On The Irrationality Of A Certain q Series

Peter B. Borwein and Ping Zhou
Department of Mathematics and Statistics
Simon Fraser University
Burnaby, B.C.
Canada, V5A 1S6
email: pborwein@cecm.sfu.ca, pzhou@cecm.sfu.ca

August 1997

Abstract

We prove that if q is an integer greater than one, r and s are any positive rationals such that $1 + q^m r - q^{2m} s \neq 0$ for all integers $m \geq 0$, then

$$\sum_{j=0}^{\infty} \frac{1}{1 + q^j r - q^{2j} s}$$

is irrational and is not a Liouville number.

AMS Class: Primary 11J72.

1 Introduction and Results

For q a positive integer greater than one and r a non-zero rational ($r \neq -q^m$ for all integers $m \geq 0$), the irrationality of the series

$$\sum_{n=1}^{\infty} \frac{1}{q^n + r}$$

was proved by Borwein [3] in 1991. This generalized the result of the irrationality of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1} = \sum_{n=1}^{\infty} \frac{d(n)}{2^n},$$

where $d(n)$ is the divisor function, which was proved by Erdös [5] in 1948. It also makes the unresolved claim, by Erdös and Graham [6] in 1980, of the irrationality of

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 3}$$

resolved. The technique employed in [3] to prove the above result is to examine the Padé approximants to an appropriate function, and to show, with some modification, that they

provide a rational approximation that is too rapid to be consistent with rationality. This is a general approach that has been explored to prove irrationality by Mahler [8], Chudnovsky and Chudnovsky [4], and Walliser [9]. But it is not often the case that one can explicitly construct or completely analyze the Padé approximants or rational approximants that are required. The situation becomes more complicated when we deal with two variable functions. By using a similar but more technical method, the second auther generalized in [10] the irrationality of the infinite product

$$\prod_{n=1}^{\infty}(1+\frac{\alpha}{q^n}),$$

to the two variable case

$$\prod_{j=0}^{\infty}(1+q^{-j}r+q^{-2j}s)$$

where q is an integer greater than one, and r, s are any positive rationals. In this paper, we use the general approach to generalize the previously mentioned results of [3] to the two variable case:

Theorem 1.1: If q is an integer greater than one, r and s are any positive rationals such that $1+q^m r-q^{2m}s\neq 0$ for all integers $m\geq 0$, then

$$\sum_{j=0}^{\infty}\frac{1}{1+q^j r-q^{2j}s}$$

is irrational.

We need here the standard q analogues of factorials and binomial coefficients. The $q-$*factorial* is

$$[n]_q!:=[n]!:=\frac{(1-q^n)(1-q^{n-1})\cdots(1-q)}{(1-q)^n}, \tag{1.1}$$

where $[0]_q!:=1$. The $q-$*binomial coefficient* is

$$\begin{bmatrix}n\\k\end{bmatrix}_q:=\begin{bmatrix}n\\k\end{bmatrix}:=\frac{[n]!}{[k]!\cdot[n-k]!}. \tag{1.2}$$

We note that

$$\prod_{\substack{h=0\\h\neq k}}^{n}(q^k-q^h)=(-1)^k q^{k(2n-k-1)/2}[n-k]![k]!(1-q)^n, \tag{1.3}$$

and (see Gasper and Rahman [7]) for $|t|<1$,

$$\frac{1}{\prod_{k=0}^{n}(t-q^k)}=(-1)^{n+1}q^{-n(n+1)/2}\sum_{l=0}^{\infty}\begin{bmatrix}n+l\\l\end{bmatrix}q^{-nl}t^l, \tag{1.4}$$

and (the Cauchy binomial theorem)

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}q^{k(k+1)/2}x^k=\prod_{k=1}^{n}(1+q^k x). \tag{1.5}$$

We prove some properties of approximants to the function

$$F(x,y)=\sum_{j=0}^{\infty}\frac{1}{1+q^jx-q^{2j}xy},\quad |q|>1$$

in section 2, and use those properties to prove Theorem 1.1 in section 3.

