RUDIN-SHAPIRO LIKE POLYNOMIALS IN L_4

PETER BORWEIN AND MICHAEL MOSSINGHOFF

ABSTRACT. We examine sequences of polynomials with $\{+1,-1\}$ coefficients constructed using the iterations $p(x) \to p(x) \pm x^{d+1}p^*(-x)$, where d is the degree of p and p^* is the reciprocal polynomial of p. If $p_0 = 1$ these generate the Rudin-Shapiro polynomials. We show that the L_4 norm of these polynomials is explicitly computable. We are particularly interested in the case where the iteration produces sequences with smallest possible asymptotic L_4 norm (or, equivalently, with largest possible asymptotic merit factor). The Rudin-Shapiro polynomials form one such sequence.

We determine all p_0 of degree less than 40 that generate sequences under the iteration with this property. These sequences have asymptotic merit factor 3. The first really distinct example has a p_0 of degree 19.

1. Introduction

We are interested the L_4 norm of a polynomial with coefficients $\{+1,-1\}$ (or some other fixed set of coefficients), with the most interesting case being when the norm is small. The norm is the L_α norm on the boundary of the unit disc defined by

$$\|p\|_\alpha = \left(\frac{1}{2\pi}\int_0^{2\pi} |p(e^{i\theta})|^\alpha \, d\theta\right)^{1/\alpha}.$$

We call a polynomial with coefficients $\{+1,-1\}$ of degree n a Littlewood polynomial of degree n and denote this class by $\mathcal{L}_n$.

The L_2 norm of any element of $\mathcal{L}_{n-1}$ is $\sqrt{n}$ and this is, of course, a lower bound for the L_4 norm. There are two natural measures of smallness for the L_4 norm of a polynomial p in $\mathcal{L}_{n-1}$. One is the ratio of the L_4 norm to the L_2 norm, $\|p\|_4/\sqrt{n}$. The other (equivalent) measure is the merit factor, defined by

$$\mathrm{MF}(p) = \frac{\|p\|_2^4}{\|p\|_4^4 - \|p\|_2^4} = \frac{n^2}{\|p\|_4^4 - n^2}.$$

The expected L_4 norm of an element of $\mathcal{L}_n$ is computed in [Ne-90] (see also [Bor-A3]); the expected merit factor is 1. The L_4 norms of the Rudin-Shapiro polynomials are explicitly computed by Littlewood [Li-68] (see also [Ne-90]); their merit factors tend to 3. We also compute this in this paper.

In §2 we analyse the Rudin-Shapiro like polynomials generated by the iterations

$$p(x) \to p(x) \pm x^{d+1}p^*(-x).$$

1991 *Mathematics Subject Classification.* 11J54, 11B83, 12-04.
Key words and phrases. Restricted coefficients; $-1, 0, 1$ coefficients; Rudin-Shapiro polynomials; Littlewood conjectures.

We show that the merit factors of the polynomials generated by these iterations with initial polynomial p_0 approach

$$\frac{1}{4\gamma/3 - 1}$$

where

$$\gamma = \frac{\|p_0\|_4^4 + \|p_0(z)p_0^*(-z)\|_2^2}{2\|p_0\|_2^4} \geq 1.$$

Note that the maximum possible asymptotic merit factor is 3 and this occurs when $\gamma = 1$. In §3 we address the problem of determining when $\gamma = 1$, and we find all p_0 with this property of degree less than 40.

It is possible to construct sequences with asymptotic merit factor 6. Golay [Go-83] gives a heuristic argument that a sequence of polynomials explored by Turyn has limiting merit factor 6 and this is proved rigorously in [Hø-88]. Turyn's polynomials are constructed by cyclically permuting the coefficients of the Fekete polynomials

$$f_q(z) := \sum_{k=0}^{q-1} \left(\frac{k}{q}\right) z^k$$

by approximately $q/4$. Here, q is a prime number and $\left(\frac{\cdot}{q}\right)$ is the Legendre symbol. The Fekete polynomials themselves have asymptotic merit factor 3/2 and different amounts of cyclic permutations can give rise to any asymptotic merit factor between 3/2 and 6.

