LITTLEWOOD-TYPE PROBLEMS ON SUBARCS OF THE UNIT CIRCLE

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Abstract. The results of this paper show that many types of polynomials cannot be small on subarcs of the unit circle in the complex plane. A typical result of the paper is the following. Let $\mathcal{F}_n$ denote the set of polynomials of degree at most n with coefficients from $\{-1,0,1\}$. There are absolute constants $c_1>0$, $c_2>0$, and $c_3>0$ such that

$$\exp(-c_1/a) \le \inf_{0\ne p\in\mathcal{F}_n} \|p\|_{L_1(A)}\,, \qquad \inf_{0\ne p\in\mathcal{F}_n} \|p\|_A \le \exp(-c_2/a)$$

for every subarc A of the unit circle $\partial D := \{z\in\mathbb{C} : |z|=1\}$ with length $0<a<c_3$.

The lower bound results extend to the class of f of the form

$$f(z)=\sum_{j=m}^{n} a_j z^j\,, \quad a_j\in\mathbb{C}\,, \quad |a_j|\le M\,, \quad |a_m|=1$$

with varying nonnegative integers $m\le n$. It is shown that functions f of the above form cannot be arbitrarily small uniformly on subarcs of the circle. However, this does not extend to sets of positive measure. It shown that it is possible to find a polynomial of the above form that is arbitrarily small on as much of the boundary (in the sense of linear Lebesgue measure) as one likes.

An easy to formulate corollary of the results of this paper is the following.

Corollary. *Let A be a subarc of the unit circle with length $\ell(A)=a$. If (p_k) is a sequence of monic polynomials that tends to 0 in $L_1(A)$, then the sequence $H(p_k)$ of heights tends to ∞.*

The results of this paper are dealing with (extensions of) classes much studied by Littlewood and many others in regards to the various conjectures of Littlewood concerning growth and flatness of unimodular polynomials on the unit circle ∂D. Hence the title of the paper.

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1. Introduction

Littlewood's well-known and now resolved conjecture of around 1948 concerns polynomials of the form

$$p(z) := \sum_{j=1}^{n} a_j z^{k_j},$$

where the coefficients a_j are complex numbers of modulus at least 1 and the exponents k_j are distinct non-negative integers. It states that such polynomials have L_1 norms on the unit circle

$$\partial D := \{z \in \mathbb{C} : |z| = 1\}$$

that grow at least like $c \log n$ with an absolute constant $c > 0$. This was proved by Konjagin [Ko-81] and independently by McGehee, Pigno, and Smith [MPS-81].

Pichorides, who contributed essentially to the proof of the Littlewood conjecture, observed in [Pi-83] that the original Littlewood conjecture (when all the coefficients are from $\{0, 1\}$ would follow from a result on the L_1 norm of such polynomials on sets $E \subset \partial D$ of measure π. Namely if

$$\int_E \Big| \sum_{j=0}^{n} z^{k_j} \Big| \, |dz| \geq c$$

for any subset $E \subset \partial D$ of measure π with an absolute constant $c > 0$, then the original Littlewood conjecture holds. Throughout the paper the measure of a set $E \subset \partial D$ is the linear Lebesgue measure of the set

$$\{t \in [-\pi, \pi) : e^{it} \in E\}.$$

Konjagin [Ko-96] gives a lovely probabilistic proof that this hypothesis fails. He does however conjecture the following: for any *fixed* set $E \subset \partial D$ of positive measure there exists a constant $c = c(E) > 0$ depending only on E such that

$$\int_E \Big| \sum_{j=0}^{n} z^{k_j} \Big| \, |dz| \geq c(E).$$

In other words the sets $E_\epsilon \subset \partial D$ of measure π in his example where

$$\int_{E_\epsilon} \Big| \sum_{j=0}^{n} z^{k_j} \Big| \, |dz| < \epsilon$$

must vary with $\epsilon > 0$.

We show, among other things, that Konjagin's conjecture holds on subarcs of the unit circle ∂D.

Additional material on Littlewood's conjecture and related problems concerning the growth of polynomials with unimodular coefficients in various norms on the unit disk is to be found, for example, in [Bou-86], [Be-95], [Ka-85], [Li-86], [Ma-63], [Ne-90], [Od-93], and [So-95].

All the results of this paper concern how small polynomials of the above and related forms can be in the L_p norms on subarcs of the unit disk. For $1 \le p \le \infty$ the results are sharp, at least up to a constant in the exponent.

An interesting related result is due to Nazarov [Na-93]. One of its simpler versions states that there is an absolute constant $c > 0$ such that

$$\max_{z \in I} |p(z)| \le \left(\frac{c\, m(I)}{m(A)} \right)^n \max_{z \in A} |p(z)|$$

for every polynomial p of the form $p(z) = \sum_{j=0}^n a_j z^{k_j}$ with $k_j \in \mathbb{N}$ and $a_j \in \mathbb{C}$ and for every $A \subset I$, where I is a subarc of ∂D with length $m(I)$ and A is measurable with Lebesgue measure $m(A)$. This extends a result of Turán [Tu-84] called *Turán's Lemma*, where $I = \partial D$ and A is a subarc.

2. Notation

For $M > 0$ and $\mu \ge 0$, let $\mathcal{S}_M^\mu$ denote the collection of all analytic functions f on the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ that satisfy

$$|f(z)| \le \frac{M}{(1 - |z|)^\mu}\,, \qquad z \in D\,.$$

We define the following subsets of $\mathcal{S}_1^1$. Let

$$\mathcal{F}_n := \left\{ f : f(x) = \sum_{j=0}^n a_j x^j\,,\ \ a_j \in \{-1, 0, 1\} \right\}$$

and denote the set of all polynomials with coefficients from the set $\{-1, 0, 1\}$ by

$$\mathcal{F} := \bigcup_{n=0}^\infty \mathcal{F}_n\,.$$

More generally we define the following classes of polynomials. For $M > 0$ and $\mu \ge 0$ let

$$\mathcal{K}_M^\mu := \left\{ f : f(x) = \sum_{j=0}^n a_j x^j\,,\ \ a_j \in \mathbb{C},\ \ |a_j| \le M j^\mu\,,\ \ |a_0| = 1\,,\ \ n \in \mathbb{N} \right\}\,.$$

On occasion we let $\mathcal{S} := \mathcal{S}_1^1$, $\mathcal{S}_M := \mathcal{S}_M^1$, and $\mathcal{K}_M := \mathcal{K}_M^0$.

