MERIT FACTORS OF CHARACTER POLYNOMIALS

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Abstract. Let q be a prime and χ be a non-principal character mod q. Let

$$f_\chi^t(z) := \sum_{n=0}^{q-1} \chi(n+t) z^n$$

for $1 \le t \le q$ be the character polynomial associated to χ (cyclically permuted t places).

Our principal result is the following.

Theorem. *For any non-principal and non-real character χ modulo q and $1 \le t \le q$, we have*

$$\|f_\chi^t(z)\|_4^4 = \frac{4}{3}q^2 + O(q^{3/2}\log^2 q)$$

where the implicit constant is independent of t and q. Here $\|\cdot\|_4$ denotes the L_4 norm on the unit circle.

It follows from this that all cyclically permuted character polynomials associated with non-principal and non-real characters have merit factors that approach 3. This compliments and completes results of Golay, Høholdt and Jensen, and Turyn (and others). These results show that the merit factors of cyclically permuted character polynomials associated with non-principal real characters vary asymptotically between 3/2 and 6.

We also compute the averages of the L_4 norms:

Theorem. *Let q be a prime number. We have*

$$\sum_{\chi \pmod q} \|f_\chi^t\|_4^4 = (2q-3)(q-1)^2$$

where the summation is over all characters modulo q.

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1. Introduction

The problem we address in this paper is the computation of the L_4 norm, or equivalently the merit factor, of cyclically permuted character polynomials associated with non-principal and non-real characters. Let q be a prime and χ be a non-principal character mod q. Let

$$f_\chi^t(z) := \sum_{n=0}^{q-1} \chi(n+t) z^n$$

for $1 \le t \le q$ be a permutation of the character polynomial associated to χ. As usual the L_p norm on the boundary of the unit disc is defined by by

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right)^{1/p}.$$

Any polynomial of degree n with coefficients of modulus 1 (as is the case for the character polynomials we consider) has L_2 norm $\sqrt{q}$. We are particularly interested in polynomials of the above form with small L_4 norm. These arise naturally in a variety of signal processing problems and have been much studied. See [BC-98, Go-77, Go-83, Bo-98, Ne-90]. The question is how close can the L_4 norm be made to the L_2 norm.

There are two natural measures of smallness for the L_4 norm of a polynomial f. One is the ratio of the L_4 norm to the L_2 norm, $\|f\|_4/\|f\|_2$. The other (obviously equivalent) measure is the merit factor, defined by

$$\mathrm{MF}(f) = \frac{\|f\|_2^4}{\|f|_4^4 - \|f\|_2^4}.$$

The merit factor is a useful normalization. It tends to give interesting sequences integer limits and makes the expected merit factor of a polynomial with ± 1 coefficients 1.

For polynomials with real coefficients of modulus 1 (that is coefficients ± 1) it is conjectured that the merit factor is bounded. The best asymptotic bound known is 6, which is approached, for q prime, by

$$R_q(z) := \sum_{k=0}^{q-1} \left(\frac{k + [q/4]}{q} \right) z^k$$

where $[\cdot]$ denotes the nearest integer. Here $\left(\frac{\cdot}{q}\right)$ denotes the Legendre symbol. This is an old observation of Turyn that was first proved in [Hø-88]. (The asymptotic bound of 6 has been conjectured to be best possible, though not, in the authors' opinion for any compelling reason.) In [BC-98] we show that the merit factor is explicitly given in terms of the class number. That is

$$\|R_q\|_4^4 = \frac{7q^2}{6} - q - \frac{1}{6} - \gamma_q$$

where

$$\gamma_q := \begin{cases} h(-q)(h(-q)-4) & \text{if} \quad q \equiv 1,5 \pmod 8, \\ 12(h(-q))^2 & \text{if} \quad q \equiv 3 \pmod 8, \\ 0 & \text{if} \quad q \equiv 7 \pmod 8. \end{cases}$$

and $h(-q)$ is the class number of $\mathbb{Q}(\sqrt{-q})$.

For polynomials with complex coefficients of modulus 1 it is possible to have asymptotically unbounded merit factors as the following example mostly due to Littlewood shows. Let

$$L_n(z) := \sum_{k=0}^{n-1} e^{\frac{k(k+1)\pi i}{n}} z^k$$

$$\|L_n\|_4^4 = n^2 + \frac{2n^{3/2}}{\pi} + \delta_n \frac{n^{1/2}}{3} + O(n^{-1/2})$$

where

$$\delta_n := \begin{cases} -2 & \text{if} \quad n \equiv 0, 1 \pmod 4, \\ 1 & \text{if} \quad n \equiv 2, 3 \pmod 4. \end{cases}$$

Another old problem of a similar flavor is due to Littlewood [Li-68]. See [Ka-80, Be-91, Saf-90]. It asks whether it is possible to have a sequence of polynomials with modulus 1 coefficients so that the L_∞ norms are asymptotic to the L_2 norm. Erdős conjectures that this is not possible if the coefficients are real (this has now been open for over forty years) though in a remarkable paper Kahane [Ka-80] shows that it is possible if the coefficients are complex and of modulus 1.

As mentioned in the abstract results of Golay, Høholdt and Jensen, and Turyn (and others) show that the merit factors of cyclically permuted character polynomials associated with non-principal real characters (the Legendre symbol) vary asymptotically between 3/2 and 6.

The main result of this paper, Theorem 3.2, shows that for non-real characters the asymptotic merit factor is always 3. The underlying methods are based on an interpolation formula of Høholdt and Jensen for the L_4 norm [Hø-88].

