$$

See [4] for a more detailed and precise discussion of these results.

In this paper, we are interested in finding exact formulae for $\mu_n(m)$. This is the content of the first section where results like the following are obtained.

Theorem 1.1. *For $n \geq 0$, we have*

$$\mu_n(2) = n + 1,$$
$$\mu_n(4) = 2n^2 + 3n + 1,$$
$$\mu_n(6) = 6n^3 + 9n^2 + 4n + 1$$

and

$$\mu_n(8) = 24n^4 + 30n^3 + 4n^2 + 5n + 4 - 3(-1)^n.$$

That $\mu_n(2) = n+1$ is trivial since $\|P\|_2 = n+1$ for each $P \in \mathfrak{L}_n$. The above result for $\mu_n(4)$ is due to Newman and Byrnes [9] though we offer a different proof. The results for $\mu_n(6)$ and $\mu_n(8)$ are new and are the tip of an iceberg that we explore further, but by no means exhaustively, in this paper by extending the results to: height H polynomials; reciprocal polynomials; polynomials with various roots of unity as coefficients. What is striking, and perhaps surprising, is that such exact formulae exist at all.

One interesting generalization is the following. Let

$$\mathfrak{F}_n(H) := \left\{ \sum_{i=0}^{n} a_i z^i : |a_i| \leq H, a_i \in \mathbb{Z} \right\}$$

be the set of all the polynomials of height H of degree $\leq n$. Let

$$\beta_n(m, H) := \frac{1}{(2H+1)^{n+1}} \sum_{P \in \mathfrak{F}_n(H)} \|P\|_m^m$$

which is now the average over all polynomials of degree n and height at most H of the mth power of the L_m norm. For this class we get results like

$$\beta_n(4, H) = \frac{2}{9} H^2 (H+1)^2 n^2 + \frac{1}{45} H(H+1)(19H^2 + 19H - 3)n + \frac{1}{15} H(H+1)(3H^2 + 3H - 1).$$

There is a considerable literature on the maximum and minimum norms of polynomials in $\mathfrak{L}_n$. In the L_4 norm this problem is often called Golay's "Merit Factor" problem. See [5, 3, 10]. The specific old and difficult problem is to find the minimum possible L_4 norm of a polynomial in $\mathfrak{L}_n$. The cognate problem in the supremum norm is due to Littlewood [7, 6]. Both of these problems are at least 50 years old and neither is solved though there has been significant progress on both. For further information on these and related problems see [2].

Exact formulae for $\mu_n(m)$ for larger values for m are obtained in [8].

2. Average of L_2, L_4 and L_6

As in the previous section let $n \geq 0$ be any integer and

$$\mathfrak{L}_n := \left\{ \sum_{i=0}^{n} a_i z^i : a_i = \pm 1 \right\}$$

be the set of all Littlewood polynomials of degree n. Let

$$\mu_n(m) := \frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \|P\|_m^m$$

be the average of the mth power of the L_m norm over $\mathfrak{L}_n$. We are interested in finding exact formulae for $\mu_n(m)$.

For any complex number z on the unit circle and any real number h, we have

$$\left.\begin{aligned} &|z+h|^2 + |z-h|^2 = 2(|z|^2 + h^2); \\ &|z+h|^4 + |z-h|^4 = 2(|z|^4 + 4h^2|z|^2 + h^4 + h^2(z^2 + \overline{z}^2)); \\ &|z+h|^6 + |z-h|^6 = 2(|z|^6 + 9|z|^4h^2 + 9|z|^2h^4 + h^6) \\ &\qquad\qquad\qquad\qquad\qquad + 6(|z|^2 + h^2)h^2(z^2 + \overline{z}^2). \end{aligned}\right\} \tag{2.1}$$

Hence for any polynomial $P(z)$,

$$\left.\begin{aligned} &\|zP(z)+h\|_2^2 + \|zP(z)-h\|_2^2 = 2(\|P(z)\|_2^2 + h^2); \\ &\|zP(z)+h\|_4^4 + \|zP(z)-h\|_4^4 = 2(\|P(z)\|_4^4 + 4h^2\|P(z)\|_2^2 + h^4); \\ &\|zP(z)+h\|_6^6 + \|zP(z)-h\|_6^6 = 2(\|P(z)\|_6^6 + 9\|P(z)\|_4^4 h^2 \\ &\quad + 9\|P(z)\|_2^2 h^4 + h^6) + \tfrac{6h^2}{2\pi}\textstyle\int_0^{2\pi} |P(z)|^2(z^2P(z)^2 + \overline{z}^2 P(\overline{z})^2)d\theta. \end{aligned}\right\} \tag{2.2}$$