We also employ the following standard notations. We denote by $\mathcal{P}_n$ the set of all polynomials of degree at most n with real coefficients. We denote by $\mathcal{P}_n^c$ the set of all polynomials of degree at most n with complex coefficients. The height of a polynomial

$$p_n(z) := \sum_{j=0}^{n} a_j z^j, \quad a_j \in \mathbb{C}, \quad a_n \neq 0,$$

is defined by

$$H(p_n) := \max\left\{\frac{|a_j|}{|a_n|} : \ j = 0, 1, \dots, n\right\}.$$

Also,

$$\|p\|_A := \sup_{z \in A} |p(z)|$$

and

$$\|p\|_{L_q(A)} := \left(\int_A |p(z)|^q \, |dz|\right)^{1/q}$$

are used throughout this paper for measurable functions (in this paper usually polynomials) p defined on a measurable subset of the unit circle or the real line, and for $q \in (0, \infty)$.

3. New Results

The first two results concern lower bounds on subarcs in the supremum norm.

Theorem 3.1. *Let $0 < a < 2\pi$ and $M \geq 1$. Let A be a subarc of the unit circle with length $\ell(A) = a$. Then there is an absolute constant $c_1 > 0$ such that*

$$\|f\|_A \geq \exp\left(\frac{-c_1(1 + \log M)}{a}\right)$$

for every $f \in \mathcal{S}_M (:= \mathcal{S}_M^1)$ that is continuous on the closed unit disk and satisfies $|f(z_0)| \geq \frac{1}{2}$ for every $z_0 \in \mathbb{C}$ with $|z_0| = \frac{1}{4M}$.

Corollary 3.2. *Let $0 < a < 2\pi$ and $M \geq 1$. Let A be a subarc of the unit circle with length $\ell(A) = a$. Then there is an absolute constant $c_1 > 0$ such that*

$$\|f\|_A \geq \exp\left(\frac{-c_1(1 + \log M)}{a}\right)$$

for every $f \in \mathcal{K}_M (:= \mathcal{K}_M^1)$.

The next two results show that the previous results are, up to constants, sharp.

Theorem 3.3. *Let $0 < a < 2\pi$. Let A be the subarc of the unit circle with length $\ell(A) = a$. Then there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\inf_{0 \neq f \in \mathcal{F}} \|f\|_A \leq \exp\left(\frac{-c_1}{a}\right)$$

whenever $\ell(A) = a \leq c_2$.

Theorem 3.4. *Let $0 < a < 2\pi$ and $M \geq 1$. Let A be the subarc of the unit circle with length $\ell(A) = a$. Then there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\inf_{0 \neq f \in \mathcal{K}_M} \|f\|_A \leq \exp\left(\frac{-c_1(1 + \log M)}{a}\right)$$

whenever $\ell(A) = a \leq c_2$.

The next two results extend the first two results to the L_1 norm (and hence to all L_p norms with $p \geq 1$).

Theorem 3.5. *Let $0 < a < 2\pi$, $M \geq 1$, and $\mu = 1, 2, \ldots$. Let A be a subarc of the unit circle with length $\ell(A) = a$. Then there is an absolute constant $c_1 > 0$ such that*

$$\|f\|_{L_1(A)} \geq \exp\left(\frac{-c_1(\mu + \log M)}{a}\right)$$

for every $f \in \mathcal{S}_M^\mu$ that is continuous on the closed unit disk and satisfies $|f(z_0)| \geq \frac{1}{2}$ for every $z_0 \in \mathbb{C}$ with $|z_0| \leq \frac{1}{4M2^\mu}$.

Corollary 3.6. *Let $0 < a < 2\pi$, $M \geq 1$, and $\mu = 1, 2, \ldots$. Let A be a subarc of the unit circle with length $\ell(A) = a$. Then there is an absolute constant $c_1 > 0$ such that*

$$\|f\|_{L_1(A)} \geq \exp\left(\frac{-c_1(1 + \mu \log \mu + \log M)}{a}\right)$$

for every $f \in \mathcal{K}_M^\mu$.

The following is an interesting consequence of the preceding results.

Corollary 3.7. *Let A be a subarc of the unit circle with length $\ell(A) = a$. If (p_k) is a sequence of monic polynomials that tends to 0 in $L_1(A)$, then the sequence $H(p_k)$ of heights tends to ∞.*

The final result shows that the theory does not extend to arbitrary sets of positive measure.

Theorem 3.8. *For every $\epsilon > 0$ there is a polynomial $p \in \mathcal{K}_1$ such that $|p(z)| < \epsilon$ everywhere on the unit circle except possibly in a set of linear measure at most ϵ.*

The above results should be compared with earlier result of the authors [Bor-96] on approximation on the interval $[0,1]$. These state that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$\exp\left(-c_1\sqrt{n}\right) \le \inf_{0\ne p\in\mathcal{F}_n} \|p\|_{[0,1]} \le \exp\left(-c_2\sqrt{n}\right) .$$

4. Lemmas

Lemma 4.1. *Let $0 < a < \pi$ and $M \ge 1$. Let Γ_M be the circle with diameter $\left[-1+\frac{1}{2M}, 1\right]$. Let J be the subarc of Γ_M with length $\ell(J) = a$ which is symmetric with respect to the real line and contains 1. Then there is an absolute constant $c_3 > 0$ such that*

$$\|g\|_J \ge \exp\left(\frac{-c_4(1+\log M)}{a}\right)$$

for all $g \in \mathcal{S}_{4M}$ that is continuous on the closed unit disk and satisfies $\left|g\left(\frac{1}{4M}\right)\right| \ge \frac{1}{4}$.

Our next lemma is known as (a version) of the three-line-theorem. It may be found, for example, in [Zy-59, p. 93]. Its proof is so short and simple that we present it for the sake of completeness.