2. Explicit Formula for L_4 Norm

As before let q be a prime and χ be a non-principal character mod q. Let

$$f_\chi(z) := \sum_{n=0}^{q-1} \chi(n) z^n$$

be the character polynomial associated to χ. Let $\omega := e^{2\pi i/q}$ and $\tau(\chi)$ be the Gaussian sum defined by

$$\tau(\chi) := \sum_{n=1}^{q-1} \chi(n) \omega^n.$$

Since χ is primitive,

$$f_\chi(\omega^k) = \tau(\chi)\overline{\chi}(k). \tag{2.1}$$

for $k = 0, 1, \cdots, q-1$. Also we have, $|\tau(\chi)|^2 = q$ and $\overline{\tau(\chi)} = \chi(-1)\tau(\overline{\chi})$ (see Chapter 8 in [Ap-80]). Let

$$f_\chi^t(z) := \sum_{n=0}^{q-1} \chi(n+t) z^n$$

for $1 \le t \le q$. Thus, $f_\chi^t(z)$ is the character polynomial obtained by shifting the coefficients of $f_\chi(z)$ to the left by t. In particular, $f_\chi^q(z) = f_\chi(z)$. It should be noted that since not all the coefficients of $f_\chi^t(z)$ are non-zero, in particular, $f_\chi^1(z)$ has degree $q-2$ while $f_\chi^t(z)$ has degree $q-1$ for all $2 \le t \le q$. If χ is real and non-principal, then χ is just the Legendre symbol and $f_\chi(z)$ becomes the Fekete polynomial and we have shown in [BC-98] that if $\chi(n) = \left(\frac{n}{q}\right)$, then

$$\|f_\chi^t\|_4^4 = \frac{1}{3}(5q^2+3q+4) + 8t^2 - 4qt - 8t - \frac{8}{q^2}\left(1 - \frac{1}{2}\left(\frac{-1}{q}\right)\right)\left|\sum_{n=1}^{q-1} n\left(\frac{n+t}{q}\right)\right|^2, \tag{2.2}$$

and $\|f_\chi^{q-t+1}\|_4^4 = \|f_\chi^t\|_4^4$ if $1 \le t \le (q+1)/2$. From now on, we suppose χ is a non-principal and non-real character modulo q. The main result in this section is the following.

Theorem 2.1. *Let q be a prime and χ be a non-principal and non-real character modulo q. Then*

$$\|f_\chi^t\|_4^4 = \frac{1}{3q}(7q^3 - 21q^2 + 2q + 24qt - 12t^2) - \frac{8}{q^2}\left|\sum_{n=1}^{q-1} n\chi(n+t)\right|^2 - \frac{4}{q^3}\Re\left\{\tau(\chi)^2\tau(\overline{\chi}^2)H(\chi,t)\right\} \tag{2.3}$$

and $\|f_\chi^{q-t+1}\|_4^4 = \|f_\chi^t\|_4^4$ for $1 \le t \le (q+1)/2$. Here

$$H(\chi,t) = \sum_{n,m=1}^{q-1} nm\overline{\chi}(n+t)\overline{\chi}(m+t)\chi^2(m+n+2t).$$

Lemma 2.2. *For any $1 \le t \le q$, we have*

$$\sum_{b=1}^{q-1}\chi(b)\frac{\omega^{bt}}{\omega^b - 1} = \frac{\tau(\chi)}{q}\sum_{n=1}^{q-1} n\overline{\chi}(n+t).$$

Proof. Since

$$\frac{1}{\omega^j - 1} = \frac{1}{q}\sum_{n=1}^{q-1} n\omega^{jn}, \tag{2.4}$$

for $j = 1, \cdots, q-1$, it follows that

$$\begin{aligned}\sum_{b=1}^{q-1}\chi(b)\frac{\omega^{bt}}{\omega^b - 1} &= \sum_{b=1}^{q-1}\chi(b)\omega^{bt}\frac{1}{q}\sum_{n=1}^{q-1} n\omega^{bn} \\ &= \frac{1}{q}\sum_{n=1}^{q-1} n\sum_{b=1}^{q-1}\chi(b)\omega^{b(n+t)} \\ &= \frac{1}{q}\sum_{n=1}^{q-1} n f_\chi(\omega^{n+t}).\end{aligned}$$

Lemma 2.2 now follows from (2.1). □

Lemma 2.3. *If $1 \le k \le q$, then*

$$\sum_{\substack{n,m=1 \\ k+n+m \equiv 0 \pmod q}}^{q-1} nm = \frac{q}{6}(q^2 - 6q - 1 + 6k + 3qk - 3k^2). \tag{2.5}$$

Proof. This is Lemma 2 in [BC-98]. □

Lemma 2.4. *If $1 \le k \le q$, then*

$$\sum_{n=1}^{q-1} \frac{\omega^{nk}}{(\omega^n - 1)^2} = -\frac{1}{12}(q^2 + 6q + 5 - 12k - 6qk + 6k^2).$$

Proof. This is (3.6) in [BC-98]. □

It is easy to see that

$$f_\chi^t(\omega^k) = \omega^{-tk} f_\chi(\omega^k) \tag{2.6}$$

for any $0 \le k \le q-1$. We let

$$c_k := \sum_{\substack{n,m=0 \\ n-m=k}}^{q-1} \chi(n+t)\overline{\chi(m+t)}$$

then

$$\begin{aligned} \|f_\chi^t\|_4^4 = \int_0^1 |f_\chi^t(e^{2\pi i\theta})|^4 d\theta &= \int_0^1 \left| \sum_{k=-(q-1)}^{q-1} c_k e^{2\pi ik\theta} \right|^2 d\theta \\ &= \sum_{k=-(q-1)}^{q-1} |c_k|^2 = |c_0|^2 + 2\sum_{k=1}^{q-1} |c_k|^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{1}{q}\sum_{k=0}^{q-1} |f_\chi^t(\omega^k)|^4 &= \sum_{k=0}^{q-1} \left| \sum_{\substack{n,m=0 \\ n-m \equiv k \pmod q}}^{q-1} \chi(n+t)\overline{\chi(m+t)} \right|^2 \\ &= |c_0|^2 + \sum_{k=1}^{q-1} |c_k + \overline{c_{q-k}}|^2 \end{aligned}$$

and similarly

$$\frac{1}{q}\sum_{k=0}^{q-1} |f_\chi^t(-\omega^k)|^4 = |c_0|^2 + \sum_{k=1}^{q-1} |c_k - \overline{c_{q-k}}|^2.$$

Thus, we have

$$\|f_\chi^t\|_4^4 = \frac{1}{2q}\left\{\sum_{k=0}^{q-1}|f_\chi^t(\omega^k)|^4 + \sum_{k=0}^{q-1}|f_\chi^t(-\omega^k)|^4\right\}. \tag{2.7}$$

In view of (2.1) and (2.6), the first summation in (2.7) is $q^2(q-1)$. We are going to evaluate the second summation in (2.7).

