Müntz's Theorem on Compact Subsets of Positive Measure

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Abstract

The principal result of this paper is a Remez-type inequality for Müntz polynomials:

$$p(x) := \sum_{i=-n}^{n} a_i x^{\lambda_i},$$

or equivalently for Dirichlet sums:

$$P(t) := \sum_{i=-n}^{n} a_i e^{-\lambda_i t},$$

where $(\lambda_i)_{i=-\infty}^{\infty}$ is a sequence of distinct real numbers. The most useful form of this inequality states that for every sequence $(\lambda_i)_{i=-\infty}^{\infty}$ satisfying

$$\sum_{\substack{i=-\infty \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty$$

there is a constant c depending only on $(\lambda_i)_{i=-\infty}^{\infty}$, A, α, and β (and not on n or A) so that the inequality

$$\|p\|_{[\alpha,\beta]} \leq c \, \|p\|_A$$

holds for every Müntz polynomial p, as above, associated with $(\lambda_i)_{i=-\infty}^{\infty}$, for every set $A \subset [0, \infty)$ of positive Lebessgue measure, and for every

$$[\alpha, \beta] \subset (\text{ess inf } A, \text{ess sup } A).$$

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Here $\|\cdot\|_A$ denotes the supremum norm on A.

This Remez-type inequality allows us to resolve several problems. Most notably we show that the Müntz-type theorems of Clarkson, Erdős, and Schwartz on the denseness of

$$\mathrm{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}, \qquad \lambda_i \in \mathbb{R} \text{ distinct}$$

on $[a, b]$, $a > 0$, remain valid with $[a, b]$ replaced by an arbitrary compact set $A \subset (0, \infty)$ of positive Lebesgue measure. This extends earlier results of the authors under the assumption that the numbers λ_i are nonnegative.

1 Introduction

Müntz's classical theorem characterizes sequences $\Lambda := (\lambda_i)_{i=0}^{\infty}$ with

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

for which the Müntz space

$$\mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $C[0, 1]$. Here, and in what follows, $\mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ denotes the collection of finite linear combinations of the functions $x^{\lambda_0}, x^{\lambda_1}, \dots$ with real coefficients, and $C(A)$ is the space of all real-valued continuous functions on $A \subset [0, \infty)$ equipped with the uniform norm. Müntz's Theorem [11, 18, 27, 30] states the following.

Theorem 1.1 *Suppose $(\lambda_i)_{i=0}^{\infty}$ is an increasing sequence of nonnegative real numbers with $\lambda_0 = 0$. The Müntz space $\mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $C[0, 1]$ if and only if $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$.*

The original Müntz Theorem proved by Müntz [18] in 1914, by Szász [27] in 1916, and anticipated by Bernstein [3] was only for sequences of exponents tending to infinity. The point 0 is special in the study of Müntz spaces. Even replacing $[0, 1]$ by an interval $[a, b] \subset (0, \infty)$ in Müntz's Theorem is a non-trivial issue. This is, in large measure, due to Clarkson and Erdős [12] and Schwartz [24] whose works include the result that if $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, then every function belonging to the uniform closure of $\mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ on $[a, b]$ can be extended analytically throughout the region

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < b\}.$$

There are many generalizations and variations of Müntz's Theorem [1, 4, 5, 6, 7, 8, 9, 16, 17, 19, 24, 26, 28, 29]. There are also still many open problems. For example, the proper generalizations to many variables are still open.

Schwartz [24] extended the results of Clarkson and Erdős to sequences $(\lambda_i)_{i=-\infty}^{\infty}$ of arbitrary real numbers. His main results in this direction are formulated by the next two theorems.

Theorem 1.2 *Suppose $(\lambda_i)_{i=-\infty}^{\infty}$ is a sequence of distinct real numbers. Suppose $0 < a < b$, and $q \in (0, \infty)$. Then*

$$\operatorname{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$$

is dense in $L^q[a,b]$ if and only if

$$\sum_{\substack{i=-\infty \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} = \infty.$$

The same conclusion is valid with $L^q[a,b]$ replaced by $C[a,b]$.