2 Some Results On A Relevant Function

Let $|q|>1$,

$$F(x,y):=\sum_{j=0}^{\infty}\frac{1}{1+q^jx-q^{2j}xy}, \tag{2.1}$$

and

$$F_n(x,y):=\sum_{j=n+1}^{\infty}\frac{1}{1+q^jx-q^{2j}xy}. \tag{2.2}$$

Then for $k\geq 0$ integer,

$$\begin{aligned} F(q^kx,q^ky) &= \sum_{j=0}^{\infty}\frac{1}{1+q^{j+k}x-q^{2j+2k}xy} \\ &= F(x,y)-\sum_{j=0}^{k-1}\frac{1}{1+q^jx-q^{2j}xy} \\ &=: F(x,y)-S_k(x,y), \end{aligned} \tag{2.3}$$

and for $0\leq k\leq n$ integer,

$$\begin{aligned} F_n(q^{-k}x,q^{-k}y) &= \sum_{j=n+1}^{\infty}\frac{1}{1+q^{j-k}x-q^{2j-2k}xy} \\ &= F(x,y)-\sum_{j=0}^{n-k}\frac{1}{1+q^jx-q^{2j}xy} \\ &=: F(x,y)-S_{n-k+1}(x,y), \end{aligned} \tag{2.4}$$

where

$$S_k(x,y):=\sum_{j=0}^{k-1}\frac{1}{1+q^jx-q^{2j}xy}. \tag{2.5}$$

From (2.3) we can see that for rationals q,x and y, the irrationality of $F(q^kx,q^ky)$ is equivalent to the irrationality of $F(x,y)$. So we assume $|x|,|y|>2$ throughout this paper, as we can replace x and y by q^mx and q^my respectively for some integers $m\geq 1$. Now let

$$z_1:=\frac{-x+\sqrt{x^2+4xy}}{2}\quad\text{and}\quad z_2:=\frac{-x-\sqrt{x^2+4xy}}{2}. \tag{2.6}$$

Then

$$\min\{|z_1|,|z_2|\}>1, \tag{2.7}$$

as we assume $|x|, |y| > 2$.

Theorem 2.1: Let $F(x,y), F_n(x,y)$ be defined by (2.1) and (2.2) respectively, $n \geq 2$ be an even integer, and

$$I(x,y) := \frac{1}{2\pi i} \int_{|t|=q^{n/2+1}} \frac{F_{n/2}(x/t, y/t)dt}{\left(\prod_{k=0}^{n/2}(t-q^k)\right) t^{n/2+1}}. \tag{2.8}$$

Let

$$Q(q) := \frac{1}{(1-q)^{n/2}[n/2]!} \sum_{k=0}^{n/2} (-1)^k \begin{bmatrix} n/2 \\ k \end{bmatrix} q^{k(k-1)/2-nk}, \tag{2.9}$$

and

$$\begin{aligned} P(x,y) \quad : \quad &= \frac{1}{(1-q)^{n/2}[n/2]!} \sum_{k=0}^{n/2} (-1)^k \begin{bmatrix} n/2 \\ k \end{bmatrix} q^{k(k-1)/2-nk} S_{n/2-k+1}(x,y) \\ &+ \frac{1}{(n/2)!} \frac{d^{n/2}}{dt^{n/2}} \left\{ \frac{F(x/t, y/t)}{\prod_{k=0}^{n/2}(t-q^k)} \right\}_{t=0}. \end{aligned} \tag{2.10}$$

Then
(i)

$$I(x,y) = Q(q)F(x,y) + P(x,y); \tag{2.11}$$

(ii)

$$q^{n(3n+2)/8} \left(\prod_{j=1}^{n/2} (q^j - 1) \right) Q(q) \in \mathbb{Z}[q], \tag{2.12}$$

where $\mathbb{Z}[q]$ is the set of polynomials in q with integer coefficients;

(iii) If we let

$$R_{n/2}(x,y) := \prod_{j=0}^{n/2} (1 + q^j x - q^{2j} xy), \tag{2.13}$$

then

$$q^{n(3n+2)/8} x^{n/2} y^{n/2} R_{n/2}(x,y) \left(\prod_{j=1}^{n/2} (q^j - 1) \right) P(x,y) \in \mathbb{Z}[q,x,y]; \tag{2.14}$$

(iv) For $n \in \mathbb{N}$ fixed,

$$|I(x,y)| \leq \frac{c_q}{q^{n(2n+1)/2}}, \tag{2.15}$$

where c_q is a constant depending only on q.