For $1 \le t \le q$ and $0 \le k \le q-1$, we have

$$\begin{aligned}
f_\chi^{q-t+1}(-\omega^k) &= \sum_{n=0}^{q-1}\chi(n+q-t+1)(-\omega^k)^n \\
&= (-\omega^k)^{-(q-t+1)}\left\{-\sum_{n=1}^{q-t}+\sum_{n=q-t+1}^{q-1}\right\}\chi(n)(-\omega^k)^n \\
&= (-\omega^k)^{-(q-t+1)}\left\{-\sum_{n=t}^{q-1}+\sum_{n=1}^{t-1}\right\}\chi(q-n)(-\omega^k)^{q-n} \\
&= (-\omega^k)^{-(q+1)}\chi(-1)(-\omega^k)^t\left\{-\sum_{n=1}^{t-1}+\sum_{n=t}^{q-1}\right\}\chi(n)(-\omega^k)^n \\
&= \omega^{-k}\chi(-1)\overline{f_\chi^t(-\omega^k)}.
\end{aligned}$$

In particular, we have $|f_\chi^t(-\omega^k)| = |f_\chi^{q-t+1}(-\omega^k)|$ for $0 \le k \le q-1$ and hence from now on we may assume $1 \le t \le (q+1)/2$.

We employ an interpolation formula as in [Hø-88, BC-98] which is

$$\sum_{k=0}^{q-1}|f_\chi^t(-\omega^k)|^4 = \frac{16}{q^4}(A+B+C+D) \tag{2.8}$$

where

$$A = \frac{1}{48}q^2(q^2+2)\sum_{a=0}^{q-1}|f_\chi^t(\omega^a)|^4$$

$$B = \frac{q^2}{4}\sum_{\substack{a,b=0\\a\neq b}}^{q-1}|f_\chi^t(\omega^a)|^2(\overline{f_\chi^t(\omega^a)}f_\chi^t(\omega^b)\omega^b + f_\chi^t(\omega^a)\overline{f_\chi^t(\omega^b)}\omega^a)\left(\frac{\omega^a+\omega^b}{(\omega^a-\omega^b)^2}\right)$$

$$\begin{aligned}
C = &-\frac{q^2}{4}\sum_{\substack{a,b,c=0\\a\neq b\neq c}}^{q-1}2|f_\chi^t(\omega^a)|^2\left(\frac{f_\chi^t(\omega^b)\overline{f_\chi^t(\omega^c)}\omega^{a+b}+f_\chi^t(\omega^c)\overline{f_\chi^t(\omega^b)}\omega^{a+c}}{(\omega^b-\omega^a)(\omega^c-\omega^a)}\right) \\
&-\frac{q^2}{4}\sum_{\substack{a,b,c=0\\a\neq b\neq c}}^{q-1}\frac{f_\chi^t(\omega^a)^2\overline{f_\chi^t(\omega^b)}\,\overline{f_\chi^t(\omega^c)}\omega^{2a}+\overline{f_\chi^t(\omega^a)^2}f_\chi^t(\omega^b)f_\chi^t(\omega^c)\omega^{b+c}}{(\omega^b-\omega^a)(\omega^c-\omega^a)}
\end{aligned}$$

$$D = -\frac{q^2}{4}\sum_{\substack{a,b=0\\a\neq b}}^{q-1}\frac{4|f_\chi^t(\omega^a)|^2|f_\chi^t(\omega^b)|^2\omega^{a+b}+f_\chi^t(\omega^b)^2\overline{f_\chi^t(\omega^a)^2}\omega^{2b}+f_\chi^t(\omega^a)^2\overline{f_\chi^t(\omega^b)^2}\omega^{2a}}{(\omega^a-\omega^b)^2}.$$

We can further simplify the terms B, C and D. For the term B, we have

$$\begin{aligned} B &= \frac{q^2}{2}\Re\left\{\sum_{\substack{a,b=0\\a\neq b}}^{q-1}|f_\chi^t(\omega^a)|^2\frac{f_\chi^t(\omega^a)\overline{f_\chi^t(\omega^b)}\omega^a(\omega^a+\omega^b)}{(\omega^a-\omega^b)^2}\right\} \\ &= -\frac{q^2}{2}\Re\left\{\sum_{a=0}^{q-1}|f_\chi^t(\omega^a)|^2 f_\chi^t(\omega^a)\sum_{\substack{b=0\\b\neq a}}^{q-1}\frac{\overline{f_\chi^t(\omega^b)}(\omega^{a-b}+1)}{|\omega^{a-b}-1|^2}\right\} \\ &= -\frac{q^2}{2}\Re\left\{\sum_{a=0}^{q-1}|f_\chi^t(\omega^a)|^2 f_\chi^t(\omega^a)\sum_{k=1}^{q-1}\frac{\overline{f_\chi^t(\omega^{a-k})}(\omega^k+1)}{|\omega^k-1|^2}\right\}. \end{aligned} \tag{2.9}$$

Similarly, we have

$$\begin{aligned} C = q^2\sum_{a=0}^{q-1}|f_\chi^t(\omega^a)|^2\left\{\left|\sum_{k=1}^{q-1}\frac{f_\chi^t(\omega^{a-k})}{\omega^k-1}\right|^2-\sum_{k=1}^{q-1}\frac{|f_\chi^t(\omega^{a-k})|^2}{|\omega^k-1|^2}\right\} \\ -\frac{q^2}{2}\Re\left\{\sum_{a=0}^{q-1}\overline{(f_\chi^t(\omega^a))^2}\left\{\left(\sum_{k=1}^{q-1}\frac{f_\chi^t(\omega^{a-k})}{\omega^k-1}\right)^2-\sum_{k=1}^{q-1}\frac{(f_\chi^t(\omega^{a-k}))^2}{(\omega^k-1)^2}\right\}\right\} \end{aligned}$$

and

$$D = q^2\sum_{a=0}^{q-1}|f_\chi^t(\omega^a)|^2\sum_{k=1}^{q-1}\frac{|f_\chi^t(\omega^{a-k})|^2}{|\omega^k-1|^2}-\frac{q^2}{2}\Re\left\{\sum_{a=0}^{q-1}\overline{(f_\chi^t(\omega^a))^2}\sum_{k=1}^{q-1}\frac{(f_\chi^t(\omega^{a-k}))^2}{(\omega^k-1)^2}\right\}.$$

