MATRIX TRANSFORMATIONS OF SERIES OF ORTHOGONAL POLYNOMIALS*

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Abstract. For a sequence of polynomials (P_n) orthonormal on the interval $[-1,1]$, we consider the sequence of transforms (g_n) of the series $\sum_{k=0}^{\infty} a_k P_k(u)$ given by $g_n(u) := \sum_{k=0}^{\infty} b_{nk} a_k P_k(u)$. We establish necessary and sufficient conditions on the matrix (b_{nk}) for the sequence (g_n) to converge uniformly on compact subsets of the interior of an appropriate ellipse to a function holomorphic on that interior.

1. Introduction. Suppose throughout that $1 < P \le \infty$, $1 < R < \infty$, and that all sequences and matrices are complex with indices running through $0, 1, 2, \dots$. We make the following definitions:

$\mathbb{C}$ is the finite complex plane;

γ_R is the ellipse with foci ± 1 and half-axes $a := \frac{1}{2}(R+R^{-1})$, $b := \frac{1}{2}(R-R^{-1})$. Note that an ellipse with foci ± 1 having R as the sum of its two half-axes is necessarily γ_R;

D_R^γ is the interior of the ellipse γ_R, and $D_\infty^\gamma := \mathbb{C}$;

(P_n) is an orthonormal sequence of polynomials with respect to a fixed non-negative weight function w on the interval $[-1,1]$. That is, P_n is a polynomial of degree n, and

$$\int_{-1}^{1} P_n(u) P_m(u) w(u)\,du = \delta_{nm}\,.$$

We assume throughout that

$$w \in L(-1,1) \text{ and } w^{-\epsilon} \in L(-1,1) \text{ for some } \epsilon > 0.$$

The first of these integrability conditions is standard, and the second is imposed for the purposes of the present paper. The classical Jacobi polynomials, for which $w(u) = (u-1)^\alpha (u+1)^\beta$ with $\alpha, \beta > -1$, satisfy the conditions.

$\mathcal{E}$ is the set of all sequences $\mathbf{a} \equiv (a_n)$ such that $\lim |a_n|^{\frac{1}{n+1}} = 0$;

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$\mathcal{E}^\beta$ is the set of all sequences $\mathbf{a} \equiv (a_n)$ such that $\limsup |a_n|^{\frac{1}{n+1}} < \infty$;

$\mathcal{E}_R$ is the set of all sequences $\mathbf{a} \equiv (a_n)$ such that $\sum_{n=0}^{\infty} |a_n| R^n < \infty$;

$\mathbf{A}_R$ is the set of all sequences $\mathbf{a} \equiv (a_n)$ such that $\limsup |a_n|^{\frac{1}{n+1}} = \dfrac{1}{R}$;

The following lemma, the proof of which appears in [1], shows that $\mathcal{E}^\beta$ is the β-dual of $\mathcal{E}$.

Lemma 1. *A sequence* $\mathbf{b}$ *has the property that* $\sum_{n=0}^{\infty} b_n a_n$ *is convergent for each* $\mathbf{a} \in \mathcal{E}$ *if and only if* $\mathbf{b} \in \mathcal{E}^\beta$.

The following are the first three of eight theorems we shall prove concerning matrix transformations of series of orthogonal polynomials. They are analogues of Theorems 1, 2 and 3 in [1] concerning matrix transformations of power series.

Theorem 1. *A matrix* $\mathbf{B} \equiv (b_{nk})$ *has the property that whenever the sequence* $\mathbf{a} \equiv (a_n) \in \mathcal{E}_R$ *the sequence of functions* (g_n) *given by*

$$g_n(u) := \sum_{k=0}^{\infty} b_{nk} a_k P_k(u), \ \ n = 0, 1, \dots ,$$

converges uniformly on every compact subset of D_P^γ, *each series* $\sum_{k=0}^{\infty} b_{nk} a_k P_k(u)$ *of orthogonal polynomials being convergent on* D_P^γ, *if and only if*

(i) $\lim_{n\to\infty} b_{nk} =: b_k$ *for* $k = 0, 1, \dots$;

(ii) $M(p) := \sup_{n\geq 0, k\geq 0} |b_{nk}| \left(\dfrac{p}{R}\right)^k < \infty$ *whenever* $1 < p < P$.

And then $\lim_{n\to\infty} g_n(u) = \sum_{k=0}^{\infty} b_k a_k P_k(u)$ *on* D_P^γ.