Theorem 1.3 *Suppose $(\lambda_i)_{i=-\infty}^{\infty}$ is a sequence of distinct real numbers satisfying*

$$\sum_{\substack{i=-\infty \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty$$

with $\lambda_i < 0$ for $i < 0$ and $\lambda_i \geq 0$ for $i \geq 0$. Suppose $0 < a < b$, and $q \in (0, \infty)$. Then $\operatorname{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$ is not dense in $L^q[a,b]$.

Suppose the gap condition

$$\inf\{\lambda_i - \lambda_{i-1} : i \in \mathbb{Z}\} > 0$$

holds. Then every function $f \in L^q[a,b]$ belonging to the $L^q[a,b]$ closure of

$$\operatorname{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$$

can be represented as

$$f(x) = \sum_{i=-\infty}^{\infty} a_i x^{\lambda_i}, \qquad x \in (a,b).$$

If the above gap condition does not hold, then every function $f \in L^q[a,b]$ belonging to the $L^q[a,b]$ closure of $\operatorname{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$ can still be represented as an analytic function on

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : a < |z| < b\}.$$

The same conclusion is valid with $L^q[a,b]$ replaced by $C[a,b]$.

In [8] the authors extended Theorem 1.1 and other related results by replacing $[0,1]$ by an arbitrary compact set $A \subset [0, \infty)$ of positive Lebesgue measure. The main results of this paper, Theorems 3.6 and 3.7, extend Theorems 1.2 and 1.3 to arbitrary compact sets $A \subset (0, \infty)$ of positive Lebesgue measure. Moreover, Theorems 3.6 and 3.7 extend to weighted $L_w^q(A)$ spaces, where w is a nonnegative integrable weight function on A with $\int_A w > 0$.

Theorems 3.6 and 3.7 can be proved fairly simply, once one has established the bounded Remez-type inequality of Theorem 3.1 for non-dense Müntz spaces

$$\mathrm{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}\,.$$

This is the central result of the paper, and is a result we believe should be a basic tool for dealing with problems about Müntz spaces.

Let $\mathcal{P}_n$ denote the set of all algebraic polynomials of degree at most n with real coefficients. For a fixed $s \in (0,1)$ let

$$\mathcal{P}_n(s) := \{p \in \mathcal{P}_n : m(\{x \in [0,1] : |p(x)| \le 1\}) \ge s\}$$

where $m(\cdot)$ denotes linear Lebesgue measure. The classical Remez inequality concerns the problem of bounding the uniform norm of a polynomial $p \in \mathcal{P}_n$ on $[0,1]$ given that its modulus is bounded by 1 on a subset of $[0,1]$ of Lebesgue measure at least s. That is, how large can $\|p\|_{[0,1]}$ (the uniform norm of p on $[0,1]$) be if $p \in \mathcal{P}_n(s)$? The answer is given in terms of the Chebyshev polynomials. The extremal polynomials for the above problem are the Chebyshev polynomials $\pm T_n(x) := \pm\cos(n \arccos h(x))$, where h is a linear function that scales $[0,s]$ or $[1-s,1]$ onto $[-1,1]$.

For various proofs, extensions, and applications, see [13, 14, 15, 22, 23].

Our bounded Remez-type inequality of Theorem 3.1 states the following. If $(\lambda_i)_{i=-\infty}^{\infty}$ is a sequence of distinct real numbers satisfying

$$\sum_{\substack{i=-\infty \\ \lambda_i \ne 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty,$$

then there is a constant c depending only on $(\lambda_i)_{i=-\infty}^{\infty}$, A, α, and β (and not on the number of terms in p) so that

$$\|p\|_{[\alpha,\beta]} \le c\,\|p\|_A$$

for every Müntz polynomial $p \in \mathrm{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$, for every set $A \subset [0,\infty)$ of positive Lebesgue measure, and for every $[\alpha,\beta] \subset (\operatorname{ess\,inf} A, \operatorname{ess\,sup} A)$.

This extends the Remez-type inequality of the authors [8], where the exponents λ_i are nonnegative. One might note that the existence of such a bounded Remez-type inequality for a Müntz space $\mathrm{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$ is equivalent to the non-denseness of $\mathrm{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$ in $C[a,b]$, $0 < a < b$.