Here we have used that $\int_0^{2\pi} z^2 P(z)^2 d\theta = 0$. Since every polynomial in $\mathfrak{L}_n$ can be written in the form of $zP(z) + h$ with $h = \pm 1$ and $P \in \mathfrak{L}_{n-1}$, we have

$$\sum_{P \in \mathfrak{L}_n} \|P\|_m^m = \sum_{P \in \mathfrak{L}_{n-1}} \left(\|zP(z)+1\|_m^m + \|zP(z)-1\|_m^m\right). \tag{2.3}$$

Theorem 2.1. *For $n \geq 0$, we have*

$$\mu_n(2) = n + 1, \tag{2.4}$$

$$\mu_n(4) = 2n^2 + 3n + 1 \tag{2.5}$$

and

$$\mu_n(6) = 6n^3 + 9n^2 + 4n + 1. \tag{2.6}$$

Proof. Formula (2.4) is trivial because every Littlewood polynomial of degree n has constant L_2 norm $\sqrt{n+1}$. However we give a simple inductive proof because it is indicative of the basic method behind all the proofs. Formula (2.5) was first proved by Byrnes and Newman in [9] but we give a simpler proof here.

Using (2.3) with $m=2$ and (2.2), we have

$$\begin{aligned}\mu_n(2) &= \frac{1}{2^{n+1}} \sum_{P\in\mathfrak{L}_{n-1}} \left(\|zP(z)+1\|_2^2 + \|zP(z)-1\|_2^2\right) \\ &= \frac{1}{2^{n+1}} \sum_{P\in\mathfrak{L}_{n-1}} 2(\|P\|_2^2+1) \\ &= \mu_{n-1}(2)+1 \qquad (2.7)\end{aligned}$$

for any $n \geq 1$. It is clear that $\mu_0(m)=1$ for any m. Thus from (2.7) we have

$$\mu_n(2) = \mu_0(2) + n = n+1.$$

This proves (2.4).

Similarly, using (2.2) and (2.3) with $m=4$, we have

$$\begin{aligned}\mu_n(4) &= \frac{1}{2^{n+1}} \sum_{P\in\mathfrak{L}_{n-1}} 2\left(\|P\|_4^4 + 4\|P\|_2^2 + 1\right) \\ &= \mu_{n-1}(4) + 4\mu_{n-1}(2) + 1\end{aligned}$$

for any $n \geq 1$. Using (2.4) and $\mu_0(4)=1$, we get (2.5).

To prove (2.6), we need the following two lemmas.

Lemma 2.2. *For $m \neq 0$, we have*

$$\sum_{P\in\mathfrak{L}_n} \int_0^{2\pi} |P(z)|^2 z^m \, d\theta = 0.$$

Proof. We prove this result by induction on n. When $n=0$, we have

$$\sum_{P\in\mathfrak{L}_0} \int_0^{2\pi} |P(z)|^2 z^m \, d\theta = 2\int_0^{2\pi} z^m \, d\theta = 0$$

because $\int_0^{2\pi} z^m d\theta = 0$ when $m \neq 0$. Now, by writing every polynomial in $\mathfrak{L}_n$ as $zP(z) \pm 1$ for some $P \in \mathfrak{L}_{n-1}$ and using (2.1) and the induction assumption, we have

$$\begin{aligned}\sum_{P\in\mathfrak{L}_n} \int_0^{2\pi} |P(z)|^2 z^m \, d\theta &= \sum_{P\in\mathfrak{L}_{n-1}} \int_0^{2\pi} \left\{|zP(z)+1|^2 + |zP(z)-1|^2\right\} z^m \, d\theta \\ &= \sum_{P\in\mathfrak{L}_{n-1}} \int_0^{2\pi} \left\{2|P(z)|^2 + 2\right\} z^m \, d\theta \\ &= 2\sum_{P\in\mathfrak{L}_{n-1}} \int_0^{2\pi} |P(z)|^2 z^m \, d\theta = 0.\end{aligned}$$

□

Lemma 2.3. *For $m \geq 1$, we have*

$$\sum_{P \in \mathfrak{L}_n} \int_0^{2\pi} |P(z)|^2 z^m P(z)^2 d\theta = 0.$$

Proof. We again prove the lemma by induction on n. Like Lemma 3.2, the case $n = 0$ is easy. Now

$$\begin{aligned} &\sum_{P \in \mathfrak{L}_n} \int_0^{2\pi} |P(z)|^2 z^m P(z)^2 d\theta \\ = &\sum_{P \in \mathfrak{L}_{n-1}} \int_0^{2\pi} \left\{ |zP(z)+1|^2 z^m (zP(z)+1)^2 + |zP(z)-1|^2 z^m (zP(z)-1)^2 \right\} d\theta. \end{aligned}$$

