On the Irrationality of $\sum_{i=0}^{\infty} q^{-i} \prod_{j=0}^{i}(1+q^{-j}r+q^{-2j}s)$

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Abstract. We prove that if q is an integer greater than one, r and s are any positive rationals, then

$$\sum_{i=0}^{\infty} q^{-i} \prod_{j=0}^{i}(1+q^{-j}r+q^{-2j}s)$$

is irrational and is not a Liouville number.

§1. Introduction and Result

By using the the residue theorem and functional equation methods, we have proved the irrationality of various multivariate functions whose univeriate versions have been extensively investigated (see [2], [3], [5], [6], [7], [8], [9]. [10]), namely $\prod_{j=0}^{\infty}(1+q^{-j}r+q^{-2j}s)$; $\sum_{j=0}^{\infty} \frac{1}{1+q^j r-q^{2j}s}$; $\sum_{i,j=0}^{\infty} \frac{r^i s^j}{q^{i+j+1}-1}$, and $\sum_{j_1,\cdots,j_m=0}^{\infty} \frac{r_1^{j_1}\cdots r_m^{j_m}}{q^{j_1+\cdots+j_m+1}-1}$, where q is an integer greater than one, r, s and $r_1, \cdots, r_m$ are any positive rationals, see Borwein-Zhou [4], Zhou-Lubinsky [11] and Zhou [12] for details. The general approach has been to examine the Padé approximants to the appropriate functions and to show, with some modifications, that they provide rational approximations that are too rapid to be consistent with rationality. In this paper, we use similar techniques to prove a new irrationality result:

Theorem 1.1. *If q is an integer greater than one, r and s are any positive rationals, then*

$$\sum_{i=0}^{\infty} q^{-i} \prod_{j=0}^{i}(1+q^{-j}r+q^{-2j}s)$$

is irrational.

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We refer to Borwein [2] to the concepts of standard q analogues of factorials $[n]!$ and binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}$. We note that (see Zhou [12])

$$\prod_{h=0,h\neq k}^{n} (q^{-k} - q^{-h}) = (-1)^k q^{-k(k-1)/2-n(n+1)/2}[n]![k]!(1-q)^n, \tag{1.1}$$

and for $|t| < |q|^{-n}$,

$$\frac{1}{\prod_{k=0}^{n}(t-q^{-k})} = (-1)^{n+1} q^{n(n+1)/2} \sum_{l=0}^{\infty} \begin{bmatrix} n+l \\ l \end{bmatrix} t^l. \tag{1.2}$$

We prove some properties of approximants to the function

$$F(x,y) = \sum_{i=0}^{\infty} q^{-i} \prod_{j=0}^{i} (1 + q^{-j}x + q^{-2j}xy) \tag{1.3}$$

in section 2, and use those properties to prove Theorem 1.1 in section 3.

§2. Some Results On the Relevant Function

Theorem 2.1. *Let $|q| > 1$, and $F(x,y)$ be defined by (1.3), then $F(x,y)$ is entire in $\mathbb{C} \times \mathbb{C}$. If we write*

$$F(x,y) = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j, \tag{2.1}$$

and

$$\prod_{j=0}^{\infty} (1 + q^{-j}x + q^{-2j}xy) = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j, \tag{2.2}$$

then

$$\left(\prod_{j=1}^{r+s} (q^j - 1) \right) a_{rs} \in \mathbb{Z}[q], \quad r,s = 0,1,2\cdots, \tag{2.3}$$

and

$$\left(\prod_{j=1}^{r+s} (q^j - 1) \right) c_{rs} \in \mathbb{Z}[q], \quad r,s = 0,1,2\cdots, \tag{2.4}$$

where $\mathbb{Z}[q]$ is the set of polynomials in q with integer coefficients.

Proof: See Section 2 of Zhou-Lubinsky [11] for a proof of (2.3) and (2.4) follows immediately from (2.3). □