Lemma 4.2. *Let $a > 0$ and*

$$E_a := \{z \in \mathbb{C} : \ 0 \le \mathrm{Im}(z) \le a\} .$$

Suppose g is an analytic function in the interior of E_a, and suppose g is continuous on $E_a \cup \infty$. Then

$$\max_{\{z:\,\mathrm{Im}(z)=a/2\}} |g(z)| \le \left(\max_{\{z:\,\mathrm{Im}(z)=0\}} |g(z)|\right)^{1/2} \left(\max_{\{z:\,\mathrm{Im}(z)=a\}} |g(z)|\right)^{1/2} .$$

The next lemma, that plays a crucial role in the proof of Theorem 3.1, can be easily derived from Lemma 4.2.

Lemma 4.3. *Let $0 < a < \pi$, $\alpha := \cos(a/2) + i\sin(a/2)$, $\beta := \cos(a/2) - i\sin(a/2)$. Let*

$$I_t := \left\{z \in \mathbb{C} : \ \arg\left(\frac{\alpha - z}{z-\beta}\right) = t\right\} .$$

Note that I_a is the smaller arc on the unit circle with endpoints α and β, and I_0 is the line segment between α and β. Suppose g is an analytic function in the open region bounded by I_0 and I_a, and suppose g is continuous on the closed region between I_0 and I_a. Then

$$\max_{z\in I_{a/2}} |g(z)| \le \left(\max_{z\in I_0} |g(z)|\right)^{1/2} \left(\max_{z\in I_a} |g(z)|\right)^{1/2} .$$

To prove Theorem 3.3, we need some corollaries of the

Hadamard Three Circles Theorem. *Suppose f is regular inside and on*

$$\{z \in \mathbb{C} : r_1 \le |z| \le r_2\}.$$

For $r \in [r_1, r_2]$, let

$$M(r) := \max_{|z|=r} |f(z)|.$$

Then

$$M(r)^{\log(r_2/r_1)} \le M(r_1)^{\log(r_2/r)} M(r_2)^{\log(r/r_1)}.$$

Corollary 4.4. *Let $a \in (0, 1/8]$. Suppose f is regular inside and on the ellipse E_a with foci at $1-8a$ and 1 and with major axis*

$$[(1-4a) - 17a\,, (1-4a) + 17a]\,.$$

Let $\widetilde{E}_a$ be the ellipse with foci at $1-a$ and 1 and with major axis

$$[(1-4a) - 10a\,, (1-4a) + 10a]\,.$$

Then

$$\max_{z \in \widetilde{E}_a} |f(z)| \le \left(\max_{z \in [1-8a,1]} |f(z)| \right)^{1/2} \left(\max_{z \in E_a} |f(z)| \right)^{1/2}.$$

Corollary 4.5. *We have*

$$\max_{z \in \widetilde{E}_a} |f(z)| \le ((n+1)\exp(13na))^{1/2} \left(\max_{z \in [1-8a,1]} |f(z)| \right)^{1/2}$$

for every $f \in \mathcal{F}_n$ and $a \in (0, 1/8]$.

In one of our proofs of Theorem 3.3 we will need the upper bound of the result below proved in [Bor-96]. An application of this lemma makes the proof of Theorem 3.3 shorter in the special case when the subarc A of the unit circle is symmetric with respect to the real line and contains 1. Our second proof of Theorem 3.3 is longer. This self-contained second proof does not use the lemma below and eliminates the extra assumption on the subarc A.

Lemma 4.6. *There are absolute constants $c_4 > 0$ and $c_5 > 0$ such that*

$$\exp\left(-c_4\sqrt{n}\right) \le \inf_{0 \ne p \in \mathcal{F}_n} \|p\|_{[0,1]} \le \exp\left(-c_5\sqrt{n}\right).$$

To prove Theorem 3.4 we need two lemmas.

Lemma 4.7. *Suppose*

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \le 9\,, \quad a_j \in \mathbb{C}\,,$$

$$p(x) = (x-1)^k q(x)\,, \qquad q(x) = \sum_{j=0}^{n-k} b_j x^j\,, \quad b_j \in \mathbb{C}\,.$$

Then

$$\|q\|_{\partial D} \le \sum_{j=0}^{n-k} |b_j| \le 9(n+1)e\left(\frac{en}{k}\right)^{k-1} \le 9(n+1)\left(\frac{en}{k}\right)^{k}$$

(where ∂D denotes the unit circle). As a consequence, if A denotes the subarc of the unit circle that is symmetric to the real line, contains 1, and has length $2k/(9n)$, then

$$\|p\|_A \le 9(n+1)\left(\frac{e}{9}\right)^{k}\,.$$

To prove Theorem 3.4 our main tool is the next lemma due to Halász [Tu-84].

Lemma 4.8. *For every $k \in \mathbb{N}$, there exists a polynomial $h \in \mathcal{P}_k$ such that*

$$h(0) = 1\,, \qquad h(1) = 0\,, \qquad |h(z)| < \exp\left(\frac{2}{k}\right) \quad \text{for} \quad |z| \le 1\,.$$

Lemma 4.9. *Let $0 < a < \pi$ and $M \ge 1$. Let $\Gamma_{a,M,\mu}$ be the circle with diameter $\left[-1 + \frac{1}{2M2^\mu}, 1 - \frac{1-\cos(a/4)}{M2^\mu}\right]$. Let I be the subarc of $\Gamma_{a,M,\mu}$ with length $\ell(I) = a$ which is symmetric with respect to the real line and contains $1 - \frac{1-\cos(a/4)}{M2^\mu}$. Then there is an absolute constant $c_4 > 0$ such that*

$$\|g\|_I \ge \exp\left(\frac{-c_4(\mu + \log M)}{a}\right)$$

for every $g \in \mathcal{S}^{\mu}_{M4^\mu}$ that satisfies

$$\left|g\left(\tfrac{1}{4M2^\mu} - \tfrac{1-\cos(a/4)}{2M2^\mu}\right)\right| \ge \tfrac{1}{4}\,.$$