Golay [Go-83] speculates that 6 may be the largest possible asymptotic merit factor. He writes "the eventuality must be considered that no systematic synthesis will ever be found which will yield higher merit factors." Newman and Byrnes [Ne-90], apparently independently, make a similar conjecture. Computations by a number of people (including the authors) on polynomials up to degree 200 lead us to believe that higher merit factors are probably possible and to doubt these conjectures. See [Go-77], [Me-96], [Re-93], and the web page of A. Reinholz at http://borneo.gmd.de/∼andy/ACR.html.

All of these explorations are closely related to Littlewood's conjecture that it is possible to find $p_n \in \mathcal{L}_{n-1}$ so that

$$C_1\sqrt{n} \leq |p_n(z)| \leq C_2\sqrt{n}$$

and a related conjecture of Erdős [Er-62] that the constant C_2 is bounded away from 1 (independently of n). See [Li-68] and [Er-62]. These conjectures are all still open.

The Rudin-Shapiro polynomials (which some argue should be called the Shapiro polynomials) satisfy the upper bound in Littlewood's conjecture. No sequence is known which satisfies the lower bound.

The Fekete polynomial f_q takes the value $\sqrt{q}$ at each qth root of unity and one might hope that they also satisfy the upper bound in Littlewood's conjecture but Montgomery [Mo-80] shows that this is not the case.

2. The Iteration

Let p^* denote the reciprocal polynomial of p: $p^*(z) = z^d p(1/z)$, where d is the degree of p. We consider the following construction.

Iteration 1. *Let $p_0(z)$ be a polynomial of degree $D-1$ with coefficients in a set A. Let*

$$p_{n+1}(z) = p_n(z) + z^{d+1} p_n^*(-z)$$

where d is the degree of p_n. Then p_n is a polynomial of degree $2^n D - 1$ with all coefficients in $A \cup -A$. Furthermore, if

$$R_n := p_n(z) \ \text{and} \ S_n := p_n^*(-z)$$

then

$$R_{n+1} = R_n + z^{d+1} S_n$$

and

$$S_{n+1} = (-1)^d (R_n - z^{d+1} S_n).$$

Proof. Most of this is simple calculation. Observe that

$$p_{n+1}(z) = p_n(z) + (-1)^d z^{2d+1} p_n(-1/z)$$

so

$$p_{n+1}(-1/z) = p_n(-1/z) - (-1)^d z^{-2d-1} p_n(z)$$

and multiplying this equation by $-z^{2d+1}$ yields the second form of the iteration. □

Lemma 1. *In the notation of Iteration 1,*

$$|R_n(z)|^2 + |S_n(z)|^2 = 2^n(|p_0(z)|^2 + |p_0^*(-z)|^2)$$

provided $|z| = 1$. Furthermore,

$$\frac{|R_n(z)|^2}{\|R_n\|_2^2} + \frac{|S_n(z)|^2}{\|S_n\|_2^2} = \frac{|p_0(z)|^2}{\|p_0\|_2^2} + \frac{|p_0^*(-z)|^2}{\|p_0\|_2^2}.$$

Proof. The first statement follows from the parallelogram law for complex numbers:

$$\begin{aligned} |R_{n+1}(z)|^2 + |S_{n+1}(z)|^2 &= |R_n(z) + z^{d+1} S_n(z)|^2 + |R_n(z) - z^{d+1} S_n(z)|^2 \\ &= 2(|R_n(z)|^2 + |S_n(z)|^2). \end{aligned}$$

The second statement follows on observing that $\|R_{n+1}\|_2^2 = 2\|R_n\|_2^2$ and $\|S_{n+1}\|_2^2 = 2\|S_n\|_2^2$. □

We wish to compute the L_4 norm of p_n. For this we follow Littlewood [Li-68].

Theorem 1. *In the notation of Iteration 1, let $y_n = \|p_n\|_4^4 / \|p_n\|_2^4$ for $n \geq 0$, and let*

$$\gamma = \frac{\|p_0\|_4^4 + \|p_0(z) p_0^*(-z)\|_2^2}{2\|p_0\|_2^4}.$$

Then

$$y_n = \frac{4\gamma}{3} + \left(y_0 - \frac{4\gamma}{3} \right) \left(-\frac{1}{2} \right)^n.$$

Proof. With R_n and S_n as in Iteration 1, let

$$x_n := \|R_n\|_4^4 = \|S_n\|_4^4$$

and

$$w_n := \|R_n S_n\|_2^2.$$

Then, with $z = e^{i\theta}$ and $d = \deg(R_n)$,

$$\begin{aligned} 2x_{n+1} &= \|R_{n+1}\|_4^4 + \|S_{n+1}\|_4^4 \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(|R_n(z) + z^{d+1}S_n(z)|^4 + |R_n(z) - z^{d+1}S_n(z)|^4\right) d\theta. \end{aligned}$$