Proof of Theorem 2.1: Proof of (i). As

$$F_{n/2}(x/t, y/t) = \sum_{j=n/2+1}^{\infty} \frac{t^2}{t^2 + q^j xt - q^{2j} xy}, \tag{2.16}$$

the poles of $F_{n/2}(x/t, y/t)$ are

$$t = z_1 q^j \quad \text{and} \quad t = z_2 q^j, \quad \text{for } j = \frac{n}{2}+1, \frac{n}{2}+2, \cdots,$$

where z_1, z_2 are defined in (2.6). From (2.7) we can see that the integrand in (2.8) has simple poles at $t = 1, q^1, q^2, \cdots, q^{n/2}$, and a pole of order $n+1$ at $t = 0$, inside the circle $\{t : |t| = q^{n/2+1}\}$. By the residue theorem and the functional equation (2.4), and (1.3), we have

$$\begin{aligned}
I(x,y) &= \frac{1}{2\pi i}\int_{|t|=q^{n/2+1}} \frac{F_{n/2}(x/t, y/t)dt}{\left(\prod_{k=0}^{n/2}(t-q^k)\right) t^{n/2+1}} \\
&= \sum_{k=0}^{n/2} \frac{F_{n/2}(q^k x, q^k y)}{\left(\prod_{\substack{h=0 \\ h\neq k}}^{n/2}(q^k - q^h)\right) q^{k(n/2+1)}} + \frac{1}{(n/2)!}\frac{d^{n/2}}{dt^{n/2}}\left\{\frac{F_{n/2}(x/t, y/t)}{\prod_{k=0}^{n/2}(t-q^k)}\right\}_{t=0} \\
&= \frac{1}{(1-q)^{n/2}[n/2]!}\sum_{k=0}^{n/2}(-1)^k \begin{bmatrix} n/2 \\ k \end{bmatrix} q^{k(k-1)/2-nk}\left(F(x,y) - S_{n/2-k+1}(x,y)\right) \\
&\quad + \frac{1}{(n/2)!}\frac{d^{n/2}}{dt^{n/2}}\left\{\frac{F_{n/2}(x/t, y/t)}{\prod_{k=0}^{n/2}(t-q^k)}\right\}_{t=0} \\
&= : Q(q)F(x,y) + P(x,y).
\end{aligned}$$

Proof of (ii). As $\begin{bmatrix} n/2 \\ k \end{bmatrix}$ is a polynomial in q with integer coefficients, and the lowest power of q in $\sum_{k=0}^{n/2}(-1)^k \begin{bmatrix} n/2 \\ k \end{bmatrix} q^{k(k-1)/2-nk}$ is when $k = n/2, -n(3n+2)/8$, (2.12) holds.

Proof of (iii). For $|t| \le q^{n/2+1}$,

$$\begin{aligned}
F_{n/2}(x/t, y/t) &= \sum_{j=n/2+1}^{\infty} \frac{t^2}{t^2 + q^j x t - q^{2j} xy} \\
&= \sum_{j=n/2+1}^{\infty} \frac{-t^2}{q^{2j}xy\left(1 - (t^2 + q^j xt)/(q^{2j}xy)\right)} \\
&= \sum_{j=n/2+1}^{\infty} \frac{-t^2}{q^{2j}xy} \sum_{i=0}^{\infty}\left(\frac{t^2 + q^j xt}{q^{2j}xy}\right)^i \\
&= -\sum_{j=n/2+1}^{\infty}\sum_{i=0}^{\infty} \frac{t^{i+2}(q^j x)^i}{(q^{2j}xy)^{i+1}}(1 + \frac{t}{q^j x})^i \\
&= -\sum_{j=n/2+1}^{\infty}\sum_{i=0}^{\infty} \frac{t^{i+2}}{q^{j(i+2)}xy^{i+1}} \sum_{k=0}^{i}\binom{i}{k}\left(\frac{t}{q^j x}\right)^k \\
&= -\sum_{j=n/2+1}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{i}\binom{i}{k}\frac{t^{i+k+2}}{q^{j(i+k+2)}x^{k+1}y^{i+1}} \\
&= -\sum_{i=0}^{\infty}\sum_{k=0}^{i}\binom{i}{k}\frac{t^{i+k+2}}{x^{k+1}y^{i+1}}\frac{q^{-n(i+k+2)/2}}{q^{i+k+2}-1}, \qquad (2.17)
\end{aligned}$$