Thus,

$$C+D = q^2\sum_{a=0}^{q-1}|f_\chi^t(\omega^a)|^2\left|\sum_{k=1}^{q-1}\frac{f_\chi^t(\omega^{a-k})}{\omega^k-1}\right|^2-\frac{q^2}{2}\Re\left\{\sum_{a=0}^{q-1}\overline{(f_\chi^t(\omega^a))^2}\left(\sum_{k=1}^{q-1}\frac{f_\chi^t(\omega^{a-k})}{\omega^k-1}\right)^2\right\}. \tag{2.10}$$

We now evaluate A, B and $C+D$ separately. Using (2.1) and (2.6), we have

$$A = \frac{q^4(q-1)(q^2+2)}{48} \tag{2.11}$$

and

$$\begin{aligned}
B &= -\frac{q^2}{2}|\tau(\chi)|^4 \Re\left\{\sum_{a=0}^{q-1}\overline{\chi(a)}\sum_{k=1}^{q-1}\frac{\omega^{-tk}\chi(a-k)(\omega^k+1)}{|\omega^k-1|^2}\right\} \\
&= -\frac{q^4}{2}\Re\left\{\sum_{k=1}^{q-1}\frac{\omega^{-tk}(\omega^k+1)}{|\omega^k-1|^2}\sum_{a=0}^{q-1}\overline{\chi(a)}\chi(a-k)\right\} \\
&= \frac{q^4}{2}\Re\left\{\sum_{k=1}^{q-1}\frac{\omega^{-tk}(\omega^k+1)}{|\omega^k-1|^2}\right\} \\
&= -\frac{q^4}{2}\Re\left\{\sum_{k=1}^{q-1}\frac{\omega^{kt}(\omega^k+1)}{(\omega^k-1)^2}\right\} \\
&= -\frac{q^4}{2}\Re\left\{\sum_{k=1}^{q-1}\frac{\omega^{kt}+\omega^{k(t+1)}}{(\omega^k-1)^2}\right\} \\
&= \frac{q^4}{12}(q^2+3q+2-6t-6qt+6t^2) \qquad (2.12)
\end{aligned}$$

by (2.1), (2.6) ,(2.9) and Lemma 2.4. Here we have used the fact (see Lemma 2 in [CGP-98]) that

$$\sum_{a=0}^{q-1}\overline{\chi(a)}\chi(a-k) = \begin{cases} q-1 & \text{if } k\equiv 0 \pmod q \\ -1 & \text{if } k\not\equiv 0 \pmod q. \end{cases} \qquad (2.13)$$

Using (2.1), (2.4), (2.6), (2.13) and Lemma 2.2, the first term of (2.10) equals

$$\begin{aligned}
& q^2|\tau(\chi)|^4\sum_{a=0}^{q-1}|\chi(a)|^2\left|\sum_{k=1}^{q-1}\frac{\omega^{kt}\chi(a-k)}{\omega^k-1}\right|^2 \\
=& q^4\sum_{a=1}^{q-1}\left|\sum_{k=1}^{q-1}\frac{\omega^{kt}\chi(a-k)}{\omega^k-1}\right|^2 \\
=& q^4\sum_{k,l=1}^{q-1}\frac{\omega^{t(k-l)}}{(\omega^k-1)(\omega^{-l}-1)}\left\{\sum_{a=0}^{q-1}\chi(a-k)\overline{\chi(a-l)}-\chi(k)\overline{\chi(l)}\right\} \\
=& q^5\sum_{k=1}^{q-1}\frac{1}{|\omega^k-1|^2}-q^4\left|\sum_{k=1}^{q-1}\frac{\omega^{tk}}{\omega^k-1}\right|^2-q^4\left|\sum_{k=1}^{q-1}\frac{\omega^{tk}\chi(k)}{\omega^k-1}\right|^2 \\
=& \frac{q^5(q^2-1)}{12}-q^4\left(q-t-\frac{q-1}{2}\right)^2-q^3\left|\sum_{n=1}^{q-1}n\chi(n+t)\right|^2 \\
=& \frac{q^4(q-1)(q^2-2q+3)}{12}+q^4(q-t)(t-1)-q^3\left|\sum_{n=1}^{q-1}n\chi(n+t)\right|^2. \qquad (2.14)
\end{aligned}$$

It remains to evaluate the second term in (2.10) and using (2.1), (2.4) and (2.6) again, this is equal

to

$$-\frac{q^2}{2}|\tau(\chi)|^4\Re\left\{\sum_{a=0}^{q-1}\chi^2(a)\left(\sum_{k=1}^{q-1}\frac{\omega^{tk}\overline{\chi(a-k)}}{\omega^k-1}\right)^2\right\}$$
$$=-\frac{q^4}{2}\Re\left\{\sum_{a=0}^{q-1}\chi^2(a)\left(\frac{1}{q}\sum_{n=1}^{q-1}n\sum_{k=1}^{q-1}\overline{\chi(a-k)}\omega^{k(t+n)}\right)^2\right\}.$$

Using (2.1), this equals

$$=-\frac{q^4}{2}\Re\left\{\sum_{a=0}^{q-1}\chi^2(a)\left(\frac{1}{q}\sum_{n=1}^{q-1}n\sum_{k=0}^{q-1}\overline{\chi(k)}\omega^{(a-k)(t+n)}-\frac{q-1}{2}\overline{\chi(a)}\right)^2\right\}$$
$$=-\frac{q^4}{2}\Re\left\{\sum_{a=0}^{q-1}\chi^2(a)\left(\frac{1}{q}\sum_{n=1}^{q-1}n\omega^{a(t+n)}\overline{f_\chi(\omega^{t+n})}-\frac{q-1}{2}\overline{\chi(a)}\right)^2\right\}$$
$$=-\frac{q^4}{2}\Re\left\{\sum_{a=0}^{q-1}\chi^2(a)\left(\frac{\overline{\tau(\chi)}}{q}\sum_{n=1}^{q-1}n\omega^{a(t+n)}\chi(t+n)-\frac{q-1}{2}\overline{\chi(a)}\right)^2\right\}.$$