If we use the identity for complex numbers

$$|u+v|^4 + |u-v|^4 = 2(|u|^4 + |v|^4) + 4\mathrm{Re}(u\overline{v})^2 + 8|uv|^2$$

with $u := z^{d+1}S_n(z)$ and $v := R_n(z)$, we deduce that

$$2x_{n+1} = 4x_n + 8w_n + \frac{4}{2\pi}\int_0^{2\pi} \mathrm{Re}(R_n(z)\overline{z^{d+1}S_n(z)})^2\, d\theta.$$

Now $R_n(z)\overline{z^{d+1}S_n(z)} = R_n^*(1/z)S_n(1/z)/z$, a polynomial in $1/z$ with constant term 0, so the integral above is 0. Thus

$$x_{n+1} = 2x_n + 4w_n. \tag{1}$$

We now observe that, with Lemma 1,

$$\begin{aligned} 2x_n + 2w_n &= \frac{1}{2\pi}\int_0^{2\pi} \left(|R_n(z)|^2 + |S_n(z)|^2\right)^2 d\theta \\ &= \frac{2^{2n}}{2\pi}\int_0^{2\pi} \left(|p_0(z)|^2 + |p_0^*(-z)|^2\right)^2 d\theta \\ &= \frac{2^{2n+2}}{2\pi}\int_0^{2\pi} \frac{|p_0(z)|^4 + |p_0(z)p_0^*(-z)|^2}{2}\, d\theta \\ &= 2^{2n+2}\left(\frac{\|p_0\|_4^4 + \|p_0(z)p_0^*(-z)\|_2^2}{2}\right). \end{aligned}$$

From this and (1) we deduce

$$x_{n+1} = -2x_n + 2^{2n+3}\left(\frac{\|p_0\|_4^4 + \|p_0(z)p_0^*(-z)\|_2^2}{2}\right).$$

Since $\|p_{n+1}\|_2^4 = 4\|p_n\|_2^4$, this yields

$$y_{n+1} = -\frac{y_n}{2} + 2\gamma$$

which simply solves to give the result. □

An immediate consequence of this is the following.

Corollary 1. *The sequence $p_n(z)$ generated by Iteration 1 satisfies*

$$\lim_{n\to\infty} \frac{\|p_n\|_4}{\|p_n\|_2} = \left(\frac{4\gamma}{3}\right)^{1/4}$$

and

$$\lim_{n\to\infty} \mathrm{MF}(p_n) = \frac{1}{4\gamma/3 - 1}$$

where

$$\gamma = \frac{\|p_0\|_4^4 + \|p_0(z)p_0^*(-z)\|_2^2}{2\|p_0\|_2^4} \geq 1.$$

Proof. The only part needing proof is that $\gamma \geq 1$. For this note that

$$\begin{aligned}\|p\|_4^4 + \|p(z)p^*(-z)\|_2^2 &= \frac{2}{2\pi}\int_0^{2\pi} \left(\frac{|p(z)|^2 + |p^*(-z)|^2}{2}\right)^2 d\theta \\ &\geq 2\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{|p(z)|^2 + |p^*(-z)|^2}{2}\, d\theta\right)^2 \\ &= 2\|p\|_2^4.\end{aligned}$$

Here we have used the fact that $L_2(q) \geq L_1(q)$. □

It is easy to check that the same results hold for the iteration $p_{n+1}(x) = p_n(x) - x^{d+1}p_n^*(-x)$.

Define

$$\gamma(p) = \frac{\|p\|_4^4 + \|p(x)p^*(-x)\|_2^2}{2\|p\|_2^4}$$

and let

$$T_{\pm}(p) = p(x) \pm x^{d+1}p^*(-x).$$

By Corollary 1, we have $\gamma(T_{\pm}(p)) = \gamma(p)$. Thus, if $\{q_n\}$ is a sequence of polynomials generated by $q_{n+1} = T_{\pm}(q_n)$ for some choice of signs, then

$$\lim_{n\to\infty} \frac{\|q_n\|_4}{\|q_n\|_2} = \left(\frac{4\gamma(q_0)}{3}\right)^{1/4}.$$

We remark that the usual Rudin-Shapiro polynomials satisfy the recurrence

$$P_{n+1}(x) = P_n(x) - (-1)^n x^{2^n} P_n^*(-x)$$

and

$$Q_{n+1}(x) = P_n(x) + (-1)^n x^{2^n} P_n^*(-x)$$

for $n \geq 1$, so

$$\{P_{n+1}, Q_{n+1}\} = \{T_+(P_n), T_-(P_n)\}.$$

The interesting question now becomes for which p is $\gamma(p) = 1$.