The key to the proof of Theorem 3.1 is Theorem 3.2. This theorem states that for the "positive and negative parts" p^+ and p^- of a $p \in \mathrm{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$, the inequalities

$$\|p^+\|_A \le c\,\|p\|_A$$

and

$$\|p^-\|_A \le c\,\|p\|_A$$

hold with a constant c depending only on $(\lambda_i)_{i=-\infty}^{\infty}$ and A (but not on the number of terms in p).

Yet another remarkable consequence of the bounded Remez-type inequality of Theorem 3.1 is that the pointwise and locally uniform convergence of a sequence $(p_n)_{n=1}^{\infty} \subset \mathrm{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$ on $(0,1)$ are equivalent whenever

$$\sum_{\substack{i=-\infty \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty.$$

See Theorem 3.5. In fact, one can characterize the non-dense Müntz spaces within the Müntz spaces $\mathrm{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$ as exactly those in which locally uniform and pointwise convergence on $(0,1)$ are equivalent.

2 Notation

The notations

$$\begin{aligned} \|p\|_A &:= \sup_{x \in A} |p(x)|, \\ \|p\|_{L_w^q(A)} &:= \left(\int_A |p(x)|^q w(x)\, dx\right)^{1/q}, \end{aligned}$$

and

$$\|p\|_{L^q(A)} := \left(\int_A |p(x)|^q\, dx\right)^{1/q}$$

are used throughout this paper for measurable functions p defined on a measurable set $A \subset [0,\infty)$, for nonnegative measurable weight functions w defined on A, and for $q \in (0,\infty)$. The space of all real-valued continuous functions on a set $A \subset [0,\infty)$ equipped with the uniform norm is denoted by $C(A)$. If $A := [a,b]$ is a finite closed inerval, then the notation $C[a,b] := C([a,b])$ will be used.

The space $L_w^q(A)$ is defined as the collection of equivalence classes of real-valued measurable functions for which $\|f\|_{L_w^q(A))} < \infty$. The equivalence classes are defined by the equivalence relation $f \sim g$ if $fw = gw$ almost everywhere on A. When $A := [a,b]$ is a finite closed interval, we use the notation $L_w^q[a,b] := L_w^q(A)$. When $w := 1$, we use the notation $L^q[a,b] := L_w^q[a,b]$. Again, it is always our understanding that the space $L_w^q(A)$ is equipped with the $L_w^q(A)$ norm.

The nonnegative-valued functions x^{λ_i} are well-defined on $[0,\infty)$. For a fixed sequence $(\lambda_i)_{i=0}^{\infty}$, the collection of Müntz polynomials

$$p(x) = \sum_{i=0}^{n} a_i x^{\lambda_i}, \qquad a_i \in \mathbb{R}, \;\; n \in \mathbb{N}$$

is denoted by

$$\mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}.$$

Similarly, for a fixed sequence $(\lambda_i)_{i=-\infty}^{\infty}$, the collection of Müntz polynomials

$$p(x) = \sum_{i=-n}^{n} a_i x^{\lambda_i}, \qquad a_i \in \mathbb{R}, \;\; n \in \mathbb{N}$$

is denoted by

$$\mathrm{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}.$$

The above spaces are called Müntz spaces.

For a measurable set $A \subset \mathbb{R}$, we use the notation

$$\text{ess inf } A := \sup\{x \in \mathbb{R} : m((-\infty, x] \cap A) = 0\}$$

and

$$\text{ess sup } A := \sup\{x \in \mathbb{R} : m([x, \infty) \cap A) = 0\}$$

where $m(\cdot)$ denotes the one-dimensional Lebesgue measure.

3 New Results

The central result of this paper is the following theorem.

Theorem 3.1 *Suppose $(\lambda_i)_{i=-\infty}^{\infty}$ is a sequence of distinct real numbers satisfying*

$$\sum_{\substack{i=-\infty \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty \,.$$

Then there is a constant c depending only on $(\lambda_i)_{i=-\infty}^{\infty}$, A, α, and β (and not on the number of terms in p) so that

$$\|p\|_{[\alpha,\beta]} \le c\,\|p\|_A$$

for every Müntz polynomial $p \in \mathrm{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$, for every set $A \subset (0,\infty)$ of positive Lebesgue measure, and for every $[\alpha, \beta] \subset (\text{ess inf } A, \text{ess sup } A)$.