Theorem 2.2. *Let $q > 1$ be an integer, and $F(x, y)$ be defined by (1.3). Let $n \geq 0$ be integer and*

$$I(x,y) := \frac{1}{2\pi i} \int_\Gamma \frac{F(tx,ty)dt}{\left(\prod_{k=0}^{n}(t-q^{-k})\right)t^{n+1}}, \tag{2.5}$$

where Γ is a circular contour containing $0, q^{-n}, \cdots, q^{-1}, q^0$. Let

$$R_k(x,y) := \prod_{j=0}^{k-1}(q^{2j}+q^jx+xy), \tag{2.6}$$

$$S_k(x,y) := q^{-2k}\sum_{i=1}^{k} q^{-i(2k-i-2)} \prod_{j=k-i}^{k}(q^{2j}+q^jx+xy), \tag{2.7}$$

$$Q(x,y) := \frac{q^{n(n+1)/2}}{(1-q)^n[n]!}\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{k(n+k)+k(k+1)/2}R_k^{-1}(x,y), \tag{2.8}$$

and

$$\begin{aligned} P(x,y) := \frac{q^{n(n+1)/2}}{(1-q)^n[n]!}\sum_{k=0}^{n}(-1)^{k+1}\begin{bmatrix} n \\ k \end{bmatrix} q^{nk+k(k+1)/2}S_k(x,y) \\ +\frac{1}{n!}\frac{d^n}{dt^n}\left\{\frac{F(tx,ty)}{\prod_{k=0}^{n}(t-q^{-k})}\right\}_{t=0}. \end{aligned} \tag{2.9}$$

Then

i)

$$I(x,y) = Q(x,y)F(x,y) + P(x,y); \tag{2.10}$$

ii)

$$I(x,y) = \sum_{i+j\geq 2n+1} d_{ij}x^iy^j, \quad d_{ij} \in \mathbb{C}; \tag{2.11}$$

iii)

$$q^{-n(n+1)/2}\left(\prod_{j=1}^{n}(q^j-1)\right)R_n(x,y)Q(x,y) \in \mathbb{Z}[q,x,y]; \tag{2.12}$$

iv)

$$q^{-n(n+1)/2}\left(\prod_{j=1}^{n}(q^j-1)\right)P(x,y) \in \mathbb{Z}[q,x,y]; \tag{2.13}$$

v) *For any integer l, $0 \leq l < n$, we have*

$$q^{-n(n+1)/2}\left(\prod_{j=1}^{n}(q^j-1)\right)\cdot q^{l(l+1)}R_n\left(\frac{x}{q^l},\frac{y}{q^l}\right)Q\left(\frac{x}{q^l},\frac{y}{q^l}\right) \in \mathbb{Z}[q,x,y]; \tag{2.14}$$

$$q^{-n(n+1)/2}\left(\prod_{j=1}^{n}(q^j-1)\right)\cdot q^{nl}P(\frac{x}{q^l},\frac{y}{q^l})\in \mathbb{Z}[q,x,y]; \qquad (2.15)$$

vi) *For* $n\in\mathbb{N}$ *fixed and*

$$(x,y)\in \mathbb{P}:=(x,y)\in\mathbb{C}\times\mathbb{C}:0<|x|,|y|\leq 1/2\}, \qquad (2.16)$$

$$|I(x,y)|\leq c_q\cdot\frac{(n+1)v_{x,y}^{2n+1}}{q^{n^2}}, \qquad (2.17)$$

where c_q *is a constant depending only on* q, *and*

$$v_{x,y}:=\max\{|x|,|y|\}. \qquad (2.18)$$

Proof: Proof of (i). From (1.3), we have the functional relations for $F(x,y)$ for integers $k\geq 1$:

$$\begin{aligned} F(q^{-k}x,q^{-k}y) &= \sum_{i=0}^{\infty} q^{-i+k}\prod_{j=0}^{i-k}(1+q^{-j-k}x+q^{-2j-2k}xy) \\ &\qquad -\sum_{i=1}^{k} q^{i}\prod_{j=k-i}^{k}(1+q^{-j}x+q^{-2j}xy) \qquad (2.19) \\ &= q^{k^2}R_k^{-1}(x,y)F(x,y)-S_k(x,y). \end{aligned}$$

By the residue theorem and the functional equation (2.19), and (1.1), we have

$$\begin{aligned} &I(x,y) \\ &= \sum_{k=0}^{n}\frac{F(q^{-k}x,q^{-k}y)}{\left(\prod_{h=0,h\neq k}^{n}(q^{-k}-q^{-h})\right)q^{-k(n+1)}}+\frac{1}{n!}\frac{d^n}{dt^n}\left\{\frac{F(tx,ty)}{\prod_{k=0}^{n}(t-q^{-k})}\right\}_{t=0} \\ &= \frac{q^{n(n+1)/2}}{(1-q)^n[n]!}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{k(n+k)+k(k+1)/2}R_k^{-1}(x,y)F(x,y) \\ &\quad -\frac{q^{n(n+1)/2}}{(1-q)^n[n]!}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{nk+k(k+1)/2}S_k(x,y)+\frac{1}{n!}\frac{d^n}{dt^n}\left\{\frac{F(tx,ty)}{\prod_{k=0}^{n}(t-q^{-k})}\right\}_{t=0} \\ &= Q(x,y)F(x,y)+P(x,y). \end{aligned}$$