Lemma 4.10. *Let $w_1 \ne w_2 \in \mathbb{C}$ and let $z_0 := \frac{1}{2}(w_1 + w_2)$. Assume that J_1 is an arc that connects w_1 and w_2. Let J_2 be the arc that is the symmetric image of J_1 with respect to the z_0. Let $J := J_1 \cup J_2$ be positively oriented. Suppose that g is an analytic function inside and on J. Suppose that the region inside J contains the disk centered at z_0 with radius $\delta > 0$. Let $|g(z)| \le K$ for $z \in J_2$. Then*

$$|g(z_0)|^2 \le (\pi\delta)^{-1} K \int_{J_1} |g(z)|\,|dz|\,.$$

5. Proofs of Theorems 3.1 – 3.6

Proof of Lemma 4.1. Suppose $g \in \mathcal{S}_{4M}$ is continuous on the closed unit disk and $\left|g\left(\frac{1}{4M}\right)\right| \geq \frac{1}{4}$. Let $2m \geq 4$ be the smallest even integer not less than $4\pi/a$. Let

$$\xi := \exp\left(\frac{2\pi i}{2m}\right)$$

be the first $(2m)$th root of unity. We define $2m$ equally spaced points on Γ by

$$\eta_k := \tfrac{1}{4M} + \left(1 - \tfrac{1}{4M}\right)\xi^k\,, \qquad k = 0, 1, \ldots, 2m-1\,.$$

Then there is an absolute constant $c_5 > 0$ such that

$$1 - |z| \geq c_5 M^{-1}(ka)^2\,, \qquad k = 1, 2, \ldots m-1\,,$$

whenever z is on the smaller subarc of the circle Γ with endpoints η_k and η_{k+1} or with endpoints η_{2m-k} and η_{2m-k-1}, respectively. We define the function

$$h(z) := \prod_{j=0}^{2m-1} g\left(\tfrac{1}{4M} + \left(1 - \tfrac{1}{4M}\right)\xi^j\left(z - \tfrac{1}{4M}\right)\right).$$

Since $g \in \mathcal{S}_{4M}$, we obtain

$$\begin{aligned}
\max_{z\in\Gamma}|h(z)| &\leq \|g\|_J^2 \prod_{k=1}^{m-1}\left(4M\left(c_4 M^{-1}(ka)^2\right)^{-1}\right)^2 \leq \left(\frac{2}{c_5 a}\right)^{4m-4}\frac{M^{4m-4}}{((m-1)!)^4}\|g\|_J^2 \\
&\leq \left(\frac{m}{\pi c_5}\right)^{4m-4}\left(\frac{Me}{m-1}\right)^{4m-4}\|g\|_J^2 \leq e^4\left(\frac{eM}{\pi c_5}\right)^{4m-4}\|g\|_J^2 \\
&\leq \exp\left(\frac{c_6(1+\log M)}{a}\right)\|g\|_J^2
\end{aligned}$$

with an absolute constant $c_6 > 0$. Now the Maximum Principle yields that

$$\left|g\left(\tfrac{1}{4M}\right)\right|^{2m} = \left|h\left(\tfrac{1}{4M}\right)\right| \leq \max_{z\in\Gamma_M}|h(z)| \leq \exp\left(\frac{c_6(1+\log M)}{a}\right)\|g\|_J^2\,.$$

Since $2m \leq 2 + 4\pi/a$ and $\left|g\left(\frac{1}{4M}\right)\right| \geq \frac{1}{4}$, we obtain

$$\begin{aligned}
\|g\|_J^2 &\geq \exp\left(\frac{-c_6(1+\log M)}{a}\right)\left|g\left(\tfrac{1}{4M}\right)\right|^{2m} \geq \exp\left(\frac{-c_6(1+\log M)}{a}\right)\left(\frac{1}{4}\right)^{2m} \\
&\geq \exp\left(\frac{-c_7(1+\log M)}{a}\right)
\end{aligned}$$

with an absolute constant $c_7 > 0$. This finishes the proof. $\square$

Proof of Lemma 4.2. Let z_0 be a complex number with $\mathrm{Im}(z) = a/2$. We want to show that

$$|g(z_0)|^2 \le \left(\max_{\{z:\, \mathrm{Im}(z)=0\}} |g(z)| \right) \left(\max_{\{z:\, \mathrm{Im}(z)=a\}} |g(z)| \right) .$$

Without loss of generality we may assume that $\mathrm{Re}(z_0) = 0$; the general case follows by a linear transformation. So let $z_0 := ia/2$. Applying the Maximum Principle to $h(z) := g(z)g(ia - z)$ on $E_a \cup \infty$, we obtain

$$\begin{aligned} |g(ia/2)|^2 &= |g(ia/2)g(ia - ia/2)| = |h(ia/2)| \le \max_{z \in \partial E_a} |h(z)| \\ &= \max_{z \in \partial E_a} |g(z)g(ia - z)| \le \left(\max_{\{z:\, \mathrm{Im}(z)=0\}} |g(z)| \right) \left(\max_{\{z:\, \mathrm{Im}(z)=a\}} |g(z)| \right) . \end{aligned}$$

and the proof is finished. □

Proof of Lemma 4.3. This follows from Lemma 4.2 by the substitution

$$w := \log \left(\frac{\alpha - z}{z - \beta} \right) .$$

□

Proof of Theorem 3.1. Without loss of generality we may assume that the arc A is the subarc of the unit circle with length $\ell(A) = a < \pi$ which is symmetric with respect to the real line and contains 1. Suppose $f \in \mathcal{S}_M$ is continuous on the open unit disk, and suppose $\left| f\left(\frac{1}{4M}\right) \right| \ge \frac{1}{2}$. Using the notation of Lemma 4.3, let $g(z) := (z - \alpha)(z - \beta)f(z)$. Then a straightforward geometric argument yields that

$$|g(z)| \le \frac{M\,|(z - \alpha)(z - \beta)|}{1 - |z|} \le \frac{2M}{\sin(a/2)}, \qquad z \in I_0$$

(note that I_0 is the line segment between α and β). Hence, with $L := \|g\|_A$ (note that $A = I_a$), we conclude by Lemma 4.3 that

$$\max_{z \in I_{a/2}} |g(z)| \le \left(\frac{2ML}{\sin(a/2)} \right)^{1/2} .$$

Denote by G_a the open region bounded by $I_{a/2}$ and I_a. By the Maximum Principle

$$\max_{z \in G_a} |g(z)| \le \max \left\{ L, \left(\frac{2L}{\sin(a/2)} \right)^{1/2} \right\} .$$