Theorem 3.2 *Suppose $(\lambda_i)_{i=-\infty}^{\infty}$ is a sequence of distinct real numbers satisfying*

$$\sum_{\substack{i=-\infty \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty$$

with $\lambda_i < 0$ for $i < 0$ and $\lambda_i \ge 0$ for $i \ge 0$. Associated with

$$p(x) := \sum_{i=-n}^{n} a_i x^{\lambda_i}, \qquad n = 0, 1, \ldots$$

let

$$p^-(x) := \sum_{i=-n}^{-1} a_i x^{\lambda_i} \quad \text{and} \quad p^+(x) := \sum_{i=0}^{n} a_i x^{\lambda_i}.$$

Let $A \subset (0,\infty)$ be a set of positive Lebesgue measure. Then there exists a constant c depending only on $(\lambda_i)_{i=-\infty}^{\infty}$ and A (and not on the number of terms in p) so that

$$\|p^+\|_A \le c\,\|p\|_A$$

and

$$\|p^-\|_A \le c\,\|p\|_A$$

for every $p \in \operatorname{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$.

Theorem 3.3 *Suppose* $(\lambda_i)_{i=-\infty}^{\infty}$ *is a sequence of distinct real numbers satisfying*

$$\sum_{\substack{i=-\infty \\ \lambda_i \ne 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty$$

with $\lambda_i < 0$ *for* $i < 0$ *and* $\lambda_i \ge 0$ *for* $i \ge 0$. *Suppose* $A \subset (0,\infty)$ *is a compact set of positive Lebesgue measure. Let* $a := \operatorname{ess\,inf} A$ *and* $b := \operatorname{ess\,sup} A$. *Let* $f \in C(A)$, *and suppose there exist* $p_n \in \operatorname{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$ *of the form*

$$p_n(x) = \sum_{i=-k_n}^{k_n} a_{i,n} x^{\lambda_i}, \qquad n = 1, 2, \ldots$$

so that $\lim_{n\to\infty} \|p_n - f\|_A = 0$.

Suppose the gap condition

$$\inf\{\lambda_i - \lambda_{i-1} : i \in \mathbb{Z}\} > 0$$

holds. Then f *is of the form*

$$f(x) = \sum_{i=-\infty}^{\infty} a_i x^{\lambda_i}, \qquad x \in (a,b),$$

where

$$f^+(x) := \textstyle\sum_{i=0}^{\infty} a_i x^{\lambda_i}, \qquad x \in [0,b),$$
$$f^-(x) := \textstyle\sum_{i=-\infty}^{-1} a_i x^{\lambda_i}, \qquad x \in (a,\infty), \qquad \lim_{x\to\infty} f^-(x) = 0.$$

Furthermore, f *can be extended analytically throughout the region*

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : a < |z| < b\},$$

and

$$\lim_{n\to\infty} a_{i,n} = a_i, \qquad i \in \mathbb{Z}.$$

If the above gap condition does not hold then f *can still be extended analytically throughout the region*

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : a < |z| < b\}.$$

Theorem 3.4 *Suppose* $(\lambda_i)_{i=-\infty}^{\infty}$ *is a sequence of distinct real numbers. Suppose* $A \subset (0,\infty)$ *is a compact set of positive Lebesgue measure. Then*

$$\operatorname{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$$

is dense in $C(A)$ *if and only if*

$$\sum_{\substack{i=-\infty \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty.$$

Theorem 3.5 *Suppose* $(\lambda_i)_{i=-\infty}^{\infty}$ *is a sequence of distinct real numbers satisfying*

$$\sum_{\substack{i=-\infty \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty.$$

Let $A \subset [0, \infty)$ *be a set of positive Lebesgue measure, and let* $a := \text{ess inf } A$ *and* $b := \text{ess sup } A$. *Assume* $(p_n)_{n=1}^{\infty} \subset \text{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$ *and*

$$p_n(x) \to f(x), \qquad x \in A.$$

Then $(p_n)_{n=1}^{\infty}$ *converges uniformly on every closed subinterval of* (a, b).