3. Littlewood polynomials with $\gamma = 1$

Polynomials which satisfy $\gamma(p) = 1$ are of special interest in that they give rise to sequences of polynomials (under iteration by $T_{\pm}$) that satisfy

$$\lim_{n\to\infty} \frac{\|p_n\|_4}{\|p_n\|_2} = \left(\frac{4}{3}\right)^{1/4},$$

the smallest possible limit under the process. The interesting observation is that many such p exist. Indeed there are 128 distinct such p of degree 19 which we list later in this section. One example is

$$1+x-x^2+x^3+x^4+x^5-x^6+x^7-x^8+x^9-x^{10}-x^{11}+x^{12}+x^{13}+x^{14}-x^{15}-x^{16}-x^{17}-x^{18}+x^{19}.$$

We describe an algorithm for determining all Littlewood polynomials p of degree d having $\gamma(p) = 1$. We first require some preliminary lemmas.

Lemma 2. *Let $p(x) = \sum_{k=0}^{d} x^k$. Then $\|p\|_4^4 = (d+1)(2d^2+4d+3)/3$ and $\|p(x)p^*(-x)\|_2^2 = d+1$.*

Proof. Since $p(x)^2 = (d+1)x^d + \sum_{k=0}^{d-1}(k+1)(x^k + x^{2d-k})$, the first identity follows easily from Parseval's formula. For the second identity, we have

$$\begin{aligned} p(x)p^*(-x) &= \frac{x^{d+1}-1}{x-1} \cdot \frac{x^{d+1}+(-1)^d}{x+1} \\ &= \begin{cases} \sum_{k=0}^{d} x^{2k}, & d \text{ even} \\ (x^{d+1}-1)\sum_{k=0}^{(d-1)/2} x^{2k}, & d \text{ odd} \end{cases} \end{aligned}$$

and the formula follows. □

Lemma 3. *Let p be a Littlewood polynomial of degree d. The coefficient of x^d in $p(x)p^*(-x)$ is 0 if d is odd and 1 if d is even.*

Proof. Write $p(x) = \sum_{k=0}^{d} a_k x^k$ so that $p^*(-x) = (-1)^d \sum_{k=0}^{d} a_{d-k}(-1)^k x^k$. The coefficient of x^d in the product is therefore

$$(-1)^d \sum_{i+j=d} a_i a_{d-j}(-1)^j = \sum_{i=0}^{d} (-1)^i a_i^2 = \sum_{i=0}^{d} (-1)^i$$

and the result follows. □

Lemma 4. *Suppose p and q are Littlewood polynomials of degree d. Then $\|p\|_4^4 \equiv \|q\|_4^4 \pmod 8$, and $\|p(x)p^*(-x)\|_2^2 \equiv \|q(x)q^*(-x)\|_2^2 \pmod 8$.*

Proof. Let $p(x) = \sum_{k=0}^{d} a_k x^k$ and $p(x)^2 = \sum_{k=0}^{2d} b_k x^k$. It is enough to prove the statement for the case where p and q are identical except for one coefficient, so assume that $q(x) = p(x) - 2a_m x^m$ for some m. Write $q(x)^2 = \sum_{k=0}^{2d} \beta_k x^k$. Then

$$\beta_k = \begin{cases} b_k - 4a_m a_{k-m}, & m \le k \le m+d,\ k \ne 2m \\ b_k & \text{otherwise.} \end{cases}$$

Therefore

$$\|q\|_4^4 = \|p\|_4^4 + 16d - 8a_m \sum_{\substack{m \le k \le m+d \\ k \ne 2m}} a_{k-m} b_k \tag{2}$$

and the first assertion of the theorem follows. For the second assertion, let $p(x)p^*(-x) = \sum_{k=0}^{2d} c_k x^k$ and $q(x)q^*(-x) = \sum_{k=0}^{2d} \delta_k x^k$. Now

$$q(x)q^*(-x) = (p(x) - 2a_m x^m)\left(p^*(-x) - 2a_m(-1)^m x^{d-m}\right) \tag{3}$$

so $\delta_k^2 \equiv c_k^2 \pmod 4$ for each k. Because $\delta_d = c_d$ by Lemma 3 and $\delta_k = \pm\delta_{2d-k}$, it follows that $\|p(x)p^*(-x)\|_2^2 \equiv \|q(x)q^*(-x)\|_2^2 \pmod 8$. □

We immediately deduce the following theorem.