Theorem 2. *A matrix* $\mathbf{B} \equiv (b_{nk})$ *has the property that whenever the sequence* $\mathbf{a} \equiv (a_n) \in \mathbf{A}_R$ *the sequence of functions* (g_n) *given by*

$$g_n(u) := \sum_{k=0}^{\infty} b_{nk} a_k P_k(u), \ \ n = 0, 1, \dots ,$$

converges uniformly on every compact subset of D_P^γ*, each series* $\sum_{k=0}^{\infty} b_{nk}a_kP_k(u)$ *of orthogonal polynomials being convergent on* D_P^γ*, if and only if*

(i) $\lim_{n\to\infty} b_{nk} =: b_k$ *for* $k = 0, 1, \ldots$;

(ii) $M(p) := \sup_{n\geq 0, k\geq 0} |b_{nk}| \left(\frac{p}{R}\right)^k < \infty$ *whenever* $1 < p < P$.

And then $\lim_{n\to\infty} g_n(u) = \sum_{k=0}^{\infty} b_k a_k P_k(u)$ *on* D_P^γ.

Theorem 3. *A matrix* $\mathbf{B} \equiv (b_{nk})$ *has the property that whenever the sequence* $\mathbf{a} \equiv (a_n) \in \mathcal{E}$ *the sequence of functions* (g_n) *given by*

$$g_n(u) := \sum_{k=0}^{\infty} b_{nk}a_kP_k(u), \ n = 0, 1, \ldots ,$$

converges uniformly on every compact subset of $\mathbb{C}$*, each series* $\sum_{k=0}^{\infty} b_{nk}a_kP_k(u)$ *of orthogonal polynomials being convergent on* $\mathbb{C}$*, if and only if*

(i) $\lim_{n\to\infty} b_{nk} =: b_k$ *for* $k = 0, 1, \ldots$;

(ii) $M := \sup_{n\geq 0, k\geq 0} |b_{nk}|^{\frac{1}{k+1}} < \infty$.

And then $\lim_{n\to\infty} g_n(u) = \sum_{k=0}^{\infty} b_k a_k P_k(u)$ *on* $\mathbb{C}$.

These theorems show that if the series-to-sequence transform given by $\mathbf{B}$ is regular, then it is necessary in each case that $\lim_{n\to\infty} b_{nk} = b_k = 1$ for $k = 0, 1, \ldots$, and this in turn implies that $P \leq R$ in Theorems 1 and 2 (i.e., the sequence (g_n) cannot converge uniformly in the interior of any ellipse γ_P with $P > R$). Regular sequence-to-sequence transforms of power series have been considered by Peyerimhoff [8] and Luh [7] among others. One of the novel features of our approach is that we deal with series-to-sequence transforms rather than sequence-to-sequence transforms.

Let (B_n) be a sequence of non-zero complex numbers. The associated Nörlund series-to-sequence matrix $\mathbf{N}_B$ is the triangular matrix (b_{nk}) with

$$b_{nk} := \begin{cases} \dfrac{B_{n-k}}{B_n} & \text{if} \quad 0 \leq k \leq n, \\ 0 & \quad\quad \text{otherwise.} \end{cases}$$

The following theorem is an immediate consequence of Theorem 1.

Theorem N. *The Nörlund matrix* $\mathbf{N}_B$ *has the property that whenever the sequence* $\mathbf{a} \equiv (a_n) \in \mathcal{E}_R$ *the sequence of functions* (g_n) *given by*

$$g_n(u) := \frac{1}{B_n} \sum_{k=0}^{n} B_{n-k} a_k P_k(u), \ \ n = 0, 1, \dots ,$$

converges uniformly on every compact subset of D_P^γ, *if and only if*

$$\lim_{n\to\infty} \frac{B_{n-1}}{B_n} = b \quad \text{with} \quad |b| = \frac{R}{P}.$$

And then $\lim_{n\to\infty} g_n(u) = \sum_{k=0}^{\infty} b^k a_k P_k(u)$ *on* D_P^γ.

Note. In view of Theorem 2, Theorem N remains true if $\mathcal{E}_R$ is replaced by $\mathbf{A}_R$.