Theorem 3.6 *Suppose* $(\lambda_i)_{i=-\infty}^{\infty}$ *is a sequence of distinct real numbers satisfying*

$$\sum_{\substack{i=-\infty \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty$$

with $\lambda_i < 0$ *for* $i < 0$ *and* $\lambda_i \geq 0$ *for* $i \geq 0$. *Suppose* $A \subset [0, \infty)$ *is a set of positive Lebesgue measure with* $\inf A > 0$, w *is a nonnegative-valued, integrable weight function on* A *with* $\int_A w > 0$, *and* $q \in (0, \infty)$. *Then*

$$\text{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$$

is not dense in $L_w^q(A)$.

Suppose the gap condition

$$\inf\{\lambda_i - \lambda_{i-1} : i \in \mathbb{Z}\} > 0$$

holds. Then every function $f \in L_w^q(A)$ *belonging to the* $L_w^q(A)$ *closure of*

$$\text{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$$

can be represented as

$$f(x) = \sum_{i=-\infty}^{\infty} a_i x^{\lambda_i}, \qquad x \in A \cap (a_w, b_w),$$

where

$$a_w := \inf\left\{y \in [0, \infty) : \int_{A \cap (0,y)} w(x)\, dx > 0\right\}$$

and

$$b_w := \sup\left\{y \in [0,\infty) : \int_{A\cap(y,\infty)} w(x)\,dx > 0\right\}.$$

If the above gap condition does not hold, then every function $f \in L^q_w(A)$ *belonging to the* $L^q_w(A)$ *closure of*

$$\operatorname{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$$

can still be represented as an analytic function on

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : a_w < |z| < b_w\}$$

restricted to A.

Theorem 3.7 *Suppose* $(\lambda_i)_{i=-\infty}^{\infty}$ *is a sequence of distinct real numbers. Suppose* $A \subset (0,\infty)$ *is a bounded set of positive Lebesgue measure,* $\inf A > 0$, w *is a nonnegative-valued integrable weight function on* A *with* $\int_A w > 0$, *and* $q \in (0,\infty)$. *Then*

$$\operatorname{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$$

is dense in $L^q_w(A)$ *if and only if*

$$\sum_{\substack{i=-\infty \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty.$$

4 Tools

In this section we collect various previously known results concerning Müntz spaces with exponents of the same sign. In Section 5 the proof of the new results from Section 3, which deal with Müntz spaces with arbitrary exponents, will be reduced to the results of this section. Our most important tool is the following Remez-type inequality established in [8].

Theorem 4.1 *Let* $(\lambda_i)_{i=0}^{\infty}$ *be a sequence of distinct nonnegative exponents satisfying*

$$\sum_{\substack{i=0 \\ \lambda_i \neq 0}}^{\infty} \frac{1}{\lambda_i} < \infty.$$

Then there exists a constant c *depending only on* $(\lambda_i)_{i=-\infty}^{\infty}$, s, *and* $\sup A$ *(and not on* A *or the number of terms in* p*) so that*

$$\|p\|_{[0,\inf A]} \le c\,\|p\|_A$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ *and for every compact set* $A \subset (0,\infty)$ *of Lebesgue measure at least* $s > 0$.

By the substitution $y = x^{-1}$ Theorem 4.1 implies the following.

Theorem 4.2 *Let $(\lambda_i)_{i=0}^{\infty}$ be a sequence of distinct nonpositive exponents satisfying*

$$\sum_{i=0 \, \lambda_i \neq 0}^{\infty} \frac{1}{|\lambda_i|} < \infty .$$

Then there exists a constant c depending only on $(\lambda_i)_{i=-\infty}^{\infty}$, s, and $\inf A$ (and not on A or the number of terms in p) so that

$$\|p\|_{[\sup A, \infty)} \leq c \, \|p\|_A$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and for every compact set $A \subset (0, \infty)$ of Lebesgue measure at least $s > 0$.