We now multiply the inner square out and interchange the order of summation and get

$$=-\frac{q^2}{2}\Re\left\{\overline{\tau(\chi)}^2\sum_{n,m=1}^{q-1}nm\chi(n+t)\chi(m+t)\sum_{a=0}^{q-1}\chi^2(a)\omega^{a(n+m+2t)}\right\}$$
$$-\frac{q^4(q-1)^3}{8}+\frac{q^3(q-1)}{2}\Re\left\{\overline{\tau(\chi)}\sum_{n=1}^{q-1}n\chi(n+t)\sum_{a=0}^{q-1}\chi(a)\omega^{a(n+t)}\right\}$$
$$=-\frac{q^2}{2}\Re\left\{\tau(\chi)^2\tau(\overline{\chi}^2)H(\chi,t)\right\}-\frac{q^4(q-1)^3}{8}+\frac{q^4(q-1)}{2}\sum_{n=1}^{q-1}n|\chi(n+t)|^2$$
$$=-\frac{q^2}{2}\Re\left\{\tau(\chi)^2\tau(\overline{\chi}^2)H(\chi,t)\right\}-\frac{q^4(q-1)^3}{8}+\frac{q^4(q-1)}{2}\left(\frac{q(q-1)}{2}-(q-t)\right)$$

(2.15)

$$=-\frac{q^2}{2}\Re\left\{\tau(\chi)^2\tau(\overline{\chi}^2)H(\chi,t)\right\}+\frac{q^4(q-1)(q^2-1)}{8}-\frac{1}{2}q^4(q-1)(q-t).$$

Here we have used the fact that χ^2 is not a principal character for non-real χ and hence (2.1) can be applied. Therefore, from (2.8), (2.10), (2.11), (2.12), (2.14) and (2.15) we have

$$\sum_{j=0}^{q-1}|f_\chi^t(-\omega^j)|^4=\frac{1}{3}(11q^3-39q^2+4q+48qt-24t^2)-\frac{16}{q}\left|\sum_{n=1}^{q-1}n\chi(n+t)\right|^2-\frac{8}{q^2}\Re\left\{\tau(\chi)^2\tau(\overline{\chi}^2)H(\chi,t)\right\}.$$

Finally, from (2.7), we conclude that

$$\|f_\chi^t\|_4^4=\frac{1}{3q}(7q^3-21q^2+2q+24qt-12t^2)-\frac{8}{q^2}\left|\sum_{n=1}^{q-1}n\chi(n+t)\right|^2-\frac{4}{q^3}\Re\left\{\tau(\chi)^2\tau(\overline{\chi}^2)H(\chi,t)\right\}$$

and completes the proof of Theorem 2.1.

3. Asymptotic Formula for L_4 Norm

In this section, we are going to prove the asymptotic formula for the L_4 norm of character polynomials. In view of (2.3), we need to estimate the following character sums $H(\chi, t)$.

Theorem 3.1. *For any non-principal and non-real character χ modulo q and $1 \le t \le q$, we have*

$$H(\chi, t) = \frac{q^3 \tau(\overline{\chi})^2}{4\tau(\overline{\chi}^2)} + O(q^3 \log^2 q) \tag{3.1}$$

where the implicit constant is independent of t and q.

Using (2.3), we obtain the following result immediately from (3.1).

Theorem 3.2. *For any non-principal and non-real character χ modulo q and $1 \le t \le q$, we have*

$$\|f_\chi^t(z)\|_4^4 = \frac{4}{3}q^2 + O(q^{3/2} \log^2 q)$$

where the implicit constant is independent of t and q.

Proof. Theorem 3.2 follows from Theorems 2.1 and 3.1 on observing that

$$\frac{1}{q}\sum_{n=1}^{q-1} n\chi(n+t) = \sum_{n=1}^{t-1}\chi(n) + \frac{1}{q}\sum_{n=1}^{q-1} n\chi(n).$$

This is coupled with Pólya's inequality (see Theorem 8.21 in [Ap-80])

$$\left|\sum_{n=1}^{t-1}\chi(n)\right| < q^{\frac{1}{2}}\log q$$

and on using partial summation formula, we have

$$\left|\sum_{n=1}^{q-1} n\chi(n)\right| < q^{\frac{3}{2}}\log q.$$

□

When q is large and $\frac{3-\sqrt{3}}{12} \le \frac{t}{q} \le \frac{3+\sqrt{3}}{12}$, then $\|f^t_{\left(\frac{\cdot}{q}\right)}(z)\|_4^4 < \frac{4}{3}q^2 < \|f_\chi^t(z)\|_4^4$. So the shifted character polynomial for the real character usually has the smallest L_4 norm among all the other characters modulo q.

Lemma 3.3. *For any character χ modulo q and $0 \le t \le q-1$, we have*

$$H(\chi, t) = q^2 \sum_{m=1}^{q-1} \overline{\chi}(m)\chi^2(m+1) \sum_{n=1}^{q-1} \left\{\frac{n-t}{q}\right\}\left\{\frac{nm-t}{q}\right\}$$

where $\{x\}$ is the fractional part of x.

Proof. We first observe that $q\left\{\frac{n}{q}\right\}$ is the least non-negative residue of n modulo q. It follows that

$$\begin{aligned}
H(\chi,t) &= \sum_{n,m=1}^{q-1} nm\overline{\chi}(n+t)\overline{\chi}(m+t)\chi^2(m+n+2t)\\
&= q^2\sum_{n,m=1}^{q-1}\left\{\frac{n}{q}\right\}\left\{\frac{m}{q}\right\}\overline{\chi}(n+t)\overline{\chi}(m+t)\chi^2(m+n+2t)\\
&= q^2\sum_{n,m=1}^{q-1}\left\{\frac{n-t}{q}\right\}\left\{\frac{m-t}{q}\right\}\overline{\chi}(n)\overline{\chi}(m)\chi^2(m+n)\\
&= q^2\sum_{n=1}^{q-1}\left\{\frac{n-t}{q}\right\}\overline{\chi}(n)\sum_{m=1}^{q-1}\left\{\frac{nm-t}{q}\right\}\overline{\chi}(mn)\chi^2(mn+n)\\
&= q^2\sum_{n=1}^{q-1}\left\{\frac{n-t}{q}\right\}\sum_{m=1}^{q-1}\left\{\frac{nm-t}{q}\right\}\overline{\chi}(m)\chi^2(m+1)\\
&= q^2\sum_{m=1}^{q-1}\overline{\chi}(m)\chi^2(m+1)\sum_{n=1}^{q-1}\left\{\frac{n-t}{q}\right\}\left\{\frac{nm-t}{q}\right\}
\end{aligned}$$