Proof of (ii). From (2.5) we observe that the denominator in the integral defining $I(x,y)$ is a polynomial of degree $2n+2$ in t, and any terms in

$$F(tx,ty)=\sum_{i,j=0}^{\infty}c_{ij}t^{i+j}x^iy^j$$

of order less than $2n+1$ in t vanish on integration, so (2.11) holds.

Proof of (iii). From (2.6) and (2.8), we have

$$R_n(x,y)Q(x,y) = \frac{q^{n(n+1)/2}}{(1-q)^n[n]!}\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{k(n+k)+k(k+1)/2} \times \prod_{j=k}^{n-1}(q^{2j}+q^j x + xy). \tag{2.20}$$

As $\begin{bmatrix} n \\ k \end{bmatrix}$ is a polynomial in q with integer coefficients, (2.12) holds.

Proof of (iv). From (1.2),

$$\frac{F(tx,ty)}{\prod_{k=0}^{n}(t-q^{-k})} = (-1)^{n+1}q^{n(n+1)/2}\sum_{r,s,l=0}^{\infty} c_{rs}x^r y^s t^{r+s+l}\begin{bmatrix} n+l \\ l \end{bmatrix}, \tag{2.21}$$

and

$$P(x,y) = \frac{q^{n(n+1)/2}}{(1-q)^n[n]!}\sum_{k=0}^{n}(-1)^{k+1}\begin{bmatrix} n \\ k \end{bmatrix} \times \left(\sum_{i=1}^{k} q^{k(n-i)-i(k-i)+(k^2+4i-3k)/2}\prod_{j=k-i}^{k}(q^{2j}+q^j x+xy)\right) + (-1)^{n+1}q^{n(n+1)/2}\sum_{\mu=0}^{n}\left(\sum_{r+s=\mu} c_{rs}x^r y^s\right)\begin{bmatrix} 2n-\mu \\ n-\mu \end{bmatrix}. \tag{2.22}$$

As

$$k(n-i)-i(k-i)+(k^2+4i-3k)/2 \ge 0, \tag{2.23}$$

for $0 \le k \le n$ and $1 \le i \le k$, with (2.4), we have (2.13).

Proof of (v). For $0 \le l \le n-1$,

$$R_n\left(\frac{x}{q^l},\frac{y}{q^l}\right) = q^{-l(l+1)}\left(\prod_{j=0}^{l}(q^{2l+2j}+q^{j+l}x+xy)\right). \tag{2.24}$$

So

$$q^{l(l+1)}R_n\left(\frac{x}{q^l},\frac{y}{q^l}\right) \in \mathbb{Z}[q,x,y]. \tag{2.25}$$

¿From (2.12), (2.20) and (2.25), we have (2.14) and (2.15) follows from (2.22).

Proof of (vi). See the proof of (vi) of Theorem 2.2 in Zhou [12]. □

§3. Proof of Theorem 1.1

The proof is similar to the proof of (3.1) in Zhou [12], we note that for $x, y > 0$,

$$I(x,y) > 0, \tag{3.1}$$

where $I(x,y)$ is defined by (2.5). Now let r, s be any fixed positive rational numbers, then $u := s/r$ is also a positive rational number such that

$$F(r,u) = \sum_{i=0}^{\infty} q^{-i} \prod_{j=0}^{\infty} (1 + q^{-j} r + q^{-2j} ru) > 0.$$

We now use the estimate (2.17) in (vi) of Theorem 2.2 to prove the main result. Let $n \geq 0$ be chosen so that n is a multiple of 4, and

$$X := \frac{r}{q^{n/4}} \leq 1/2 \qquad \text{and} \qquad Y := \frac{u}{q^{n/4}} \leq 1/2, \tag{3.2}$$