The following Bernstein-type inequality for non-dense Müntz spaces is also established in [8].

Theorem 4.3 *Let $(\lambda_i)_{i=0}^{\infty}$ be a sequence of distinct nonnegative exponents satisfying $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Suppose $\lambda_0 = 0$ and $\lambda_1 \geq 1$. Then for every $\varepsilon \in (0,1)$, there is a constant c_ε depending only on ε and $(\lambda_i)_{i=-\infty}^{\infty}$ (but not on the number of terms in p) so that*

$$\|p'\|_{[0,1-\varepsilon]} \leq c_\varepsilon \|p\|_{[0,1]}$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$.

Theorem 4.4 *Let $\Lambda := (\lambda_i)_{i=0}^{\infty}$ be a sequence of distinct nonnegative exponents satisfying*

$$\sum_{\substack{i=0 \\ \lambda_i \neq 0}}^{\infty} \frac{1}{\lambda_i} < \infty .$$

Let $0 \leq a < b$. Suppose

$$(p_n)_{n=1}^{\infty} \subset \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

and $\|p_n\|_{[a,b]} \leq 1$ for each n. Then there is a subsequence of $(p_n)_{n=1}^{\infty}$ that converges uniformly on every closed subinterval of $[0, b)$.

Proof: Note that the assumptions of the Arzela-Ascoli Theorem are satisfied by Theorems 4.1 and 4.3.

By the substitution $y = x^{-1}$ Theorem 4.4 implies the following.

Theorem 4.5 *Let $(\lambda_i)_{i=0}^{\infty}$ be a sequence of distinct nonpositive exponents satisfying*

$$\sum_{\substack{i=0 \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty .$$

Let $0 \leq a < b$. Suppose

$$(p_n)_{n=1}^{\infty} \subset \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

and $\|p_n\|_{[a,b]} \leq 1$ *for each* n. *Then there is a subsequence of* $(p_n)_{n=1}^{\infty}$ *that converges uniformly on every closed subinterval of* (a, ∞).

The following theorem is from Schwartz [24].

Theorem 4.6 *Let* $(\lambda_i)_{i=0}^{\infty}$ *be a sequence of distinct nonnegative exponents satisfying*

$$\sum_{\substack{i=0 \\ \lambda_i \neq 0}}^{\infty} \frac{1}{\lambda_i} < \infty.$$

Let $0 \leq a < b$. *Suppose the sequence*

$$(p_n)_{n=1}^{\infty} \subset \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

converges to a function f *uniformly on* $[a, b]$. *Then* f *can be extended analytically throughout the region*

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < b\}.$$

By the substitution $y = x^{-1}$ Theorem 4.6 implies the following.

Theorem 4.7 *Let* $(\lambda_i)_{i=0}^{\infty}$ *be a sequence of distinct nonpositive exponents satisfying*

$$\sum_{\substack{i=0 \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty.$$

Let $0 \leq a < b$. *Suppose the sequence*

$$(p_n)_{n=1}^{\infty} \subset \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

converges to a function f *uniformly on* $[a, b]$. *Then* f *can be extended analytically throughout the region*

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : a < |z|\}.$$

The following two results are also from Schwartz [24].

Theorem 4.8 *Let* $(\lambda_i)_{i=0}^{\infty}$ *be a sequence of distinct nonnegative exponents satisfying*

$$\sum_{\substack{i=0 \\ \lambda_i \neq 0}}^{\infty} \frac{1}{\lambda_i} < \infty.$$

Let $(\gamma_i)_{i=0}^{\infty}$ *be a sequence of distinct negative exponents satisfying*

$$\sum_{\substack{i=0 \\ \gamma_i \neq 0}}^{\infty} \frac{1}{|\gamma_i|} < \infty.$$

Let $0 \le a < b$. Suppose $f \in C[a,b]$ is a function so that both of the sequences

$$(p_n)_{n=1}^{\infty} \subset \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

and

$$(q_n)_{n=1}^{\infty} \subset \operatorname{span}\{x^{\gamma_0}, x^{\gamma_1}, \dots\}$$

converge to f uniformly on $[a,b]$. Then $f = 0$ on $[a,b]$.