and from (1.4),

$$\frac{1}{\prod_{k=0}^{n/2}(t-q^k)} = (-1)^{n/2+1}q^{-n(n+2)/8}\sum_{h=0}^{\infty}\begin{bmatrix} n/2+h \\ h \end{bmatrix} q^{-hn/2}t^h. \tag{2.18}$$

Combining (2.17) and (2.18), we have

$$\begin{aligned}\frac{F_{n/2}(x/t,y/t)}{\prod_{k=0}^{n/2}(t-q^k)} &= (-1)^{n/2}q^{-n(n+2)/8}\sum_{i,h=0}^{\infty}\sum_{k=0}^{i}\begin{bmatrix} n/2+h \\ h \end{bmatrix}\binom{i}{k} \\ &\quad \cdot\frac{t^{h+i+k+2}}{x^{k+1}y^{i+1}}\frac{q^{-n(h+i+k+2)/2}}{q^{i+k+2}-1}, \end{aligned}\tag{2.19}$$

and then

$$\begin{aligned} D(x,y) &:= \frac{1}{(n/2)!}\frac{d^{n/2}}{dt^{n/2}}\left\{\frac{F_{n/2}(x/t,y/t)}{\prod_{k=0}^{n/2}(t-q^k)}\right\}_{t=0} \\ &= (-1)^{n/2}q^{-n(n+2)/8}\sum_{\substack{h+i+k+2=n/2\\ 0\le h,i\le n/2-2\\ 0\le k\le i}}\begin{bmatrix} n/2+h \\ h \end{bmatrix}\binom{i}{k}\frac{1}{x^{k+1}y^{i+1}}\frac{q^{-n(h+i+k+2)/2}}{q^{i+k+2}-1} \\ &= (-1)^{n/2}q^{-n(3n+2)/8}\sum_{\substack{h+i+k+2=n/2\\ 0\le h,i\le n/2-2\\ 0\le k\le i}}\begin{bmatrix} n/2+h \\ h \end{bmatrix}\binom{i}{k}\frac{1}{x^{k+1}y^{i+1}}\frac{1}{q^{i+k+2}-1}. \end{aligned}\tag{2.20}$$

So

$$q^{n(3n+2)/8}(xy)^{n/2}\left(\prod_{j=1}^{n/2}(q^j-1)\right)D(x,y)\in\mathbb{Z}[q,x,y]. \tag{2.21}$$

So (2.14) holds by viewing (2.10), (2.12) and (2.21).

Proof of (iv). For $R := q^{n/2+1}$,

$$|I(x,y)| \le \frac{\max_{|t|=R}\left|F_{n/2}(x/t,y/t)\right|}{R^{n/2}\prod_{k=0}^{n/2}(R-q^k)}. \tag{2.22}$$

Now from (2.17), we have

$$\begin{aligned}\max_{|t|=R}\left|F_{n/2}(x/t,y/t)\right| &= \max_{|t|=R}\left|\sum_{j=n/2+1}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{i}\binom{i}{k}\frac{t^{i+k+2}}{q^{j(i+k+2)}x^{k+1}y^{i+1}}\right| \\ &\le \sum_{j=n/2+1}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{i}\binom{i}{k}\frac{q^{(n/2+1)(i+k+2)}}{q^{j(i+k+2)}x^{k+1}y^{i+1}} \\ &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{i}\binom{i}{k}\frac{1}{q^{j(i+k+2)}x^{k+1}y^{i+1}} \\ &\le \sum_{j=0}^{\infty}\frac{1}{q^j} \\ &\le 2. \end{aligned}\tag{2.23}$$