It is a simple elementary geometry to show that the arc $K := \Gamma_M \cap G_a$ has length at least $a/2$. Here, as in Lemma 4.1, Γ_M denotes the circle with diameter $\left[-1 + \frac{1}{2M}, 1\right]$. Observe that $f \in \mathcal{S}_M$ implies $g \in \mathcal{S}_{4M}$. Also, since $M \ge 1$,

$$\left| g\left(\tfrac{1}{4M}\right) \right| \ge \left(\tfrac{3}{4}\right)^2 \left| f\left(\tfrac{1}{4M}\right) \right| \ge \left(\tfrac{3}{4}\right)^2 \tfrac{1}{2} \ge \tfrac{1}{4} .$$

Hence Lemma 4.1 can be applied with $g \in \mathcal{S}_{4M}$. We conclude that

$$\max\left\{L\,,\left(\frac{2ML}{\sin(a/2)}\right)^{1/2}\right\} \geq \|g\|_K \geq \exp\left(\frac{-c_4(1+\log M)}{2a}\right)\,,$$

from which

$$\|f\|_A \geq \frac{1}{4}\,\|g\|_A = \frac{L}{4} \geq \exp\left(\frac{-c_1(1+\log M)}{a}\right)$$

with an absolute constant $c_1 > 0$. □

Proof of Corollary 3.2. Note that if $f \in \mathcal{K}_M$, then $f \in \mathcal{S}_M$ and f is continuous on the closed unit disk. Also, if $|z_0| = \frac{1}{4M}$, then

$$|f(z_0| \geq 1 - M\,\frac{|z_0|}{1-|z_0|} \geq 1 - \frac{2M}{4M} = \frac{1}{2}\,.$$

So the assumptions of Theorem 3.1 are satisfied and the corollary follows from Theorem 3.1. □

Proof of Corollary 4.4. This follows from the Hadamard Three Circles Theorem with the substitution

$$w = (1-4a) + 4a\left(\frac{z+z^{-1}}{2}\right)\,.$$

The Hadamard Three Circles Theorem is applied with $r_1 := 1, r := 2$, and $r_2 := 4$. □

Proof of Corollary 4.5. This follows from Corollary 4.4 and the Maximum Principle. □

We present two proofs of Theorem 3.3. The first one is under the additional assumption that the subarc A is symmetric with respect to the real line and contains 1.

Proof of Theorem 3.3 in the above special case. By Lemma 4.6, for every $n \in \mathbb{N}$, there is a $Q_n \in \mathcal{F}_n$ such that

$$\|Q_n\|_{[0,1]} \leq \exp\left(-c_5\sqrt{n}\right)\,.$$

Let n be chosen so that $n := \lfloor N \rfloor$, where N is defined by

$$a = \frac{c_5}{26\sqrt{N}}\,.$$

Then, by Corollary 4.5,

$$\begin{aligned}\max_{z\in\widetilde{E}_a}|Q_n(z)| &\le (n+1)^{1/2}\left(\exp\left((c_5/2)\sqrt{n}\right)\right)^{1/2}\left(\max_{z\in[1-8a,1]}|f(z)|\right)^{1/2}\\ &\le (n+1)^{1/2}\left(\exp\left((c_5/2)\sqrt{n}\right)\right)^{1/2}\left(\exp\left(-c_5\sqrt{n}\right)\right)^{1/2}\\ &\le (n+1)^{1/2}\exp\left(-(c_5/4)\sqrt{n}\right)\\ &\le \exp\left(\frac{-c_6}{a}\right)\end{aligned}$$

whenever $a \le c_7$, where $c_6 > 0$ and $c_7 > 0$ are absolute constants. Now observe that the unit circle intersects the ellipse $\widetilde{E}_a$ in an arc of length at least $c_8 a$, where $c_8 > 0$ is an absolute constant. Therefore the Maximum Principle finishes the proof. □

Our second proof of Theorem 3.3 is in the general case. In addition, it is self-contained.

Proof of Theorem 3.3. Let $A := \{e^{it} : t \in [t_1, t_2]\}$, where $0 \le t_1 < t_2 \le 2\pi$ and $t_1 - t_2 = a$. Let $t_0 := (t_1 + t_2)/2$, and $w := e^{it_0}$. Let $k := \lfloor \frac{1}{8}\sqrt{n}\rfloor$. Let

$$1 - k/(2n) =: y_0 < y_1 < \cdots < y_k := 1$$

be $k+1$ equidistant points.

We use a counting argument to find a polynomial $f \in \mathcal{F}_n$ with the property:

$$|f(y_j w)| \le 2^{1-2\sqrt{n}}, \qquad j = 0, 1, \dots, k. \tag{5.1}$$

Indeed, we can divide the $2k$-dimensional real cube

$$Q := \{(z_1, \dots, z_k) \in \mathbb{C}^k : \mathrm{Re}(z_j),\ \mathrm{Im}(z_j) \in [0, 2n)\}$$

into $(2mn)^{2k}$ subcubes by defining

$$Q_{i_1, i_2, \dots, i_{2k}}$$

$$:= \left\{(z_1, \dots, z_k) \in \mathbb{C}^k : \mathrm{Re}(z_j) \in \left[\frac{i_{2j-1}}{m}, \frac{i_{2j-1}+1}{m}\right),\ \mathrm{Im}(z_j) \in \left[\frac{i_{2j}}{m}, \frac{i_{2j}+1}{m}\right)\right\},$$

where $(i_1, i_2, \dots, i_{2k})$ are $2k$-tuples of integers with $0 \le i_j \le 2nm - 1$ for each $j = 1, 2, \dots, 2k$. Let

$$\mathcal{A}_{n-1} := \left\{f : f(x) = \sum_{j=0}^{n-1} a_j x^j,\ \ a_j \in \{0, 1\}\right\}$$

denote the set of polynomials of degree at most $n-1$ with coefficients from $\{0, 1\}$. Note that if $P \in \mathcal{A}_{n-1}$, then

$$M(P) := (\mathrm{Re}(P(y_0 w)), \mathrm{Im}(P(y_0 w)), \dots, \mathrm{Re}(P(y_k w)), \mathrm{Im}(P(y_k w))) \in Q.$$