Theorem 4.9 *Suppose $(\lambda_i)_{i=-\infty}^{\infty}$ is a set of distinct real numbers satisfying*

$$\sum_{\substack{i=-\infty \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty$$

with $\lambda_i < 0$ for $i < 0$ and $\lambda_i \ge 0$ for $i \ge 0$. Suppose $0 < a < b$. Let $f \in C[0,1]$, and suppose there exist $p_n \in \operatorname{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$ of the form

$$p_n(x) = \sum_{i=-k_n}^{k_n} a_{i,n} x^{\lambda_i}, \qquad n = 1, 2, \dots$$

so that $\lim_{n\to\infty} \|p_n - f\|_{[a,b]} = 0$.

Suppose the gap condition

$$\inf\{\lambda_i - \lambda_{i-1} : i \in \mathbb{Z}\} > 0$$

holds. Then f is of the form

$$f(x) = \sum_{i=-\infty}^{\infty} a_i x^{\lambda_i}, \qquad x \in (a,b),$$

where

$$f^+(x) := \sum_{i=1}^{\infty} a_i x^{\lambda_i}, \qquad x \in [0,b),$$
$$f^-(x) := \sum_{i=-\infty}^{-1} a_i x^{\lambda_i}, \qquad x \in (a,\infty), \qquad \lim_{x\to\infty} f^-(x) = 0,$$

f can be extended analytically throughout the region

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : a < |z| < b\},$$

and

$$\lim_{n\to\infty} a_{i,n} = a_i, \qquad i \in \mathbb{Z}.$$

If the above gap condition does not hold then f can still be extended analytically throughout the region

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : a < |z| < b\}.$$

5 Proofs

Proof of Theorem 3.2 It is sufficient to prove only the first inequality, the second inequality follows from the first one by the substitution $y = x^{-1}$. If the first inequality fails to hold then there exists a sequence $(p_n)_{n=1}^{\infty} \subset \text{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$ so that

$$\|p_n^+\|_A = 1, \quad n = 1, 2, \ldots, \quad \text{and} \quad \lim_{n\to\infty} \|p_n\|_A = 0.$$

Since $p = p^+ + p^-$, the above relations imply that

$$\|p_n^-\|_A \le K < \infty, \qquad n = 1, 2, \ldots .$$

For the sake of brevity, let $a := \text{ess inf } A$ and $b := \text{ess sup } A$. By Theorems 4.1, 4.2, 4.4, and 4.5, there exists a subsequence $(p_{n_i}^+)_{i=1}^{\infty}$ that converges uniformly to a function f on every closed subinterval of $[0, b)$, while $(p_{n_i}^-)_{i=1}^{\infty}$ converges uniformly to a function g on every closed subinterval of (a, ∞). Now $\lim_{i\to\infty} \|p_{n_i}\|_A = 0$ and $p_{n_i} = p_{n_i}^+ + p_{n_i}^-$ imply that $f + g = 0$ on $A \cap (a, b)$. By Theorem 4.6, f is analytic on $(0, b)$. By Theorem 4.7, g is analytic on (a, ∞). So $f+g$ is analytic on (a, b). Since $f+g = 0$ on $A\cap(a, b)$, and since $m(A\cap(a, b)) > 0$, we conclude by the Unicity Theorem that $f + g = 0$ on (a, b). Now Theorem 4.8 implies that $f = g = 0$ on (a, b).

Hence, for every $y \in (a, b)$,

$$\lim_{i\to\infty} \|p_{n_i}^+\|_{[\inf A,\, y]} = 0$$

and

$$\lim_{i\to\infty} \|p_{n_i}^+\|_{[y,\, \sup A]} = \lim_{i\to\infty} \|p_{n_i} - p_{n_i}^-\|_{A\cap[y,\, \sup A]} = 0.$$

Therefore

$$\lim_{i\to\infty} \|p_{n_i}^+\|_A = 0$$

which contradicts the fact that $\|p_n^+\|_A = 1, \quad n = 1, 2, \ldots$.

Proof of Theorem 3.1 The result is a straightforward consequence of Theorems 4.1, 4.2, and 3.2.