as claimed. □

Lemma 3.4. *For any non-principal character χ modulo q, we have*

$$\sum_{n=1}^{q}\overline{\chi}(n)\chi^2(n+m) = \chi(m)J(\overline{\chi},\overline{\chi}) = \begin{cases} -\left(\frac{-m}{q}\right) & \text{if } \chi(n)=\left(\frac{n}{q}\right),\\ \chi(m)\frac{\tau(\overline{\chi})^2}{\tau(\overline{\chi}^2)} & \text{otherwise,}\end{cases}$$

where the Jacobi sum (see Chapter 2 of [BEW-98]), J, is defined as

$$J(\chi,\psi) := \sum_{n=1}^{q}\chi(n)\psi(1-n)$$

for characters χ and ψ modulo q.

Proof. Let

$$G_m := \sum_{n=1}^{q-1}\overline{\chi}(n)\chi^2(n+m).$$

If $(m,q)=1$ then

$$\begin{aligned}
G_m &= \sum_{n=1}^{q-1}\overline{\chi}(nm)\chi^2(nm+m)\\
&= \chi(m)G_1.
\end{aligned}$$

Hence

$$\begin{aligned}
G_1 &= \frac{1}{q-1}\sum_{m=1}^{q-1}\overline{\chi}(m)G_m = \frac{1}{q-1}\sum_{m=1}^{q-1}\overline{\chi}(m)\sum_{n=1}^{q}\overline{\chi}(n)\chi^2(n+m)\\
&= \frac{1}{q-1}\sum_{m=1}^{q-1}\overline{\chi}(m)\sum_{n=1}^{q}\overline{\chi}(n-m)\chi^2(n) = \frac{1}{q-1}\sum_{n=1}^{q}\chi^2(n)\sum_{m=1}^{q-1}\overline{\chi}(m)\overline{\chi}(n-m)\\
&= \frac{1}{q-1}\sum_{n=1}^{q}\chi^2(n)\sum_{m=1}^{q}\overline{\chi}(mn)\overline{\chi}(n-nm) = \frac{1}{q-1}\sum_{n=1}^{q-1}\sum_{m=1}^{q}\overline{\chi}(m)\overline{\chi}(1-m)\\
&= \sum_{m=1}^{q}\overline{\chi}(m)\overline{\chi}(1-m) = J(\overline{\chi},\overline{\chi}).
\end{aligned}$$

Lemma 3.4 follows from this and Theorem 2.1.1 and 2.1.3 in [BEW-98]. □

Let $\psi(x)$ be the well-known saw-tooth function of period 1 which is defined by

$$\psi(x) := \begin{cases} \{x\} - \frac{1}{2} & \text{if } x \text{ is not an integer,}\\ 0 & \text{otherwise.}\end{cases}$$

For any coprime positive integers h and k, let

$$s(h,k) := \sum_{n=1}^{k}\psi\left(\frac{n}{k}\right)\psi\left(\frac{hn}{k}\right)$$

be the usual Dedekind sums. One of the most important properties of Dedekind sums is the reciprocity law. For estimation of the shifted character sums, we need to consider the generalized Dedekind sums which are defined by

$$s(h,k;x,y) := \sum_{n=1}^{k}\psi\left(\frac{n+y}{k}\right)\psi\left(\frac{h(n+y)}{k}+x\right)$$

for any coprime positive integers h,k and any real numbers x,y. The generalized Dedekind sums also possess a reciprocity law (e.g. p. 64 in [GR72]):

$$\begin{aligned}
s(h,k;x,y)+s(k,h;y,x) = &-\frac{1}{4}\delta(x)\delta(y)+\psi(x)\psi(y)\\
&+\frac{1}{2}\left(\frac{h}{k}\Psi_2(y)+\frac{1}{hk}\Psi_2(hy+kx)+\frac{k}{h}\Psi_2(x)\right)
\end{aligned}$$

where $\Psi_2(x) = B_2(\{x\})$. Here $B_2(z)$ is the second Bernoulli polynomial and

$$\delta(x) := \begin{cases} 1 & \text{if } x \text{ is an integer,}\\ 0 & \text{otherwise.}\end{cases}$$

As a result of the reciprocity law, we have the following lemma.

Lemma 3.5. *For any positive integer d and any real numbers x and y, we have*

$$\sum_{\substack{n=1\\(n,d)=1}}^{d} |s(n,d;x,y)| \ll d\log^2 d.$$

Here the implicit constant is independent of d, x and y.

Proof. When $x = y = 0$, Lemma 3.5 is just Lemma 6 in [CFKS-96]. For the general case, we may assume $0 \le x, y < 1$ and on using (0.6) and (1.2) in [Kn-77], we have

$$\begin{aligned}\sum_{\substack{n=1\\(n,d)=1}}^{d} |s(n,d;x,y)| &\ll \sum_{\substack{n=1\\(n,d)=1}}^{d} |\sigma(n,d;ny+dx)| + \phi(d)\\ &\ll \sum_{\substack{n=1\\(n,d)=1}}^{d} |\sigma(n,d;[ny+dx])| + \phi(d)\end{aligned}$$

where $\sigma(h,k;c) = 12\sum_{n=1}^{k} \psi\left(\frac{n}{k}\right)\psi\left(\frac{hn+c}{k}\right)$ and $[x]$ is the integral part of x. Then from Theorem 2 in [Kn-77],

$$\sum_{\substack{n=1\\(n,d)=1}}^{d} |s(n,d;x,y)| \ll \sum_{\substack{n=1\\(n,d)=1}}^{d} N(n,d) + \phi(d)$$

where $N(n,d)$ is the sum of all quotients of the continued fraction of n/d with $1 \le n \le d$ and $(n,d) = 1$. Now using the results from Knuth and Yao in [KY-75], we have

$$\sum_{\substack{n=1\\(n,d)=1}}^{d} N(n,d) \ll d\log^2 d$$

and this proves our lemma. □

In view of Lemma 3.3, $H(\chi,t)$ can be in terms of generalized Dedekind sums.