We expand the integrand out and after some simple cancellation we get that the above expression is

$$\begin{aligned} &\sum_{P \in \mathfrak{L}_{n-1}} \int_0^{2\pi} \left\{ 2|P(z)|^2 z^{m+2} P(z)^2 + 6|P(z)|^2 z^m + z^m (6z^2 P(z)^2 + 2) \right\} d\theta \\ = \; & 2 \sum_{P \in \mathfrak{L}_{n-1}} \int_0^{2\pi} |P(z)|^2 z^{m+2} P(z)^2 d\theta + 6 \sum_{P \in \mathfrak{L}_{n-1}} \int_0^{2\pi} |P(z)|^2 z^m d\theta \end{aligned}$$

because $z^m(6z^2P(z)^2 + 2)$ is a polynomial with zero constant term in z. Now the above integrals are equal to 0 by Lemma 2.2 and the induction assumption. □

We now prove (2.6). From (2.2), (2.4), (2.5) and Lemma 2.3

$$\begin{aligned} \mu_n(6) &= \frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_{n-1}} \left\{ \|zP(z)+1\|_6^6 + \|zP(z)-1\|_6^6 \right\} \\ &= \frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_{n-1}} 2\left\{ \|P\|_6^6 + 9\|P\|_4^4 + 9\|P\|_2^2 + 1 \right\} \\ &= \mu_{n-1}(6) + 9\mu_{n-1}(4) + 9\mu_{n-1}(2) + 1 \\ &= 6n^3 + 9n^2 + 4n + 1. \end{aligned}$$

This completes the proof of the theorem. □

Let $n \geq 0$ be any integer and

$$\mathfrak{F}_n := \left\{ \sum_{i=0}^{n} a_i z^i : a_i = 0, \pm 1 \right\}$$

be the set of all the polynomials of height 1 of degree n. Let

$$\beta_n(m) := \frac{1}{3^{n+1}} \sum_{P \in \mathfrak{F}_n} \|P\|_m^m.$$

As in Theorem 2.1, one can also obtain exact formulae for $\beta_n(m)$. The additional details involve observing that since

$$\|zP(z) + 0\|_m^m = \|P(z)\|_m^m$$

equations (2.2) can be extended easily to allow summing over all the height one polynomials. For example, for any polynomial $P(z)$, the middle identity of (2.2) becomes

$$\|zP(z)+h\|_4^4+\|zP(z)-h\|_4^4+\|zP(z)+0\|_4^4=3\|P(z)\|_4^4+8h^2\|P(z)\|_2^2+2h^4.$$

Theorem 2.4. *For $n \geq 0$, we have*

$$\beta_n(2)=\frac{2}{3}(n+1),$$

$$\beta_n(4)=\frac{8}{9}n^2+\frac{14}{9}n+\frac{2}{3}$$

and

$$\beta_n(6)=\frac{16}{9}n^3+4n^2+\frac{26}{9}n+\frac{2}{3}.$$

It is worth noting that the above technique can also be used to compute the averages of the norms of polynomials of height H. For example, one can show the following. Let $n \geq 0$ and $H \geq 1$ be integers and let

$$\mathfrak{F}_n(H):=\left\{\sum_{i=0}^{n}a_iz^i : |a_i|\leq H, a_i\in\mathbb{Z}\right\}$$

be the set of all the polynomials of height H and degree $\leq n$. Let

$$\beta_n(m,H):=\frac{1}{(2H+1)^{n+1}}\sum_{P\in\mathfrak{F}_n(H)}\|P\|_m^m.$$

Theorem 2.5. *For $n \geq 0$ and $H \geq 1$, we have*

$$\beta_n(2,H)=\frac{1}{3}H(H+1)(n+1),$$

$$\beta_n(4,H)=\frac{2}{9}H^2(H+1)^2n^2+\frac{1}{45}H(H+1)(19H^2+19H-3)n \\ +\frac{1}{15}H(H+1)(3H^2+3H-1)$$

and

$$\beta_n(6,H)=\frac{2}{9}H^3(H+1)^3n^3+\frac{1}{5}H^2(H+1)^2(3H^2+3H-1)n^2 \\ +\frac{1}{315}H(H+1)(164H^4+328H^3+56H^2-108H+15)n \\ +\frac{1}{21}H(H+1)(3H^4+6H^3-3H+1).$$

3. Formula for the Average of L_8

The formula for $\mu_n(8)$ is somewhat more complicated. We proceed as follows.