Also, there are exactly 2^n elements of $\mathcal{A}_{n-1}$. Therefore, if

$$(2mn)^{2k} < 2^n$$

holds, then there exist two different $P_1 \in \mathcal{A}_{n-1}$ and $P_2 \in \mathcal{A}_{n-1}$, and a subcube $Q_{i_1,i_2,\dots,i_{2k}}$ such that both

$$M(P_1) = (\mathrm{Re}(P_1(y_0 w)), \mathrm{Im}(P_1(y_0 w)), \dots, \mathrm{Re}(P_1(y_k w)), \mathrm{Im}(P_1(y_k w)))$$

and

$$M(P_2) = (\mathrm{Re}(P_2(y_0 w)), \mathrm{Im}(P_2(y_0 w)), \dots, \mathrm{Re}(P_2(y_k w)), \mathrm{Im}(P_2(y_k w)))$$

are in $Q_{i_1,i_2,\dots,i_{2k}}$, and hence for $0 \neq f := P_1 - P_2 \in \mathcal{F}_n$, we have

$$|f(y_j w)| \le \sqrt{2}\, m^{-1}, \qquad j = 0, 1, \dots, 2k.$$

Now choose $m := \lfloor 2^{2\sqrt{n}} \rfloor$. This, together with $k := \lfloor \frac{1}{8}\sqrt{n} \rfloor$, yields that the inequality $(2mn)^{2k} < 2^n$ holds, and (5.1) follows.

Associated with $f \in \mathcal{F}_n$ satisfying (5.1), we define

$$u(x) := \mathrm{Re}(f(xw)), \qquad x \in \mathbb{R},$$

and

$$v(x) := \mathrm{Im}(f(xw)), \qquad x \in \mathbb{R}.$$

Obviously

$$\|u^{(k)}\|_{[0,1]} \le n^{k+1} \qquad \text{and} \qquad \|v^{(k)}\|_{[0,1]} \le n^{k+1}.$$

Let $y \in [y_0, 1]$ be an arbitrary point different from each y_j. By a well-known formula for divided differences,

$$\frac{u(y)}{\prod_{j=0}^{k}(y - y_j)} + \sum_{i=0}^{k} \frac{u(y_i)}{(y_i - y)\prod_{j=0, j\neq i}^{k}(y_i - y_j)} = \frac{1}{(k+1)!}\, u^{(k+1)}(\xi)$$

for some $\xi \in [y_0, 1]$.

Combining our observations so far, we obtain

$$\begin{aligned}
|u(y)| &\le \frac{1}{(k+1)!}\, |u^{(k+1)}(\xi)| \left| \prod_{j=0}^{k} (y - y_j) \right| \\
&\qquad + \sum_{i=0}^{k} |u(y_i)| \left| \frac{\prod_{j=0}^{k}(y - y_j)}{(y_i - y)\prod_{j=0, j\neq i}^{k}(y_i - y_j)} \right| \\
&\le \frac{1}{(k+1)!}\, n^{k+2} \frac{(k+1)!}{(2n)^{k+1}} + 2^{-2\sqrt{n}} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \\
&\le 2^{-(k+1)} n + 2^{-2\sqrt{n}}\, 2^{(1/8)\sqrt{n}} \le \exp\left(-c_4\sqrt{n}\right)
\end{aligned}$$

with an absolute constant $c_4 > 0$. Similarly

$$|v(y)| \le \exp\left(-c_4\sqrt{n}\right) .$$

Hence

$$|f(y)| \le \sqrt{2}\exp\left(-c_4\sqrt{n}\right)$$

with an absolute constant $c_4 > 0$.

Now for $g(x) := x^n f(x)$, we have $g \in \mathcal{F}_{2n}$ and

$$\|g\|_{[0,w]} \le \exp\left(-c_2\sqrt{n}\right)$$

with an absolute constant $c_2 > 0$.

Now the proof can be finished by a trivial modification of the proof given in the special case when subarc A of the unit circle is symmetric with respect to the real line and contains 1. $\square$

Proof Lemma 4.7. We have

$$\begin{aligned}
|b_j| &= \left|\frac{1}{j!}\frac{d^j}{dx^j}(p(x)(x-1)^{-k})\Big|_{x=0}\right| \\
&= \left|\frac{1}{j!}\sum_{m=0}^{j}\binom{j}{m}(-1)^k\frac{(k+m-1)!}{(k-1)!}\,p^{(j-m)}(0)\right| \\
&= \left|\sum_{m=0}^{j}\frac{(k+m-1)!}{(k-1)!m!}\frac{1}{(j-m)!}\,p^{(j-m)}(0)\right| = \left|\sum_{m=0}^{j}\frac{(k+m-1)!}{(k-1)!m!}\,a_{j-m}\right| \\
&= \left|\sum_{m=0}^{j}\binom{k+m-1}{m}a_{j-m}\right| \le 9\binom{k+j}{k} \le 9\left(\frac{e(k+j)}{k}\right)^k \\
&\le 9\left(\frac{en}{k}\right)^k
\end{aligned}$$

which proves the lemma. $\square$

Proof of Theorem 3.4. Without loss of generality we may assume that A is the subarc of the unit circle with length $\ell(A) = a$ which is symmetric with respect to the real line and contains 1. Let $k := \lfloor \frac{2}{9a} \rfloor + 1$. Let $h \in \mathcal{P}_k$ be the polynomial with the properties of Lemma 4.8. Let $u := k^2$. Let

$$Q_u(x) := h^k(x) =: \sum_{j=0}^{u} b_j x^j .$$

Since

$$|h(z)| < \exp\left(\frac{2}{k}\right) , \qquad |z| \le 1 ,$$

$$\sum_{j=0}^{u} |b_j|^2 = \|Q_u\|^2_{L_2(\partial D)} \le \exp\left(\frac{4k}{k}\right) = e^4 .$$