Now

$$\begin{aligned} R^{n/2}\prod_{k=0}^{n/2}(R-q^k) &= R^{n+1}\prod_{k=0}^{n/2}(1-q^{-n/2-k-1}) \\ &\geq R^{2n+1}\prod_{j=0}^{\infty}(1-q^{-j}) \\ &\geq cq^{n(2n+1)/2}, \end{aligned} \tag{2.24}$$

where $c := \prod_{j=0}^{\infty}(1-q^{-j})$. Putting (2.23) and (2.24) into (2.22), we have

$$|I(x,y)| \leq \frac{c_q}{q^{n(2n+1)/2}},$$

where $c_q := 2/c$ is depending only on q. □

3 Proof of Theorem 1.1

We first prove that for $x, y > 0$,

$$|I(x,y)| > 0, \tag{3.1}$$

where $I(x,y)$ is defined by (2.8). In fact, from (2.17),

$$\begin{aligned} I(x,y) &= \frac{1}{2\pi i}\int_{|t|=q^{n/2+1}} \frac{F_{n/2}(x/t, y/t)dt}{t^{n+2}\left(\prod_{k=0}^{n/2}(1-q^k/t)\right)} \\ &= \frac{1}{2\pi i}\int_{|t|=q^{n/2+1}} \frac{F_{n/2}(x/t, y/t)}{t^{n+2}}\left(\sum_{j_0,\cdots,j_{n/2}\geq 0}\prod_{k=0}^{n/2}\left(\frac{q^k}{t}\right)^{j_k}\right)dt \\ &= \sum_{j_0,\cdots,j_{n/2}\geq 0} q^{\sum_{k=0}^{n/2} kj_k}\cdot\frac{1}{2\pi i}\int_{|t|=q^{n/2+1}}\{\frac{-1}{t^{n+2+(j_0+\cdots j_n)}} \\ &\quad \cdot\sum_{i=0}^{\infty}\sum_{k=0}^{i}\binom{i}{k}\frac{t^{i+k+2}}{x^{k+1}y^{i+1}}\frac{q^{-n(i+k+2)/2}}{q^{i+k+2}-1}\}dt \\ &= -\sum_{j_0,\cdots,j_n\geq 0} q^{\sum_{k=0}^{n} kj_k}\sum_{\substack{i+k-(n+j_0+\cdots j_n)=-1 \\ 0\leq i\leq\infty,\ 0\leq k\leq i}}\binom{i}{k}\frac{1}{x^{k+1}y^{i+1}}\frac{q^{-n(i+k+2)/2}}{q^{i+k+2}-1} \\ &\neq 0, \end{aligned}$$

as all $x, y, q > 1$. So (3.1) holds. Now let r, s be any fixed positive rational numbers, then $u := s/r$ is also a positive rational number such that

$$F(r,u) = \sum_{j=0}^{\infty}\frac{1}{1+q^{-j}r-q^{-2j}ru}. \tag{3.2}$$

Again we can assume $r, u > 2$ because of (2.3). Now let $Q(q), P(x,y)$ and $R_{n/2}(x,y)$ be defined in (2.9), (2.10), (2.13) respectively, and

$$H_n(q) := q^{n(3n+2)/8} \prod_{j=1}^{n/2} (q^j - 1), \tag{3.3}$$

then

$$0 < |H_n(q)| \leq h_q q^{n(n+1)/2}, \tag{3.4}$$

where $h_q := \prod_{j=0}^{\infty}(1 - q^{-j})$, and

$$x^{n/2} y^{n/2} H_n(q) R_{n/2}(x,y) \cdot \{Q(q), P(x,y)\} \subset \mathbb{Z}[q,x,y]. \tag{3.5}$$

Now let

$$Q^*(r,u) := r^{n/2} u^{n/2} H_n(q) R_{n/2}(r,u) Q(q), \tag{3.6}$$

and

$$P^*(r,u) := r^{n/2} u^{n/2} H_n(q) R_{n/2}(r,u) P(r,u). \tag{3.7}$$

Then

$$Q^*(r,u), P^*(r,u) \in \mathbb{Z}[q,r,u], \tag{3.8}$$

and

$$\begin{aligned} \Delta \quad &: \quad = |Q^*(r,u)F(r,u) + P^*(r,u)| \\ &= \quad (ru)^{n/2} \left|H_n(q)\right| \left|R_{n/2}(r,u)\right| \left|I(r,u)\right|. \end{aligned} \tag{3.9}$$