Lemma 3.1. *For $m \geq 0$, we have*

$$\frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} z^{2m} P(\overline{z})^2 d\theta = \begin{cases} 1 & \text{if } m \leq n; \\ 0 & \text{if } m > n. \end{cases}$$

Proof. This is clearly true for $m = 0$. Suppose $m \geq 1$. If $m \leq n$, then

$$\begin{aligned}
& \frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} z^{2m} P(\overline{z})^2 d\theta \\
= & \frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} z^{2m} \left\{ (\overline{z}P(\overline{z}) + 1)^2 + (\overline{z}P(\overline{z}) - 1)^2 \right\} d\theta \\
= & \frac{1}{2^{n}} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} z^{2m} (\overline{z}^2 P(\overline{z})^2 + 1) d\theta \\
= & \frac{1}{2^{n}} \sum_{P \in \mathfrak{L}_{n-1}} \frac{1}{2\pi} \int_0^{2\pi} z^{2(m-1)} P(\overline{z})^2 d\theta \\
= & \cdots = \frac{1}{2^{n-m+1}} \sum_{P \in \mathfrak{L}_{n-m}} \frac{1}{2\pi} \int_0^{2\pi} P(\overline{z})^2 d\theta \\
= & \frac{1}{2^{n-m+1}} \sum_{P \in \mathfrak{L}_{n-m}} 1 = 1.
\end{aligned}$$

If $m > n$, then

$$\begin{aligned}
\frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} z^{2m} P(\overline{z})^2 d\theta &= \frac{1}{2} \sum_{P \in \mathfrak{L}_0} \frac{1}{2\pi} \int_0^{2\pi} z^{2(m-n)} P(\overline{z})^2 d\theta \\
&= \frac{1}{2} \sum_{P \in \mathfrak{L}_0} \frac{1}{2\pi} \int_0^{2\pi} z^{2(m-n)} d\theta = 0.
\end{aligned}$$

□

Lemma 3.2. *For $m \geq 1$, we have*

$$\frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \frac{1}{2\pi} \int_0^{2\pi} z^{2m} |P(z)|^4 d\theta = \begin{cases} n - m + 1 & \text{if } m \leq n; \\ 0 & \text{if } m > n. \end{cases} \tag{3.1}$$

Proof. Let $B_n(m)$ be the left hand side of (3.1). By Lemma 2.2 and (2.1)

$$\begin{aligned}
B_n(m) &= \frac{1}{2^{n+1}} \sum_{P\in\mathfrak{L}_{n-1}} \frac{1}{2\pi}\int_0^{2\pi} z^{2m}(|zP(z)+1|^4+|zP(z)-1|^2)d\theta \\
&= \frac{1}{2^{n}} \sum_{P\in\mathfrak{L}_{n-1}} \frac{1}{2\pi}\int_0^{2\pi} z^{2m}(|P(z)|^4+4|p(z)|^2+1+z^2P(z)^2+\overline{z}^2P(\overline{z})^2)d\theta \\
&= B_{n-1}(m)+\frac{1}{2^{n}} \sum_{P\in\mathfrak{L}_{n-1}} \frac{1}{2\pi}\int_0^{2\pi} z^{2(m-1)}P(\overline{z})^2d\theta \\
&= B_{n-1}(m)+C_{n-1}(m-1)
\end{aligned}$$

where

$$C_n(m) := \frac{1}{2^{n+1}} \sum_{P\in\mathfrak{L}_{n}} \frac{1}{2\pi}\int_0^{2\pi} z^{2m}P(\overline{z})^2d\theta = \begin{cases} 1 & \text{if } m\le n; \\ 0 & \text{if } m>n; \end{cases}$$

by Lemma 3.1. Clearly,

$$B_0(m) = \frac{1}{2\pi}\int_0^{2\pi} z^{2m}d\theta = 0.$$

For $n\ge 1$, we have $B_n(m) = B_{n-1}(m)+C_{n-1}(m-1)$. If $m>n$, then $C_{n-1}(m-1) = 0$ and hence $B_n(m)=\cdots=B_0(m)=0$. If $m\le n$, then

$$\begin{aligned}
B_n(m) &= B_{n-1}(m)+C_{n-1}(m-1) \\
&= \cdots = C_{m-1}(m-1)+C_m(m-1)+\cdots+C_{n-1}(m-1) \\
&= n-m+1.
\end{aligned}$$