So

$$|b_0| = 1 , \qquad |b_j| \le e^2 \le 9 , \qquad j = 0, 1, \dots , u .$$

Now let A denote the subarc of the unit circle which is symmetric to the real line, contains 1, and has length $a \in \left(0, \frac{1}{9}\right)$. Since $2k/(9u) = 2/(9k) \ge a$, Lemma 4.7 implies that

$$\|Q_u\|_A \le \exp(-c_4 k) \le \exp\left(\frac{-c_5}{a}\right)$$

with some absolute constants $c_4 > 0$ and $c_5 > 0$. Now let $M \ge 1$. Without loss of generality we may assume that $M = 9^m$ with a nonnegative integer m. When $m = 0$ the theorem follows from Theorem 3.3. So let $m \ge 1$. Let $n := um$ and let

$$P_n(x) := Q_u^m(x) := h^{km}(x) =: \sum_{j=0}^{n} a_j x^j .$$

Since

$$|h(z)| < \exp\left(\frac{2}{k}\right) , \qquad |z| \le 1 ,$$

$$\sum_{j=0}^{n} |a_j|^2 = \|P_n\|^2_{L_2(\partial D)} \le \exp\left(\frac{4km}{k}\right) = e^{4m} .$$

So

$$|a_0| = 1 , \qquad |a_j| \le e^{2m} \le M , \qquad j = 0, 1, \dots , n .$$

Also, using $m \ge 1$, we obtain

$$\|P_n\|_A = \|Q_u\|_A^m = \exp\left(\frac{-c_5 m}{a}\right) \le \exp\left(\frac{-c_2(1 + \log M}{a}\right)$$

with an absolute constant $c_2 > 0$. This finishes the proof. □

Proof of Lemma 4.9. The proof is the same as that of Lemma 4.1 with trivial modifications. □

Proof of Lemma 4.10. Applying Cauchy's integral formula with

$$G(z) := g(z_0 + (z - z_0))g(z_0 - (z_0 - z))$$

on J, we obtain

$$\begin{aligned}
|g(z_0)|^2 = |G(z_0)| &= \left| \frac{1}{2\pi i} \int_J \frac{G(z)\, dz}{z - z_0} \right| \\
&= \frac{2}{2\pi} \left| \int_{J_1} \frac{G(z)\, dz}{z - z_0} \right| \le \frac{1}{\pi} \int_{J_1} \frac{|G(z)|\, |dz|}{|z - z_0|} \\
&= \frac{1}{\pi} \int_{J_1} \frac{|g(z_0 + (z - z_0))g(z_0 - (z - z_0))|\, |dz|}{|z - z_0|} \\
&\le (\pi\delta)^{-1} K \int_{J_1} |g(z)|\, |dz| .
\end{aligned}$$

□

Proof of Theorem 3.5. Without loss of generality we may assume that the arc A is the subarc of the unit circle with length $\ell(A) = a < \pi/2$ which is symmetric with respect to the real line and contains 1. Suppose $f \in \mathcal{S}_M^\mu$ and

$$\left| f\left(\tfrac{1}{4M2^\mu} - \tfrac{1-\cos(a/4)}{2M2^\mu}\right)\right| \geq \tfrac{1}{2}\,.$$

Note that this is guaranteed by the assumption of the theorem since

$$0 \leq \frac{1}{4M2^\mu} - \frac{1-\cos(a/4)}{2M2^\mu} \leq \frac{1}{4M2^\mu}\,,$$

Let the region $H_{a,M,\mu}$ be defined by

$$H_{a,M,\mu} := \left\{ z = re^{i\theta} : \ \cos(a/4) < r < 1 - \frac{1-\cos(a/4)}{M2^\mu}, \ -\frac{a}{4} < \theta < \frac{a}{4} \right\}.$$

Let $\Gamma_{a,M,\mu}$ be the circle as in Lemma 4.9. It is a simple geometric argument to show that the arc $I := \Gamma_{a,M,\mu} \cap H_{a,M,\mu}$ has length greater than $c_5 a$ with an absolute constant $c_5 > 0$. Let $z_0 \in I \subset H_{a,M,\mu}$ be fixed. Then we can choose $w_1 \in A$ and $w_2 \in A$ such that $z_0 = \frac{1}{2}(w_1 + w_2)$. Let J_1 be the arc connecting w_1 and w_2 on the unit circle. Note that J_1 is a subarc of A. Let J_2 be the arc which is the symmetric image of J_1 with respect to the line segment connecting w_1 and w_2. Let

$$g(z) := ((z-w_1)(z-w_2))^\mu f(z)\,.$$

Then it is elementary geometry again to show that

$$|g(z)| \leq \frac{M\,|(z-w_1)(z-w_2)|^\mu}{(1-|z|)^\mu} \leq \frac{M2^\mu}{\sin^\mu(a/2)}\,, \qquad z \in J_2\,.$$

By Lemma 4.10 we obtain

$$|g(z_0)|^2 \leq \left(\frac{\pi(1-\cos(a/4))}{M2^\mu}\right)^{-1} \frac{M2^\mu}{\sin^\mu(a/2)} \int_{J_1} |g(z)|\,|dz|\,. \tag{5.2}$$

Observe that $f \in \mathcal{S}_M^\mu$ implies $g \in \mathcal{S}_{M4^\mu}^\mu$. Also, since $M \geq 1$,

$$\begin{aligned}
\left| g\left(\tfrac{1}{4M2^\mu} - \tfrac{1-\cos(a/4)}{2M2^\mu}\right)\right| &\geq \left(1 - 2^{-\mu-2}\right)^{2\mu} \left| f\left(\tfrac{1}{4M2^\mu} - \tfrac{1-\cos(a/4)}{2M2^\mu}\right)\right| \\
&\geq \left(1 - \tfrac{1}{8^\mu}\right)^{2\mu} \left| f\left(\tfrac{1}{4M2^\mu} - \tfrac{1-\cos(a/4)}{2M2^\mu}\right)\right| \\
&\geq \left(\tfrac{7}{8}\right)^2 \tfrac{1}{2} \geq \tfrac{1}{4}\,.
\end{aligned}$$