Proof of Theorem 3.3 The result is a straightforward consequence of Theorems 3.1 and 4.9.

Proof of Theorem 3.4 Suppose

$$\sum_{\substack{i=-\infty \\ \lambda_i \ne 0}}^{\infty} \frac{1}{|\lambda_i|} = \infty.$$

Let $f \in C(A)$. By Tietze's Theorem there exists an $\tilde{f} \in C[\inf A, \sup A]$ so that $\tilde{f}(x) = f(x)$ for every $x \in A$. By Müntz's Theorem there is a sequence

$$(p_n)_{n=1}^{\infty} \subset \text{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$$

so that

$$\lim_{n\to\infty} \|\tilde{f} - p_n\|_{[0,1]} = 0.$$

Therefore

$$\lim_{n\to\infty} \|f - p_n\|_A = 0,$$

which finishes the trivial part of the theorem.

Suppose now that

$$\sum_{\substack{i=-\infty \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty.$$

Then Theorem 3.3 yields that

$$\operatorname{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$$

is not dense in $C(A)$.

Proof of Theorem 3.5 Let $[\alpha, \beta] \subset (a, b)$. Egoroff's Theorem and the definition of a and b imply the existence of sets $B_1 \subset A \cap (0, \alpha)$ and $B_2 \subset A \cap (\beta, \infty)$ of positive Lebesgue measure so that $(p_i)_{i=1}^{\infty}$ converges uniformly on $B := B_1 \cap B_2$, hence it is uniformly Cauchy on B. Now Theorem 3.1 yields that $(p_i)_{i=1}^{\infty}$ is uniformly Cauchy on $[\alpha, \beta]$, which proves the theorem.

Proof of Theorem 3.6 Suppose $f \in L_w^q(A)$ and suppose there is a sequence

$$(p_n)_{n=1}^{\infty} \subset \operatorname{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$$

so that

$$\lim_{n\to\infty} \|f - p_n\|_{L_w^q(A)} = 0.$$

Minkowski's Inequality (if $q \in (0, 1)$, then a multiplicative factor $2^{1/q-1}$ is needed) yields that $(p_i)_{i=1}^{\infty}$ is a Cauchy sequence in $L_w^q(A)$. The assumptions on w imply that for every $(\alpha, \beta) \subset [a, b]$ there exists a $\delta > 0$ so that the sets

$$B_1 := \{x \in A \cap (\beta, \infty) : w(x) > \delta\}$$

and

$$B_2 := \{x \in A \cap (0, \alpha) : w(x) > \delta\}$$

are of positive Lebesgue measure. Note that

$$\|p\|_{L^q(B_i)} \le \delta^{-1} \|p\|_{L_w^q(B_i)} \le \delta^{-1} \|p\|_{L_w^q(A)}, \qquad i = 1, 2,$$

for every $p \in L_w^q(A)$. Therefore, $(p_n)_{n=1}^{\infty}$ is a Cauchy sequence in $L^q(B)$, where $B := B_1 \cap B_2$. So, by Theorem 3.1, $(p_n)_{n=1}^{\infty}$ is uniformly Cauchy on $[\alpha, \beta]$. The theorem now follows from Theorem 3.3.

Proof of Theorem 3.7 Suppose

$$\sum_{\substack{i=-\infty \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} = \infty.$$

Let $f \in L^q_w(A)$. It is standard measure theory to show that for every $\varepsilon > 0$ there exists a $g \in C[\inf A, \sup A]$ so that

$$\|f - g\|_{L^q_w(A)} < \frac{varepsilon}{2}.$$

Now Müntz's Theorem implies that there exists a $p \in \operatorname{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$ so that

$$\|g - p\|_{L^q_w(A)} \le \|g - p\|_A \left(\int_A w\right)^{1/q} < \frac{\varepsilon}{2}.$$

Therefore $\operatorname{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$ is dense in $L^q_w(A)$.

Suppose now that

$$\sum_{\substack{i=-\infty \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty.$$

Then Theorem 3.6 yields that $\operatorname{span}\{x^{\lambda_i} : i \in \mathbb{Z}\}$ is not dense in $L^q_w(A)$.

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