□

Lemma 3.3. *For $m\ge 1$, we have*

$$\frac{1}{2^{n+1}} \sum_{P\in\mathfrak{L}_{n}} \frac{1}{2\pi}\int_0^{2\pi} |P(z)|^4 z^{2m}P(z)^2d\theta$$
$$= \begin{cases} 0 & \text{if } m>n; \\ \frac{3}{2}(n-m)^2+\frac{7}{2}(n-m)+\frac{3-(-1)^{m+n}}{2} & \text{if } m\le n. \end{cases}$$

Proof. Let

$$A_n(m) := \frac{1}{2^{n+1}} \sum_{P\in\mathfrak{L}_{n}} \frac{1}{2\pi}\int_0^{2\pi} |P(z)|^4 z^{2m}P(z)^2d\theta.$$

Then

$$\begin{aligned}
A_n(m) = \frac{1}{2^{n+1}} \sum_{P\in\mathfrak{L}_{n}} \frac{1}{2\pi}\int_0^{2\pi} z^{2m}\big\{&|zP(z)+1|^4(zP(z)+1)^2 \\
&+|zP(z)-1|^4(zP(z)-1)^2\big\}\,d\theta.
\end{aligned}$$

We expand the integrand out and use Lemmas 3.1 and 3.2. We derive that for $n,m\ge 1$,

$$A_n(m) = A_{n-1}(m+1)+6B_{n-1}(m)+C_{n-1}(m-1).$$

It is clear that $A_0(m) = 0$. Thus if $m > n \geq 1$, then $B_{n-1}(m) = C_{n-1}(m-1) = 0$ and hence $A_n(m) = \cdots = A_0(m+n) = 0$. Suppose $1 \leq m \leq n$. If $n + m \equiv 0 \pmod 2$, then

$$\begin{aligned} & A_n(m) \\ = \;& A_{n-2}(m+2) + 6(B_{n-1}(m) + B_{n-2}(m+1)) + (C_{n-1}(m-1) + C_{n-2}(m)) \\ = \;& A_{\frac{n+m}{2}}\left(\frac{n+m}{2}\right) + 6B_{n-1}(m) + 6B_{n-2}(m+1) + \cdots + 6B_{\frac{m+n}{2}}\left(\frac{n+m}{2} - 1\right) \\ & + C_{n-1}(m-1) + C_{n-2}(m) + \cdots + C_{\frac{m+n}{2}}\left(\frac{m+n}{2} - 2\right). \end{aligned}$$

Since $A_m(m) = 6B_{m-1}(m) + C_{m-1}(m-1) = 1$ by Lemmas 3.1 and 3.2, we see that

$$\begin{aligned} A_n(m) &= 1 + 6(2 + 4 + \cdots + (n-m)) + \frac{n-m}{2} \\ &= \frac{3}{2}(n-m)^2 + \frac{7}{2}(n-m) + 1. \end{aligned}$$

The case $n + m \equiv 1 \pmod 2$ can be proved in the same way. □

In particular, $A_0(1) = 0$ and for $n \geq 1$

$$A_n(1) = \frac{3}{2}n^2 + \frac{1}{2}n + \frac{(-1)^n - 1}{2}. \tag{3.2}$$

We now come to the proof for the formula of $\mu(8)$. Since

$$\begin{aligned} |zP(z)+1|^8 + |zP(z)-1|^8 &= 2(|P(z)|^8 + 16|P(z)|^6 + 36|P(z)|^4 + 16|P(z)|^2 + 1) \\ + 2(6|P(z)|^4 z^2 P^2(z) &+ 6|P(z)|^4 \overline{z}^2 P^2(\overline{z}) + 16|P(z)|^2 z^2 P^2(z) + 16|P(z)|^2 \overline{z}^2 P^2(\overline{z})) \\ &+ 2(6P^2(z)z^2 + 6P^2(\overline{z})\overline{z}^2 + P(z)^4 z^4 + P(\overline{z})^4 \overline{z}^4). \end{aligned}$$

It follows from Lemmas 2.3 and 3.3 that

$$\mu_n(8) = \mu_{n-1}(8) + 16\mu_{n-1}(6) + 36\mu_{n-1}(4) + 16\mu_{n-1}(2) + 1 + 12A_{n-1}(1).$$

In view of Theorem 2.1 and (3.2), we have

$$\begin{aligned} \mu_n(8) &= \mu_{n-1}(8) + 16(6n^3 - 9n^2 + 4n) + 36(2n^2 - n) + 16n + 1 \\ & \quad + 18n^2 - 30n + 6 - 6(-1)^n \\ &= 24n^4 + 30n^3 + 4n^2 + 5n + 4 - 3(-1)^n. \end{aligned}$$