Hence Lemma 4.9 can be applied with $g \in \mathcal{S}_{M4^\mu}^\mu$. We conclude that there is a point $z_0 \in I \subset H_{a,M,\mu}$ such that

$$|g(z_0)| \geq \exp\left(\frac{-c_4(\mu + \log M)}{a}\right)\,.$$

Combining this with (5.2) and $J_1 \subset A$ gives

$$\begin{aligned}\|f\|_{L_1(A)} &\geq \left(\frac{1}{4}\right)^{\mu} \|g\|_{L_1(A)} \geq \left(\frac{1}{4}\right)^{\mu} \|g\|_{L_1(J_1)} \\ &\geq \left(\frac{1}{4}\right)^{\mu} \frac{\pi(1-\cos(a/4))}{M2^{\mu}} \frac{\sin^{\mu}(a/2)}{M2^{\mu}} |g(z_0)|^2 \\ &\geq \exp\left(\frac{-c_1(\mu+\log M)}{a}\right)\end{aligned}$$

with an absolute constant $c_1 > 0$. □

Proof of Corollary 3.6. Let $f \in \mathcal{K}_M^{\mu}$. Then $f \in \mathcal{S}_{M(\mu!)^2}^{\mu}$ and f is continuous on the closed unit disk. Also, if $|z_0| \leq \frac{1}{4M(\mu!)^2 2^{\mu}}$, then

$$\begin{aligned}|f(z_0| &\geq 1 - M(\mu!)^2 \sum_{j=1}^{\infty} j^{\mu} \left(\frac{1}{4M(\mu!)^2 2^{\mu}}\right)^j \geq 1 - \frac{M(\mu!)^2}{4M(\mu!)^2} \sum_{j=1}^{\infty} \left(\frac{j}{2^j}\right)^{\mu} \\ &\geq 1 - \frac{1}{4} \sum_{j=1}^{\infty} \left(\frac{j}{2^j}\right) \geq 1 - \frac{2}{4} \geq \frac{1}{2}.\end{aligned}$$

So the assumptions of Theorem 3.5 are satisfied with M replaced by $M(\mu!)^2$, and the corollary follows from Theorem 3.5. □

Proof of Corollary 3.7. Let

$$p_k(z) = \sum_{j=0}^{n_k} a_{j,k} z^j, \qquad a_{j,k} \in \mathbb{C}, \quad a_{n_k,k} \neq 0,$$

and let $M_k := H(p_k)$. Applying Corollary 3.6 with

$$q_k(z) := |a_{n_k,k}|^{-1} z^{n_k} p_k(z^{-1}) \in \mathcal{K}_{M_k} (= \mathcal{K}_{M_k}^1)$$

and the arc $B := \{z^{-1} : z \in A\}$ of length a, we obtain the corollary. □

6. Proof of Theorem 3.8

Lemma 6.1. *For every $r \in (0, 1/2)$ there exists a trigonometric polynomial*

$$p(z) = \sum_{j=-n}^{n} c_j z^j$$

such that $c_0 = 1$, $|c_j| < r$ and $|p(z)| < r$ everywhere on the unit circle except possibly in a set of linear measure at most r.

Proof. The finite Riesz product

$$p(z) = \prod_{j=1}^{N} (1 + r z^{m_j} + r z^{-m_j})$$

with sufficiently large N and $m_j = 4^j$ is such an example. For $r \in (0, 1/2)$ and for a sequence of distinct nonnegative integers integers m_j the Riesz products tend to 0 almost everywhere on the unit circle as $N \to \infty$. See, for example, [Zy-59, p 208]. $\square$

The next lemma follows simply from the fact that the transfinite diameter of any closed proper subset of the unit circle is less than 1. (We remark that due to this fact the polynomial guaranteed by Lemma 6.2 can be chosen so that its coefficients are integers. We will not need this extra property.)

Lemma 6.2. *For every $\eta > 0$ there exists a polynomial*

$$g(z) = \sum_{k=0}^{L} b_k z^k$$

such that $b_0 = 1$ and $|g(z)| < \eta$ everywhere on the unit circle except possibly on a set of linear measure at most η.

Proof of Theorem 3.8. For $\eta := \epsilon/2$ we choose a polynomial

$$g(z) = \sum_{k=0}^{L} b_k z^k$$

with the properties of Lemma 6.2, that is, $b_0 = 1$ and

$$|g(z)| < \frac{\epsilon}{2} \quad \text{everywhere on the unit circle except possibly in a set of linear measure at most } \frac{\epsilon}{2}. \tag{6.1}$$

For every k with $|b_k| > 1$ we choose a trigonometric polynomial

$$p_k(z) = \sum_{j=-n_k}^{n_k} c_{j,k} z^j$$

so that

$$c_{0,k} = 1, \qquad |c_{j,k}| < \frac{\epsilon}{2L|b_k|}$$

and

$$|p_k(z)| < \frac{\epsilon}{2L|b_k|} \quad \text{everywhere on the unit circle except possibly in a set of linear measure at most } \frac{\epsilon}{2L}. \tag{6.2}$$

This can be done by Lemma 6.2. Now let

$$h(z) := g\big(z^A\big) = \sum_{k=0}^{L} b_k z^{Ak} \qquad \text{with} \qquad A := 1 + 2 \max_{k:|b_k|>1} n_k\,.$$

Finally we define

$$f(z) := h(z) - \sum_{\substack{k=1 \\ |b_k|>1}}^{L} b_k z^{Ak} p_k(z) .$$

It is straightforward from the construction that $f \in \mathcal{K}_1$. Also, (6.1) and the definition of h imply that

$$|h(z)| < \frac{\epsilon}{2} \quad \text{everywhere on the unit circle except possibly in a set of linear measure at most } \frac{\epsilon}{2} . \tag{6.1}$$

Finally (6.2) and the definition of f imply that

$$|f(z)| < \frac{\epsilon}{2} + \sum_{\substack{k=1 \\ |b_k|>1}}^{L} |b_k| \frac{\epsilon}{2L|b_k|} \le \frac{\epsilon}{2} + L\,\frac{\epsilon}{2L} = \epsilon$$

everywhere on the unit circle except possibly in a set of linear measure at most

$$\frac{\epsilon}{2} + L\,\frac{\epsilon}{2L} = \epsilon .$$

This finishes the proof. □

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