Lemma 3.6. *For any non-principal and non-real character χ modulo q and $0 \le t \le q-1$, we have*

$$H(\chi,t) = \frac{q^3\tau(\overline{\chi})^2}{4\tau(\overline{\chi}^2)} + q^2\sum_{m=1}^{q-1}\overline{\chi}(m)\chi^2(m+1)s\left(m,q;\frac{(m-1)t}{q},0\right) + O(q^3).$$

Proof. Since

$$\sum_{n=1}^{q-1}\psi\left(\frac{n}{q}\right) = 0$$

it follows from Lemmas 3.3 and 3.4 that

$$\begin{aligned}
H(\chi,t) &= q^2\sum_{m=1}^{q-1}\overline{\chi}(m)\chi^2(m+1)\sum_{n=1}^{q-1}\left(\psi\left(\frac{n-t}{q}\right)+\frac{1}{2}\right)\left(\psi\left(\frac{nm-t}{q}\right)+\frac{1}{2}\right)+O(q^3)\\
&= \frac{q^3\tau(\overline{\chi})^2}{4\tau(\overline{\chi}^2)}+q^2\sum_{m=1}^{q-1}\overline{\chi}(m)\chi^2(m+1)\sum_{n=1}^{q-1}\psi\left(\frac{n-t}{q}\right)\psi\left(\frac{nm-t}{q}\right)+O(q^3)\\
&= \frac{q^3\tau(\overline{\chi})^2}{4\tau(\overline{\chi}^2)}+q^2\sum_{m=1}^{q-1}\overline{\chi}(m)\chi^2(m+1)\sum_{n=1}^{q-1}\psi\left(\frac{n}{q}\right)\psi\left(\frac{m(n+t)-t}{q}\right)+O(q^3)\\
&= \frac{q^3\tau(\overline{\chi})^2}{4\tau(\overline{\chi}^2)}+q^2\sum_{m=1}^{q-1}\overline{\chi}(m)\chi^2(m+1)s\left(m,q;\frac{(m-1)t}{q},0\right)+O(q^3)
\end{aligned}$$

where $\overline{m}m\equiv 1 \pmod q$. This proves Lemma 3.5 $\square$

Theorem 3.1 now follows from Lemmas 3.5 and 3.6.

4. Explicit Formula for the Average of L_4 Norm

We consider the average of the L_4 norm of $f_\chi^t(z)$ over all the characters modulo q and prove the following result.

Theorem 4.1. *Let q be a prime number. We have*

$$\sum_{\chi \pmod q}\|f_\chi^t\|_4^4=(2q-3)(q-1)^2$$

where the summation is over all characters modulo q.

Lemma 4.2. *We have*

$$\sum_{\chi\neq\chi_0}\left|\sum_{n=1}^{q-1}n\chi(n+t)\right|^2=\frac{1}{12}q(q^3-4q^2-7q-2+12tq-12t^2+12t)$$

where $\sum_{\chi\neq\chi_0}$ is the summation over all non-principle characters modulo q.

Proof. Using the orthogonality of characters, the summation in Lemma 4.2 equals to

$$\begin{aligned}
&\sum_{\chi \pmod q}\left|\sum_{n=1}^{q-1}n\chi(n+t)\right|^2-\left|\sum_{n=1}^{q-1}n\chi_0(n+t)\right|^2\\
=&\sum_{n,m=1}^{q-1}nm\sum_{\chi \pmod q}\chi(n+t)\overline{\chi}(m+t)-\left(\sum_{\substack{n=1\\ n\neq q-t}}^{q-1}n\right)^2\\
=&(q-1)\sum_{\substack{n=1\\ n\neq q-t}}^{q-1}n^2-\left(\frac{q(q-1)}{2}-(q-t)\right)^2\\
=&\frac{1}{12}q(q^3-4q^2-7q-2+12tq-12t^2+12t)
\end{aligned}$$

as claimed. □

Lemma 4.3. *For $1 \le t \le \frac{q+1}{2}$, we have*

$$\|f^t_{\chi_0}(z)\|_4^4 = \frac{2}{3}q^3 - 2q^2 + \frac{19}{3}q - 5 - 4tq + 4t^2.$$

Proof. Let

$$c_l := \sum_{n=0}^{q-l-1} \chi_0(n+t)\chi_0(n+t+l)$$

for $0 \le l \le q-1$. If $q-t+1 \le l \le q-1$, then $\chi_0(n+t) = \chi_0(n+t+l) = 1$ for any $n = 0, \cdots, q-l-1$ and so $c_l = q-l$. Similarly, we have $c_l = q-l-1$ if $t \le l \le q-t$ and $c_l = q-l-2$ if $1 \le l \le t-1$. Hence we obtain

$$\begin{aligned}
\|f^t_{\chi_0}(z)\|_4^4 &= c_0^2 + 2\sum_{l=1}^{q-1} |c_l|^2 \\
&= (q-1)^2 + 2\left(\sum_{l=1}^{t-1}(q-l-2)^2 + \sum_{l=t}^{q-t}(q-l-1)^2 + \sum_{l=q-t+1}^{q-1}(q-l)^2\right) \\
&= \frac{2}{3}q^3 - 2q^2 + \frac{19}{3}q - 5 - 4tq + 4t^2. \quad \square
\end{aligned}$$

In view of Theorem 2.1, we need to consider the summation

$$\sum_{\substack{\chi \bmod q \\ \chi, \chi^2 \ne \chi_0}} \tau(\chi)^2 \tau(\overline{\chi}^2) H(\chi, t).$$

From (2.1), we know that if χ is non-principal, then

$$\tau(\chi)^2 \tau(\overline{\chi}^2) H(\chi, t) = \sum_{n,m=1}^{q-1} nm f_\chi(\omega^{n+t}) f_\chi(\omega^{m+t}) f_{\overline{\chi}^2}(\omega^{m+n+2t}). \tag{4.1}$$