Thus we have proved

Theorem 3.4. *For $n \geq 0$, we have*

$$\frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \|P\|_8^8 = 24n^4 + 30n^3 + 4n^2 + 5n + 4 - 3(-1)^n.$$

4. Derivative and reciprocal polynomials

If we replace z by z/w in (2.1) then we have a homogeneous form of (2.1)

$$\left.\begin{aligned} |z+hw|^2 + |z-hw|^2 &= 2(|z|^2 + h^2|w|^2); \\ |z+hw|^4 + |z-hw|^4 &= 2(|z|^4 + 4h^2|z|^2|w|^2 + h^4|w|^4 \\ & \quad + h^2|w|^4((\tfrac{z}{w})^2 + (\tfrac{\overline{z}}{\overline{w}})^2)). \end{aligned}\right\} \tag{4.1}$$

Let $P^{(m)}(z)$ be the mth order derivative of $P(z)$.

Theorem 4.1. *For $n \geq 0$, we have*

$$\frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \|P^{(m)}\|_2^2 = \begin{cases} 0 & \text{if } m > n; \\ m!^2 \sum_{l=m}^{n} \binom{l}{m}^2 & \text{if } m \leq n; \end{cases} \tag{4.2}$$

and

$$\frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \|P^{(m)}\|_4^4 = \begin{cases} 0 & \text{if } m > n; \\ 2m!^4 \left(\sum_{l=m}^{n} \binom{l}{m}^2 \right)^2 - m!^4 \sum_{l=m}^{n} \binom{l}{m}^4 & \text{if } m \leq n. \end{cases} \tag{4.3}$$

Proof. We write every polynomial in $\mathfrak{L}_n$ as $P(z) \pm z^n$ for some $P \in \mathfrak{L}_{n-1}$. In view of (4.1), we have

$$\begin{aligned} &\frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \|P^{(m)}\|_2^2 \\ = &\frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_{n-1}} \left(\|P^{(m)}(z) + (z^n)^{(m)}\|_2^2 + \|P^{(m)}(z) - (z^n)^{(m)}\|_2^2 \right) \\ = &\frac{1}{2^n} \sum_{P \in \mathfrak{L}_{n-1}} \left(\|P^{(m)}\|_2^2 + m!^2 \binom{n}{m}^2 \right) \\ = &\frac{1}{2^n} \sum_{P \in \mathfrak{L}_{n-1}} \|P^{(m)}\|_2^2 + m!^2 \binom{n}{m}^2 \\ = &m!^2 \sum_{l=m}^{n} \binom{l}{m}^2. \end{aligned}$$

This proves (4.2). For (4.3), we have

$$\begin{aligned} &\frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \|P^{(m)}\|_4^4 \\ = &\frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_{n-1}} \left(\|P^{(m)}(z) + m! \binom{n}{m} z^{n-m}\|_4^4 + \|P^{(m)}(z) - m! \binom{n}{m} z^{n-m}\|_4^4 \right) \\ = &\frac{1}{2^n} \sum_{P \in \mathfrak{L}_{n-1}} \left(\|P^{(m)}\|_4^4 + 4m!^2 \binom{n}{m}^2 \|P^{(m)}\|_2^2 + m!^4 \binom{n}{m}^4 \right) \end{aligned}$$

because $P^{(m)}(z)$ is a polynomial of degree less than $n-m-1$ for any $P \in \mathfrak{L}_{n-1}$ and hence $\int_0^{2\pi} (P^{(m)}(z)/z^{n-m})^2 d\theta = 0$. Using (4.2), we have

$$\begin{aligned} &\frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \|P^{(m)}\|_4^4 \\ = &\frac{1}{2^n} \sum_{P \in \mathfrak{L}_{n-1}} \|P^{(m)}\|_4^4 + 4m!^4 \binom{n}{m}^2 \sum_{l=m}^{n-1} \binom{l}{m}^2 + m!^4 \binom{n}{m}^4. \end{aligned}$$