Theorem 2. *If p is a Littlewood polynomial of degree d and $d \equiv 2 \pmod 4$, then $\gamma(p) > 1$.*

Proof. By Lemmas 2 and 4, we have $\|p\|_4^4 + \|p(x)p^*(-x)\|_2^2 \equiv 6 \pmod 8$, but $2\|p\|_2^4 \equiv 2 \pmod 8$, so $\gamma(p) \ne 1$. The result follows from Corollary 1. □

In searching for Littlewood polynomials p having $\gamma(p) = 1$, clearly we may assume that the coefficients of the two highest-order terms are both 1. We employ a Gray code [Ni-75] to enumerate all possible combinations of signs among the lower-order terms. This way, each polynomial considered differs in exactly one position from the previous polynomial tested, and we may use formulas (2) and (3) to compute each γ in $O(d)$ time.

Algorithm 1. *Rudin-Shapiro like polynomials in L_4.*

Input. d, a positive integer, $d \not\equiv 2 \pmod 4$.

Output. All Littlewood polynomials $p(x)$ of degree d having $\gamma(p) = 1$.

Data. a_k is the coefficient of x^k in $p(x)$, b_k in $p(x)^2$, and c_k in $p(x)p^*(-x)$.

Initialize. Set the a_k, b_k, and c_k for the polynomial $p(x) = \sum_{k=0}^{d} x^k$. Set $v_k = 0$ for $1 \le k < d$. Choose s, t, s_0, and t_0 so that $(d+1)(2d^2+4d+3)/3 = 8s + s_0$, $d+1 = 8t + t_0$, $0 \le s_0 < 8$, and $0 \le t_0 < 8$. Let $u = (2(d+1)^2 - s_0 - t_0)/8$.

Loop. Enumerate all possible combinations of signs among the lower order $d-1$ coefficients of the polynomial using a Gray code. Execute the following statements when changing the sign of the mth coefficient of the polynomial.

$s \leftarrow s + 2d - a_m \sum_{\substack{0 \le k \le d \\ k \ne m}} a_k b_{k+m}$

$b_k \leftarrow b_k - 4a_m a_{k-m}, \quad m \le k \le d+m,\ k \ne 2m$

$v_k \leftarrow (-1)^{d+m-k+1} a_m a_{m+d-k}, \quad m \le k < d$

$v_k \leftarrow v_k + (-1)^{m+1} a_m a_{m+k-d}, \quad d-m \le k < d$

$t \leftarrow t + \sum_{k=1}^{d} v_k(c_k + v_k)$

$c_k \leftarrow c_k + 2v_k, \quad 1 \le k < d$

$v_k \leftarrow 0, \quad 1 \le k < d$

$a_m \leftarrow -a_m$

If $s + t = u$ then print $p(x)$. □

Searching through degree 39, we find many polynomials with $\gamma = 1$ at the degrees of the Rudin-Shapiro polynomials, plus a number of examples of degree 19 and degree 39. The following table shows the total number n of Littlewood polynomials with $\gamma = 1$ for each degree d.

d	n
1	4
3	8
7	32
15	192
19	128
31	1536
39	1088

The coefficients of sixteen of the polynomials of degree 19 are listed below. Each one represents eight Littlewood polynomials with $\gamma = 1$ since $\gamma(p(x)) = \gamma(-p(x)) = \gamma(p(-x)) = \gamma(p^*(x))$.

