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ON SYLVESTER'S PROBLEM AND HAAR SPACES

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Given a finite set of points in the plane (with distinct x coordinates) must there exist a polynomial of degree n that passes through exactly $n + 1$ of the points? Provided that the points do not all lie on the graph of a polynomial of degree n then the answer to this question is yes. This generalization of Sylvester's Problem (the $n = 1$ case) is established as a corollary to a version of Sylvester's Problem that holds for certain finite dimensional Haar spaces of continuous functions.

If E is a finite set of points in the plane then there exists a line through exactly two points of E unless all the points of E are colinear. This attractive result was posed as a problem by J. J. Sylvester in 1893 and was proved in 1933 by T. Gallai (see [**3**]). A particularly simple solution of Sylvester's Problem, due to L. M. Kelly, may be found in [**1**]. We ask the following question: If V_n is an n-dimensional vector space of real-valued continuous functions of a real variable and if E is a finite set in the plane, must there exist $g \in V_n$ so that the graph of g passes through exactly n points of E? We show that the answer to the above question is affirmative if V_n is a uni-modal Haar space of dimension n. (See Theorem 1.)

A Haar space H_n of dimension n on an interval $[a, b]$ is an n-dimensional real vector space of real-valued continuous functions with the additional property that if $g \in H_n$ and g has n distinct zeros then g is identically zero. Haar spaces are often also called Chebychev spaces. A Haar space H_n of dimension n is uni-modal if it satisfies the following: if $g \in H_n$ has $n - 1$ distinct zeros at $a \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} \leq b$ then g has a single change of monotonicity on each of the intervals

$$[\alpha_1, \alpha_2], [\alpha_2, \alpha_3], \ldots, [\alpha_{n-2}, \alpha_{n-1}]$$

and g is monotonic on $[a, \alpha_1]$ and $[\alpha_{n-1}, b]$.

The algebraic polynomials of degree less than n form a uni-modal Haar space of dimension n on any interval. The following are other examples of uni-modal Haar spaces of dimension n on $[a, b]$:

(a) The space spanned by

$$\{1, e^{\alpha_1 x}, \ldots, e^{\alpha_{n-1} x}\}$$

where $\alpha_1, \ldots, \alpha_{n-1}$ are distinct non-zero real numbers.

(b) The space spanned by

$$\{1, x, x^2, \ldots, x^{n-2}, f(x)\}$$

where $f^{(n-1)}(x) > 0$ on $[a, b]$.

(c) The space spanned by

$$\{1, x^2, x^4, \ldots, x^{2n-2}\}$$

on an interval $[a, b]$ where $a > 0$.

We say that a finite set E contained in the plane R^2 in $\text{co}(H_n)$ if all the points of E lie on (the graph of) g where g is a single element of H_n.

We shall now prove:

THEOREM 1. *Suppose that E is a finite set of points in the strip $\{(x, y) \mid a \le x \le b\}$ and suppose that no two points of E lie on the same vertical line. Suppose that H_n is a uni-modal Haar space of dimension $n \ge 2$ on $[a, b]$. Then either there exists $g \in H_n$ so that g passes through exactly n points of E or E is* $\text{co}(H_n)$.

Proof. Our proof is motivated by L. M. Kelly's proof of Sylvester's Problem. We first note that if $x_1, x_2, \ldots, x_n$ are n distinct numbers in $[a, b]$ and if $y_1, \ldots, y_n$ are real numbers then there exists a unique $h \in H_n$ so that

$$h(x_i) = y_i \quad \text{for } i = 1, \ldots, n.$$

This interpolation property is an easy consequence of the fact that H_n is a Haar space of dimension n. (For further discussion of Haar spaces see [**2**, p. 23].) We assume that E is not $\text{co}(H_n)$ and, hence, that E contains at least $n + 1$ points. We know that there is an element of H_n that passes through any n points of E. We assume, for the sake of deriving a contradiction, that any such element in fact passes through at least $n + 1$ points of E. Let $K \subset H_n$ denote the set of elements of H_n that pass through at least n points of E. Since H_n is Haar there is a unique element of H_n passing through any n points of E. Since E is finite K must be finite also.