As before, this proves (4.3). □

A polynomial $P(z)$ of degree n is **reciprocal** if $P(z) = P^*(z)$ and **negative reciprocal** if $P(z) = -P^*(z)$ where $P^*(z) = z^n P\left(\frac{1}{z}\right)$. In view of (4.1), we have

$$\|P(z) + z^{n+1}P^*(z)\|_4^4 + \|P(z) - z^{n+1}P^*(z)\|_4^4 = 12\|P\|_4^4.$$

Since every reciprocal and negative reciprocal polynomial of degree $n = 2m+1$ can be written as $P(z) + z^{m+1}P^*(z)$ and $P(z) - z^{m+1}P^*(z)$ for some $P \in \mathfrak{L}_m$ respectively, so the average of $\|P\|_4^4$ over the reciprocal and negative reciprocal Littlewood polynomials in $\mathfrak{L}_n$ is

$$\begin{aligned} &= \frac{1}{2^{m+2}} \sum_{P\in\mathfrak{L}_m} \left\{\|P(z) + z^{m+1}P^*(z)\|_4^4 + \|P(z) - z^{m+1}P^*(z)\|_4^4\right\} \\ &= \frac{6}{2^{m+1}} \sum_{P\in\mathfrak{L}_m} \|P\|_4^4 \\ &= 6(2m^2+3m+1) = 3n^2+3n \end{aligned}$$

by Theorem 1.1. Now since

$$\begin{aligned} \sum\nolimits_{\substack{P\in\mathfrak{L}_n\\ P=P^*}} \|P\|_4^4 &= \sum_{P\in\mathfrak{L}_m} \|P(z) + z^{m+1}P^*(z)\|_4^4 \\ &= \sum_{P\in\mathfrak{L}_m} \|P(z) - z^{m+1}P^*(z)\|_4^4 = \sum\nolimits_{\substack{P\in\mathfrak{L}_n\\ P=-P^*}} \|P\|_4^4 \end{aligned}$$

we have proved that if n is odd

$$\frac{1}{2^{(n+1)/2}} \sum\nolimits_{\substack{P\in\mathfrak{L}_n\\ P=P^*}} \|P\|_4^4 = \frac{1}{2^{(n+1)/2}} \sum\nolimits_{\substack{P\in\mathfrak{L}_n\\ P=-P^*}} \|P\|_4^4 = 3n^2+3n.$$

By a similar argument, we can show that the average $\|P\|_4^4$ over the reciprocal Littlewood polynomials in $\mathfrak{L}_n$ is $3n^2+3n+1$ if n is even.

5. Polynomials with coefficients of modulus 1

The critical lemma in this analysis is Lemma 5.2 below.

Lemma 5.1. *For $m \geq l \geq 0$, we have*

$$\sum_{j=0}^{\min(l,m-l)} \binom{m}{2j}\binom{2j}{j}\binom{m-2j}{l-j} = \binom{m}{l}^2. \tag{5.1}$$

Proof. Without loss of generality, we may assume $l \le m/2$. Thus the left hand side of (5.1) becomes

$$\begin{aligned} &= \sum_{j=0}^{l} \binom{m}{2j}\binom{2j}{j}\binom{m-2j}{l-j} \\ &= \sum_{j=0}^{l} \frac{m!}{(j!)^2(l-j)!(m-l-j)!} \\ &= \binom{m}{l}\sum_{j=0}^{l}\binom{m-l}{j}\binom{l}{j} \\ &= \binom{m}{l}^2. \end{aligned}$$

□

Lemma 5.2. *Let $1 \le m \le k$ and $\zeta_k = e^{\frac{2\pi i}{k}}$. Then for any complex number z, we have*

$$\sum_{j=0}^{k-1} |z+\zeta_k^j|^{2m} = \begin{cases} k\sum_{l=0}^{m}\binom{m}{l}^2|z|^{2l}, & \text{if } k<m; \\ k\sum_{l=0}^{m}\binom{m}{l}^2|z|^{2l} + m(z^m+\overline{z}^m), & \text{if } k=m. \end{cases} \tag{5.2}$$

Proof. We first have

$$\begin{aligned} \sum_{j=0}^{k-1} |z+\zeta_k^j|^{2m} &= \sum_{j=0}^{k-1} (z+\zeta_k^j)^m(\overline{z}+\zeta_k^{-j})^m \\ &= \sum_{j=0}^{k-1} (|z|^2+1+(z\zeta_k^{-j}+\overline{z}\zeta_k^j))^m \\ &= \sum_{j=0}^{k-1}\sum_{l=0}^{m}\binom{m}{l}(|z|^2+1)^{m-l}\sum_{r=0}^{l}\binom{l}{r}(z\zeta_k^{-j})^r(\overline{z}\zeta_k^j)^{l-r} \\ &= \sum_{l=0}^{m}\sum_{r=0}^{l}\binom{m}{l}\binom{l}{r}(|z|^2+1)^{m-l}z^r\overline{z}^{l-r}\sum_{j=0}^{k-1}\zeta_k^{j(l-2r)}. \end{aligned}$$