1. + + + + + − + − + + − + + + − − − + + −
2. + + + + + − + − − − + − + + − − − + + −
3. + + + + − + − + + + + − + + − − + − − +
4. + + + + − + − + − − − + + + − − + − − +
5. + + + + − − + + + − + + + − + − − + − +
6. + + + + − − + + − + + + − + − + + − + −
7. + + + + − − + − + + + − + + + − − + − +
8. + + + + − − − + + + − + + + − + + − + −
9. + + + − + + + − + − − − + + − + − − + −
10. + + + − + + + − − + + + + + − + − − + −
11. + + − + + + − + + − + + + + + − − − − +
12. + + − + + + − + − + − − + + + − − − − +
13. + + − − − − + + + − + + + − + − + − − +
14. + + − − − − + + − + + + − + − + − + + −
15. + + − − − − + − + + + − + + + − + − − +
16. + + − − − − − + + + − + + + − + − + + −

By analyzing our data we find another operator that preserves γ.

Theorem 3. *Let $p(x)$ be a polynomial, and define $U(p) = xp(x^2) + p^*(-x^2)$. Then $\gamma(U(p)) = \gamma(p)$.*

Proof. Let $q = U(p)$. Then

$$\begin{aligned} \|q\|_4^4 &= \|(xp(x^2) + p^*(-x^2))^2\|_2^2 \\ &= \|x^2p(x^2)^2 + p^*(-x^2)^2\|_2^2 + \|2xp(x^2)p^*(-x^2)\|_2^2 \\ &= \|xp(x)^2 + p^*(-x)^2\|_2^2 + 4\|p(x)p^*(-x)\|_2^2. \end{aligned}$$

The first term is

$$\frac{1}{2\pi}\int_0^{2\pi} \left|zp(z)^2 + p^*(-z)^2\right|^2 d\theta = 2\|p\|_4^4 + \frac{1}{\pi}\int_0^{2\pi} \operatorname{Re}(z^{1-2\deg(p)}p(z)^2p(-z)^2)\, d\theta$$

with $z = e^{i\theta}$ and the integral is 0 because $p(x)p(-x)$ is an even function. Thus

$$\|q\|_4^4 = 2\|p\|_4^4 + 4\|p(x)p^*(-x)\|_2^2. \tag{4}$$

Next, we compute

$$\begin{aligned}\|q(x)q^*(-x)\|_2^2 &= \|(p^*(-x^2) + xp(x^2))(p^*(x^2) - xp(-x^2))\|_2^2 \\ &= \|p^*(x^2)p^*(-x^2) - x^2p(x^2)p(-x^2)\|_2^2 + \|x(p(x^2)p^*(x^2) - p(-x^2)p^*(-x^2))\|_2^2 \\ &= \|p^*(x)p^*(-x) - xp(x)p(-x)\|_2^2 + \|p(x)p^*(x) - p(-x)p^*(-x)\|_2^2.\end{aligned}$$

The first term equals $2\|p(x)p(-x)\|_2^2$ because $p(x)p(-x)$ is an even function. The second term is

$$\frac{1}{2\pi}\int_0^{2\pi} \left||p(z)|^2 - |p(-z)|^2\right|^2 d\theta = 2\|p\|_4^4 - 2\|p(x)p(-x)\|_2^2$$

so

$$\|q(x)q^*(-x)\|_2^2 = 2\|p\|_4^4. \tag{5}$$

Clearly, $\|q\|_2^2 = 2\|p\|_2^2$, and this fact combined with (4) and (5) proves the theorem. □

Thus, the four operators T_+, T_-, U, and U^* (the reciprocal of U, $U^*(p) = xp(-x^2) + p^*(x^2)$) in general allow us to construct four polynomials of degree $2d+1$ with $\gamma = 1$ for each polynomial of degree d with this property.

References

Be-91a. J. Beck, *The modulus of polynomials with zeros on the unit circle: A problem of Erdös*, Ann. of Math. (2) **134** (1991), 609-651.

Be-91b. J. Beck, *Flat polynomials on the unit circle – note on a problem of Littlewood*, Bull. London Math. Soc. **23** (1991), 269–277.

Bh-86. A. T. Bharucha-Reid and M. Sambandham, *Random polynomials*, Academic Press, Orlando, 1986.

Bor-A1. P. Borwein and T. Erdélyi, *Markov-Bernstein type inequalities under Littlewood-type coefficient constraints*, (under submission).

Bor-A2. P. Borwein and T. Erdélyi, *Littlewood-type problems on subarcs of the unit circle*, Indiana Univ. Math. J. (to appear).

Bor-A3. P. Borwein and R. Lockhart, *The expected L_p norm of random polynomials*, (under submission).

Boy-97. D. Boyd, *On a problem of Byrnes concerning polynomials with restricted coefficients*, Math. Comp. **66** (1997), 1697–1703.