Let P be a point in E that is vertically closest to, though not on, the graph of an element g in K. Since K and E are finite such a pair P and g must exist. We assumed that g was an element of K, thus there exist $n + 1$ points $(x_1, y_1), \ldots, (x_{n+1}, y_{n+1})$ in E through which g passes. We may suppose that

$$a \le x_1 < x_2 < \cdots < x_{n+1} \le b.$$

Write $P = (x^*, y^*)$ and suppose that the vertical distance from P to g is δ.

Case 1. $x_i < x^* < x_{i+1}$ where $2 \leq i \leq n-1$. Let $f \in K$ pass through the n points $(x_1, y_1), \ldots, (x_{i-1}, y_{i-1})$, (x^*, y^*), $(x_{i+2}, y_{i+2}), \ldots$, (x_{n+1}, y_{n+1}). Consider $f - g \in H_n$. The function $f - g$ has $n-1$ distinct zeros at $x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_{n+1}$. Since H_n is a uni-modal Haar space, $f - g$ has at most a single change of monotonicity on the interval $[x_{i-1}, x_{i+2}]$. Since, $x_{i-1} < x_i < x^* < x_{i+1} < x_{i+2}$ we must have either

$$0 < |f(x_i) - g(x_i)| < |f(x^*) - g(x^*)| = \delta$$

or

$$0 < |f(x_{i+1}) - g(x_{i+1})| < |f(x^*) - g(x^*)| = \delta.$$

This implies the contradiction that either (x_i, y_i) or (x_{i+1}, y_{i+1}) is vertically too close to $f \in K$.

Case 2. Either $x^* < x_2$ or $x_n < x^*$. We treat the case $x^* < x_2$. The other case is virtually identical. Let $f \in K$ pass through the n points

$$(x^*, y^*), (x_3, y_3), \ldots, (x_{n+1}, y_{n+1}).$$

Since $f - g \in H_n$ has $n-1$ distinct zeros at $x_3, x_4, \ldots, x_{n+1}$ we know that $f - g$ is monotonic on $[a, x_3]$. This leads to the contradiction that

$$0 < |f(x_2) - g(x_2)| < |f(x^*) - g(x^*)| = \delta. \qquad \square$$

We get a solution of Sylvester's Problem by taking H_2 in Theorem 1 to be the uni-modal Haar space of lines (it may be necessary to rotate E first to ensure that no two points of E lie on the same vertical line). We also have

COROLLARY 1. *Let E be a finite set in R^2 with no two points on the same vertical line. Suppose that the points of E do not all lie on a polynomial of degree less than $n+1$. Then there exists*:

(a) *a line through exactly two points of E.*
(b) *a parabola through exactly three points of E.*
(c) *a cubic through exactly four points of E.*

⋮

(n) *a polynomial of degree n through exactly $n+1$ points of E.*

We can construct a Haar space H_n' on $[a, b)$ where we demand that each $g \in H_n'$ be periodic with period $b - a$. To make H_n' uni-modal we require that: if $g \in H_n'$ has $n-1$ distinct zeroes at $a \le \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < b$ then g has a single change of monotonicity on each of the $n-1$ intervals

$$[\alpha_1, \alpha_2], [\alpha_2, \alpha_3], \ldots, [\alpha_{n-2}, \alpha_{n-1}], [\alpha_{n-1}, b-a+\alpha_1].$$

THEOREM 2. *Suppose that E is a finite set of points in the strip $\{(x, y) \mid a \le x < b\}$ and suppose that no two points of E lie on the same vertical line. Suppose that H_n' is a uni-modal Haar space of dimension $n \ge 2$ on $[a, b)$. Then either there exists $g \in H_n'$ so that g passes through exactly n points of E or E is* co(H_n').

Proof. The proof is exactly analogous to the proof of Theorem 1, Case 1. □

The trigonometric polynomials of degree n form a uni-modal Haar space of dimension $2n+1$ on $[-\pi, \pi)$. We can now, of course, formulate a corollary similar to Corollary 1 for trigonometric polynomials.

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