Now $\Delta > 0$ and by (2.15),

$$\begin{aligned} \Delta \quad &\leq \quad (ru)^{n+1} \left|H_n(q)\right| q^{n(n+2)/4} \left(\prod_{j=0}^{n/2} (1 + q^{-j} + q^{-2j}) \right) |I(r,u)| \\ &\leq \quad h_q f_q \, (ru)^{n+1} \, q^{n(n+1)/2} q^{n(n+2)/4} \frac{c_q}{q^{n(2n+1)/2}} \\ &= \quad h_q f_q \, (ru)^{n+1} \, \frac{c_q}{q^{n(2n-1)/4}}, \end{aligned} \tag{3.10}$$

where $f_q := \prod_{j=0}^{\infty}(1 + q^{-j} + q^{-2j})$ is a constant depending only on q. Finally, if

$$r := \frac{i}{l} \quad \text{and} \quad u := \frac{j}{m} \tag{3.11}$$

with i, j, l, m positive integers, then

$$Q^{**}(r,u) := (lm)^n \, Q^*(r,u), \tag{3.12}$$

and

$$P^{**}(r,u) := (lm)^n \, P^*(r,u), \tag{3.13}$$

are integers, and by (3.10),

$$\begin{aligned} 0 \quad &< \quad |Q^{**}(r,u)F(r,u) + P^{**}(r,u)| \\ &= \quad (lm)^n \ |Q^*(r,u)F(r,u) + P^*(r,u)| \\ &\leq \quad (lm)^n \, h_q f_q \, (ru)^{n+1} \, \frac{c_q}{q^{n(2n-1)/4}}, \end{aligned}$$

which tends to zero as $n \to \infty$, This shows that $F(r,u)$ is irrational, that is

$$\sum_{j=0}^{\infty} \frac{1}{1+q^j r - q^{2j} s}$$

is irrational for $q > 1$ integers and r, s positive rationals. This completes the proof of Theorem 1.1. □

Now by the standard methods (as in chapter 11 of Borwein and Borwein [1]), the estimates in the proof of Theorem 1.1 give that, under the assumption of the theorem,

$$\left| F(r,u) - \frac{s}{t} \right| > \frac{1}{t^{\alpha}},$$

for some constant α and all integers s and t, and hence

$$\sum_{j=0}^{\infty} \frac{1}{1+q^j r - q^{2j} s}$$

is not a Liouville number.

REFERENCES

1. J.M.Borwein and P.B.Borwein, Pi and the AGM—A Study in Analytic Number Theory and Computational Complexity, New York, 1987.
2. P.B.Borwein, Padé Approximants for the q-Elementary Functions, Constr. Approx. 4(1988): 391—402.
3. P.B.Borwein, On the Irrationality of $\sum(1/(q^n + r))$, J. of Number Theory, Vol. 37, No. 3, March 1991.
4. D.V.Chudnovsky and G.V.Chudnovsky, Padé and Rational Approximation to Systems of Functions and Their Arithmetic Applications, In "Lecture Notes in Mathematics," Vol. 1052, Springer-Verlag, Berlin, 1984.
5. P.Erdös, On Arithmetical Properties Of Lambert Series, J. Indian Math. Soc. (N.S.) 12(1948), 63—66.
6. P.Erdös and R.L.Graham, Old and New Problems and Results in Combinatorial Number Theory, Enseign. Math. Monograph, 28(1980).
7. G.Gasper and M.Rahman, Basic Hypergeometric Series, Encyclopaedia of Maths and its Applications, Vol. 35, Cambridge University Press, Cambridge, 1990.
8. K.Mahler, Zur Approximation der Exponentialfunktion und des Logarithmus, J. Reine Angew. Math. 166(1931), 118—150.
9. R.Wallisser, Rationale Approximation des q-Analogons der Exponential-funktion und Irrationalitätsaussagen für diese Funktion, Arch. Math., Vol. 44, 59—64 (1985).
10. P.Zhou, On the Irrationality of the Infinite Product $\prod_{j=0}^{\infty}(1 + q^{-j} r + q^{-2j} s)$, Math. Proc. Camb. Phil. Soci., to appear.