By-88. J. S. Byrnes and D. J. Newman, *Null steering employing polynomials with restricted coefficients*, IEEE Trans. Antennas and Propagation **36** (1988), 301–303.

Ca-77. F. W. Carrol, D. Eustice and T. Figiel, *The minimum modulus of polynomials with coefficients of modulus one*, J. London Math. Soc. **16** (1977), 76–82.

Cl-59. J. Clunie, *On the minimum modulus of a polynomial on the unit circle*, Quart. J. Math. **10** (1959), 95–98.

Er-62. P. Erdős, *An inequality for the maximum of trigonometric polynomials*, Annales Polonica Math. **12** (1962), 151–154.

Fi-70. G. T. Fielding, *The expected value of the integral around the unit circle of a certain class of polynomials*, Bull. London Math. Soc. **2** (1970), 301–306.

Go-77. M. J. Golay, *Sieves for low autocorrelation binary sequences*, IEEE Trans. Inform. Theory **23** (1977), 43–51.

Go-83. M. J. Golay, *The merit factor of Legendre sequences*, IEEE Trans. Inform. Theory **29** (1983), 934–936.

Hø-88. T. Høholdt and H. Jensen, *Determination of the merit factor of Legendre sequences*, IEEE Trans. Inform. Theory **34** (1988), 161–164.

Je-91. J. Jensen, H. Jensen and T. Høholdt, *The merit factor of binary sequences related to difference sets*, IEEE Trans. Inform. Theory **37** (1991), 617–626.

Ka-80. J-P. Kahane, *Sur les polynômes á coefficients unimodulaires*, Bull. London Math. Soc. **12** (1980), 321–342.

Ka-85. J-P. Kahane, *Some Random Series of Functions*, vol. 5, Cambridge Stud. Adv. Math., Cambridge, 1985; Second Edition.

Ko-80. S. Konjagin, *On a problem of Littlewood*, Izv. A. N. SSSR, ser. mat. **45, 2** (1981), 243–265.

Li-61. J. E. Littlewood, *On the mean value of certain trigonometric polynomials*, J. London Math. Soc. **36** (1961), 307–334.

Li-66. J. E. Littlewood, *On polynomials* $\sum^n \pm z^m$ *and* $\sum^n e^{\alpha_m i} z^m$, $z = e^{\theta i}$, J. London Math. Soc. **41** (1966), 367–376.

Li-68. J. E. Littlewood, *Some Problems in Real and Complex Analysis*, Heath Mathematical Monographs, Lexington, Massachusetts, 1968.

Me-96. S. Mertens, *Exhaustive search for low-autocorrelation binary sequences*, J. Phys. A **29** (1996), L473–L481.

Mo-80. H. L. Montgomery, *An exponential sum formed with the Legendre symbol*, Acta Arith. **37** (1980), 375–380.

Ne-90. D. J. Newman and J. S. Byrnes, *The* L^4 *norm of a polynomial with coefficients* ± 1, Amer. Math. Monthly **97** (1990), 42–45.

Ni-75. A. Nijenhuis and H. S. Wilf, *Combinatorial Algorithms*, Academic Press, New York, 1975.

Re-93. A. Reinholz, *Ein paralleler genetische Algorithmus zur Optimierung der binären Autokorrelations-Funktion*, Diplomarbeit, Rheinische Friedrich-Wilhelms-Universität Bonn, 1993.

Saf-89. B. Saffari, *Polynômes réciproques: conjecture d'Erdős en norme* L^4*, taille des autocorrélations et inexistence des codes de Barker*, C. R. Acad. Sci. Paris Sér. I Math. **308** (1989), 461–464.

Saf-90. B. Saffari, *Barker sequences and Littlewood's "two-sided conjectures" on polynomials with* ± 1 *coefficients*, Séminaire d'Analyse Harmonique. Anneé 1989/90, 139–151, Univ. Paris XI, Orsay, 1990.

Sal-54. R. Salem and A. Zygmund, *Some properties of trigonometric series whose terms have random signs*, Acta Math. **91** (1954), 254–301.

Department of Mathematics and Statistics, Simon Fraser University, Burnaby, B.C., Canada V5A 1S6 (P. Borwein)

Department of Mathematical Sciences, Appalachian State University, Boone, North Carolina 28608 USA (M. Mossinghoff)