Since $\sum_{j=0}^{k-1}\zeta_k^{ja}$ equals 0 if $a \not\equiv 0 \pmod{k}$ and equals to k otherwise, if $m<k$, then we have

$$\begin{aligned} \sum_{j=0}^{k-1} |z+\zeta_k^j|^{2m} &= k\sum_{l=0}^{m}\sum_{\substack{r=0 \\ l\equiv 2r \pmod{k}}}^{l}\binom{m}{l}\binom{l}{r}(|z|^2+1)^{m-l}z^r\overline{z}^{l-r} \\ &= k\sum_{\substack{l=0 \\ l \text{ is even}}}^{m}\binom{m}{l}\binom{l}{l/2}(|z|^2+1)^{m-l}|z|^l \\ &= k\sum_{0\le l\le m/2}\binom{m}{2l}\binom{2l}{l}(|z|^2+1)^{m-2l}|z|^{2l}. \end{aligned}$$

We then expand the term $(|z|^2+1)^{m-2l}$ to get

$$\begin{aligned}\sum_{j=0}^{k-1}|z+\zeta_k^j|^{2m} &= k\sum_{0\le l\le m/2}\binom{m}{2l}\binom{2l}{l}\sum_{r=0}^{m-2l}\binom{m-2l}{r}|z|^{2(l+r)}\\ &= k\sum_{j=0}^{m}\left\{\sum_{0\le l\le m/2}\sum_{\substack{r=0\\ r+l=j}}^{m-2l}\binom{m}{2l}\binom{2l}{l}\binom{m-2l}{r}\right\}|z|^{2l}\\ &= k\sum_{j=0}^{m}\left\{\sum_{0\le l\le\min(j,m-j)}\binom{m}{2l}\binom{2l}{l}\binom{m-2l}{j-l}\right\}|z|^{2l}\\ &= k\sum_{j=0}^{m}\binom{m}{j}^2|z|^{2j}\end{aligned}$$

by Lemma 5.1. If $m=k$, then

$$\sum_{j=0}^{k-1}|z+\zeta_k^j|^{2m}=k\sum_{0\le l\le m/2}\binom{m}{2l}\binom{2l}{l}(|z|^2+1)^{m-2l}|z|^{2l}+m(z^m+\overline{z}^m).$$

This proves our lemma. □

Let $n\ge 0$ and

$$\mathfrak{L}_{n,k}:=\left\{\sum_{i=0}^{n}a_iz^i : a_i^k=1\right\}$$

be the set of all polynomials of degree $\le n$ whose coefficients are kth root of unity. This is a generalization of Littlewood polynomials and clearly $\mathfrak{L}_n=\mathfrak{L}_{n,2}$. Then we have

Theorem 5.3. *For $n\ge 0$ and $m\le k$, we have*

$$\frac{1}{k^{n+1}}\sum_{P\in\mathfrak{L}_{n,k}}\|P\|_{2m}^{2m}=\sum_{l_1=0}^{m}\sum_{l_2=0}^{l_1}\cdots\sum_{l_n=0}^{l_{n-1}}\binom{m}{l_1}^2\binom{l_1}{l_2}^2\cdots\binom{l_{n-1}}{l_n}^2. \tag{5.3}$$

Proof. In view of Lemma 5.2, the left hand side of (5.3) is

$$\begin{aligned}&= \frac{1}{k^{n+1}}\sum_{P\in\mathfrak{L}_{n-1,k}}\sum_{j=0}^{k-1}\|zP(z)+\zeta_k^j\|_{2m}^{2m}\\ &= \sum_{l=0}^{m}\binom{m}{l}^2\left\{\frac{1}{k^n}\sum_{P\in\mathfrak{L}_{n-1,k}}\|P\|_{2l}^{2l}\right\}\\ &= \sum_{l_1=0}^{m}\sum_{l_2=0}^{l_1}\cdots\sum_{l_n=0}^{l_{n-1}}\binom{m}{l_1}^2\binom{l_1}{l_2}^2\cdots\binom{l_{n-1}}{l_n}^2\end{aligned}$$

because

$$\frac{1}{k}\sum_{P\in\mathfrak{L}_{0,k}}\|P\|_{2m}^{2m}=1.$$

This completes the proof. □

For example, for $k \geq 5$, the averages $\|P\|_m^m$ over the polynomials in $\mathfrak{L}_{n,k}$ are $n+1, 2n^2+3n+1, 6n^3+9n^2+4n+1, 24n^4+24n^3+10n^2+11n+1, 120n^5+50n^3+125n^2-44n+1$ when $m = 2, 4, 6, 8, 10$ respectively.

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