(now (X, Y) is in the set $\mathbb{P}$ defined by (2.16)) and let

$$v_{r,u} := \max\{r, u\}; \tag{3.3}$$

$$v_{X,Y} := \max\{X, Y\}. \tag{3.4}$$

Then

$$v_{X,Y} = \frac{v_{r,u}}{q^{n/4}}. \tag{3.5}$$

Now let

$$H_n(q) := q^{-n(n+1)/2} \prod_{j=1}^{n} (q^j - 1), \tag{3.6}$$

then

$$0 < |H_n(q)| \leq 1. \tag{3.7}$$

¿From the functional equation (2.19) and (3.2), and from (2.6), we have

$$F(r,u) = q^{-(n/4)^2} R_{n/4}(r,u) \left[F(X,Y) + S_{n/4}(r,u) \right], \tag{3.8}$$

$$|R_n(X,Y)| \leq \prod_{j=0}^{n-1} (1 + q^j + q^{2j}) \leq a q^{n(n-1)}, \tag{3.9}$$

where $a := \prod_{j=0}^{\infty} (1 + q^{-j} + q^{-2j})$ is a constant depending only on q. Now in order to get integer coefficient approximants to $F(r,u)$, let

$$Q^*(r,u) := q^{n(3n+2)/8} H_n(q) R_n(X,Y) Q(X,Y); \tag{3.10}$$

$$\begin{aligned} P^*(r,u) := {} & q^{n(5n+4)/16} H_n(q) R_{n/4}(r,u) R_n(X,Y) \\ & \times \left(P(X,Y) - S_{n/4}(r,u) Q(X,Y) \right). \end{aligned} \tag{3.11}$$

Then from (v) of Theorem 2.2, (2.14), (2.15) and (3.6),

$$Q^*(r,u), P^*(r,u) \in \mathbb{Z}[q,r,u], \tag{3.12}$$

and from (3.8), (3.10) and (3.11),

$$\begin{aligned} \Delta &:= |Q^*(r,u)F(r,u) + P^*(r,u)| \\ &= q^{n(3n+2)/8} \, |H_n(q)| \, |R_n(X,Y)| \left| q^{-(n/4)^2} R_{n/4}(r,u) \right| |I(X,Y)| \, . \end{aligned} \tag{3.13}$$

Then $\Delta > 0$, and from (2.17), (3.5), (3.7), (3.9) and the fact that

$$\begin{aligned} \left| q^{-(n/4)^2} R_{n/4}(r,u) \right| &\leq q^{-(n/4)^2 + (n/4)(n/4-1)} \left| \prod_{j=0}^{\infty} (1 + q^{-j}r + q^{-2j}ru) \right| \\ &=: bq^{-n/4}, \end{aligned}$$

where $b := \left| \prod_{j=0}^{\infty} (1 + q^{-j}r + q^{-2j}ru) \right|$ is a constant depending only on r, u and q, we have

$$\Delta \leq q^{3n^2/8 + n(n-1)} abc_q \frac{(n+1)v_{X,Y}^{2n+1}}{q^{n^2}} =: f_{q,r,u} \frac{(n+1)v_{r,u}^{2n+1}}{q^{n(n+10)/8}}, \tag{3.14}$$

where $f_{q,r,u} := abc_q$ is a constant depending only on q, r and u. Finally, if

$$r := \frac{i}{l} \qquad \textit{and} \qquad u := \frac{j}{m} \tag{3.15}$$

with i, j, l, m positive integers, then

$$Q^{**}(r,u) := (lm)^{4n} \, Q^*(r,u), \tag{3.16}$$

and

$$P^{**}(r,u) := (lm)^{4n} \, P^*(r,u), \tag{3.17}$$

are integers, and

$$v_{r,u} = \frac{1}{lm} v_{mi,lj} := \frac{1}{lm} \max\{mi, lj\}. \tag{3.18}$$

Then by (3.14) to (3.18),

$$\begin{aligned} 0 < |Q^{**}(r,u)F(r,u) + P^{**}(r,u)| &\leq (lm)^{4n} \, f_{q,r,u}(n+1) \frac{v_{r,u}^{2n+1}}{q^{n(n+10)/8}} \\ = f_{q,r,u}(n+1)(lm)^{2n-1} \frac{v_{mi,lj}^{2n+1}}{q^{n(n+10)/8}}, \end{aligned}$$

which tends to zero as $n \to \infty$, This shows that $F(r,u)$ is irrational. This completes the proof of Theorem 1.1. □

Now by the standard methods (as in chapter 11 of Borwein-Borwein [1]), the estimates in the proof of Theorem 1.1 gives that, under the assumption of the theorem,

$$\left| F(r,u) - \frac{s}{t} \right| > \frac{1}{t^{\alpha}},$$

for some constant α and all integers s and t, and hence $F(r,u)$ is not a Liouville number.

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