Lemma 4.4. *Let $1 \le t \le \frac{q+1}{2}$. We have*

$$\begin{aligned}
&\sum_{n,m=1}^{q-1} nm \sum_{\chi \pmod q} f_\chi(\omega^{n+t}) f_\chi(\omega^{m+t}) f_{\overline{\chi}^2}(\omega^{m+n+2t}) \\
&\qquad = \frac{q^2(q-1)}{12}(3q^3 - 7q^2 + 15q + 1 - 12qt - 12t + 12t^2).
\end{aligned}$$

Proof. Let $\sum$ be the above double summation. By interchanging the order of the summation, we have

$$\begin{aligned}
\sum &= \sum_{n,m=1}^{q-1} nm \sum_{a,b,c=1}^{q-1} \omega^{(b+a)(n+t)+(c+a)(m+t)} \sum_{\chi} \overline{\chi}(a^2)\chi(bc) \\
&= (q-1) \sum_{n,m=1}^{q-1} nm \sum_{\substack{a,b,c=1 \\ a^2 \equiv bc \pmod q}}^{q-1} \omega^{(b+a)(n+t)+(c+a)(m+t)} \\
&= (q-1) \sum_{n,m=1}^{q-1} nm \sum_{a,b=1}^{q-1} \omega^{(b+a)(n+t)+(a^2\bar{b}+a)(m+t)} \\
&= (q-1) \sum_{n,m=1}^{q-1} nm \sum_{a,b=1}^{q-1} \omega^{(b+ab)(n+t)+((ab)^2\bar{b}+ab)(m+t)} \\
&= (q-1) \sum_{n,m=1}^{q-1} nm \sum_{a,b=1}^{q-1} \omega^{b\{a^2(m+t)+a(m+n+2t)+(n+t)\}}
\end{aligned}$$

where $b\bar{b} \equiv 1 \pmod q$. Consider the congruence equation $(m+t)X^2 + (m+n+2t)X + (n+t) \equiv 0 \pmod q$. Then it is easy to show that the number of solutions of the above congruence equation in $X = 1, 2 \cdots, q-1$ is

$$N(m,n,t) := \begin{cases} 2 & \text{if } (m+t,q) = (n+t,q) = 1 \text{ and } m \neq n, \\ 1 & \text{if } (m+t,q) = (n+t,q) = 1 \text{ and } m = n, \\ 1 & \text{if } (m+t,q) \neq 1 \text{ and } (n+t,q) = 1 \text{ or } (n+t,q) \neq 1 \text{ and } (m+t,q) = 1, \\ q-1 & \text{if } (m+t,q), (n+t,q) \neq 1. \end{cases}$$

Hence

$$\begin{aligned}
\sum &= (q-1) \sum_{n,m=1}^{q-1} nm \sum_{a=1}^{q-1} \left(\sum_{b=0}^{q-1} \omega^{b\{a^2(m+t)+a(m+n+2t)+(n+t)\}} - 1 \right) \\
&= q(q-1) \sum_{n,m=1}^{q-1} nm \times N(m,n,t) - \frac{q^2(q-1)^4}{4} \\
&= q(q-1) \left(2 \sum_{\substack{n \neq m=1 \\ n,m \neq q-t}}^{q-1} nm + \sum_{\substack{n=1 \\ n \neq q-t}}^{q-1} n^2 + 2 \sum_{\substack{n=1 \\ n \neq q-t}}^{q-1} n(q-t) + (q-1)(q-t)^2 \right) - \frac{q^2(q-1)^4}{4} \\
&= \frac{q^2(q-1)}{12}(3q^3 - 7q^2 + 15q + 1 - 12qt - 12t + 12t^2). \quad \square
\end{aligned}$$

Finally, we need to evaluate the right hand side of (4.1) when $\chi(n) = \left(\frac{n}{q}\right)$ and $\chi(n) = \chi_0(n)$. If $\chi(n) = \left(\frac{n}{q}\right)$, then (see Theorem 1.5.2 in [BEW-98])

$$f_\chi(\omega^k) = \sqrt{q\left(\frac{-1}{q}\right)}\left(\frac{k}{q}\right)$$

and

$$f_{\chi_0}(\omega^k) = \begin{cases} q-1 & \text{if } k \equiv 0 \pmod q, \\ -1 & \text{if } k \not\equiv 0 \pmod q. \end{cases}$$

Thus,

$$\begin{aligned}
&\sum_{n,m=1}^{q-1} nm f_\chi(\omega^{n+t}) f_\chi(\omega^{m+t}) f_{\overline{\chi}^2}(\omega^{m+n+2t}) \\
&= \sum_{n,m=1}^{q-1} nm f_\chi(\omega^{n+t}) f_\chi(\omega^{m+t}) f_{\chi_0}(\omega^{m+n+2t}) \\
&= -\left(\sum_{n=1}^{q-1} n f_\chi(\omega^{n+t})\right)^2 + q \sum_{\substack{n,m=1 \\ n+m+2t\equiv 0 \pmod q}}^{q-1} nm |f_\chi(\omega^{n+t})|^2 \\
&= -q\left(\frac{-1}{q}\right)\left(\sum_{n=1}^{q-1} n\left(\frac{n+t}{q}\right)\right)^2 + q^2 \left(\sum_{\substack{n,m=1 \\ n+m+2t\equiv 0 \pmod q}}^{q-1} nm - (q-t)^2\right) \\
&= \frac{q^2}{6}(q^3 - 12q^2 - q + 24qt + 6q^2t - 12qt^2 - 6t^2) - q\left(\frac{-1}{q}\right)\left(\sum_{n=1}^{q-1} n\left(\frac{n+t}{q}\right)\right)^2 \qquad (4.2)
\end{aligned}$$

from Lemma 2.3. Similarly, we have

(4.3)

$$\sum_{n,m=1}^{q-1} nm f_{\chi_0}(\omega^{n+t}) f_{\chi_0}(\omega^{m+t}) f_{\overline{\chi_0}^2}(\omega^{m+n+2t}) = \frac{q^2}{12}(12q^3 - 25q^2 - 18q - 5 + 72qt + 36t - 60t^2 - 24q^2t + 12t^2q).$$

Therefore, Theorem 4.1 follows from (2.2), (2.3), Lemmas 4.2-4.4, (4.2) and (4.3).

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