UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

A CONJECTURE RELATED TO SYLVESTER'S PROBLEM

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If a finite set of points spans the plane, then there exists a line through exactly two of the points. Posed as a problem by J. J. Sylvester in 1893, this was not solved until the 1930's when various people, including P. Erdős and T. Gallai, revived interest in it. Motzkin, one of a number of mathematicians to contribute to the now considerable list of refinements and variations on Sylvester's Problem, considered higher dimensional analogues. A substantial contribution to this higher dimensional problem was provided by Hansen in 1965. (See [**5**].) The rich literature on this problem may be traced through the references in [**4**], [**6**] and [**7**].

Of a distinctly similar flavor is another result of Motzkin. Given two finite and disjoint sets of points whose union spans the plane, then there exists a line through at least two points of one of the sets that does not intersect the other. Such a line is called monochrome. Essentially different proofs of this may be found in [**4**] and [**9**]. A beguilingly simple related question asks whether two finite and disjoint sets whose union spans R^4 must generate a monochrome plane (a plane through at least three noncollinear points of one of the sets that does not intersect the other set). This result is false in R^3 and may be easily proved in R^6 using the results in [**5**]. Another higher dimensional variation on Motzkin's theme is offered by Shannon [**9**]: if n finite and disjoint sets span projective n-space P^n, then there exists a line through at least two points of one of the sets that misses all the other sets. We propose the following conjecture that, if true, would provide an umbrella for the preceding results.

CONJECTURE. *If A and B are two finite and disjoint sets whose union spans P^{n+m}, then either there exists an A-monochrome n-flat or there exists a B-monochrome m-flat.*

By an A-monochrome n-flat we mean an affine variety of dimension n spanned by points of A that contains no points of B.

In [3] this conjecture is proved for $n = 1$. This, by induction, implies Shannon's result concerning n sets in n dimensions. There are various other immediate consequences that would follow from the conjecture. For instance, n sets in $2n$ dimensions would guarantee the existence of a monochrome plane.

In [**2**] it is shown that any finite set E that spans $P^{2(n+m)}$ generates an $(n + m)$-flat spanned by and containing exactly $n + m + 1$ points of E. Thus, the conclusion of the above conjecture is true if we replace P^{n+m} by $P^{2(n+m)}$. (As is often the case in this area, finding some dimension in which the problem is tractable is much simpler than determining the "correct" dimension.) If we consider $n + m$ points in general position in P^{n+m-1} and let A be composed of n of these points and let B be composed of the remaining m points, we see that we cannot hope to replace P^{n+m} by P^{n+m-1} without violating the conjecture.

An apparently related question concerns compact sets in R^{n+m} which are also countable. More precisely, if we assume that A and B are countable and compact rather than just finite, does the conjecture still hold? If not, what dimension, if any, suffices to maintain the conclusion?

As far as we know, this question is only resolved in the case of two sets in the plane. However, Tingley [**10**] shows that two compact disjoint sets in R^3 generate a monochrome line. Perhaps by stepping up a dimension the condition that the sets be countable can in general be dropped.

A curious feature of a number of results related to the conjecture is that they seem to be more easily proved in dual formulation. The latter questions concerning compact sets do not dualize as naturally. While the easiest arguments for treating Motzkin and Sylvester-like problems for finite sets are geometrical, the proofs of results for infinite sets are topological in nature. A perhaps surprising feature of many of these problems is that combinatorial methods do not prove particularly useful.

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MISCELLANEA

106. *Metagame* (so baptized by Martin Gardner) is an ingenious new twist on the Russell paradox. (It was called "hypergame" by its discoverer, William S. Zwicker.) Here is how it goes.

A two-person game (such as tic-tac-toe or chess) is *finite* if it always ends in a finite number of moves. The first move of metagame is to pick a finite game. If, for example, you and I are playing metagame and I have the first move, I can say "Let's play chess." Then you make the first move in a chess game, and we continue playing till that game ends.

Is metagame finite? If it is, then, as my first move in a metagame, I can say "Let's play metagame." It is now your turn, and, as your first move in the metagame that I chose to play, you can say "Let's play metagame." The process can obviously go on indefinitely, contrary to the assumption that metagame is finite. The contradiction forces the conclusion that metagame is not finite. It follows that I cannot choose metagame as my first move in metagame; I must choose a finite game. Consequence: the game must end in a finite number of moves, and that contradicts the proved statement that metagame is not finite.

—Paraphrased, by permission, from a
forthcoming book by Raymond Smullyan.