2. Orthogonal polynomials.

In this section we set out some of the properties of orthogonal polynomials required in our proofs. Note that the function $u = \frac{1}{2}(z + z^{-1})$ maps the region $\{z : \ |z| > 1\}$ bijectively onto the region $\{ \ u : \ u \notin [-1, 1] \ \}$, and that each circle $|z| = R$ is mapped onto γ_R . The inverse of this function is $z = u + \sqrt{u^2 - 1}$. Here and elsewhere in the paper the sign of the square root is chosen so that $|u + \sqrt{u^2 - 1}| > 1$ when $u \notin [-1, 1]$. We then have, for $z = u + \sqrt{u^2 - 1}$, that $|z| = R$ when $u \in \gamma_R$, and $|z| < R$ when $u \in D_R^\gamma$. The function $u = \frac{1}{2}(z + z^{-1})$ maps both the top half and the bottom half of the unit circle $\{z : \ |z| = 1\}$ onto $[-1, 1]$.

Lemma 2. *For* $\epsilon > 0$ *let the non-negative weight function* $w \in L(-1, 1)$ *associated with the orthonormal sequence of polynomials* (P_n) *be such that* $w^{-\epsilon} \in L(-1, 1)$, *and let* $|z| \geq 1$ *and* $u = \frac{1}{2}(z + z^{-1})$. *Then*

$$|P_n(u)| \leq K(\epsilon)(1 + n)^{2+2/\epsilon} |z|^n \ \textit{for} \ n = 0, 1, \dots ,$$

where $K(\epsilon)$ *is a positive number independent of* n.

Proof. By Bernstein's inequality (see [5, Theorem 7])

$$|P_n(u)| \leq \max_{-1\leq t\leq 1} |P_n(t)||z|^n,$$

and by a result due to Erdéli [2, Theorem 5]

$$\max_{-1\leq t\leq 1} |P_n(t)| \leq K_1(\epsilon)(1 + n)^{2+2/\epsilon} \int_{-1}^{1} |P_n(t)|w(t)\, dt.$$

Finally, by the Cauchy-Schwarz inequality,

$$\int_{-1}^{1} |P_n(t)|w(t) dt \leq \left(\int_{-1}^{1} P_n(t)^2 w(t)\, dt \right)^{\frac{1}{2}} \left(\int_{-1}^{1} w(t)\, dt \right)^{\frac{1}{2}} = \left(\int_{-1}^{1} w(t)\, dt \right)^{\frac{1}{2}}.$$

Combining the above inequalities we get the required result. □

Lemma 3. (Expansion of a holomorphic function in terms of orthogonal polynomials). *Let the non-negative weight function* $w \in L(-1,1)$ *associated with the orthonormal sequence of polynomials* (P_n) *be such that* $w^{-\epsilon} \in L(-1,1)$ *for some* $\epsilon > 0$. *Let* $f(u)$ *be holomorphic on the closed segment* $[-1,1]$, *and let* γ_R *denote the largest ellipse with foci* ± 1 *on the interior of which* $f(u)$ *is holomorphic. The Fourier series expansion of* $f(u)$ *on* D_R^γ, *the interior of* γ_R, *is given by*

$$f(u) = \sum_{k=0}^{\infty} a_k P_k(u),$$

where

$$a_k = \int_{-1}^{1} f(t) P_k(t) w(t)\, dt.$$

The Fourier series is absolutely convergent on D_R^γ, *and is also uniformly convergent on compact subsets of* D_R^γ. *It is divergent on the exterior of* γ_R. *Further, the sum* R *of the semi-axes of the ellipse of convergence is given by*

$$\frac{1}{R} = \limsup_{k\to\infty} \, |a_k|^{\frac{1}{k}} \; .$$

Proof. All but the statement about absolute convergence follows from Theorems 12.7.3 and 12.7.4 in [11], since the conditions on the weight w are more stringent than those in the said theorems. To prove the absolute convergence part, let

$$\frac{1}{R} := \limsup_{k\to\infty} |a_k|^{\frac{1}{k}},$$

and let $u \in D_R^\gamma$. Then $R > 1$ and $u = \frac{1}{2}(z + z^{-1})$ with $1 \le |z| < R$. Let $|z| < R_0 < R$. Then $|a_k| < {R_0}^{-k}$ for all sufficiently large k. Hence, by Lemma 2,

$$|a_k P_k(u)| = (|a_k||z|^k)|z^{-k} P_k(u)| \le K(\epsilon)(1+k)^{2+2/\epsilon} \left(\frac{|z|}{R_0}\right)^k$$

for all sufficiently large k, and therefore $\sum_{k=0}^{\infty} |a_k P_k(u)|$ is convergent. □

Lemma 4. (Cauchy-type inequalities for Fourier series). *Let the non-negative weight function* $w \in L(-1,1)$ *associated with the orthonormal sequence of polynomials* (P_n) *be such that* $w^{-\epsilon} \in L(-1,1)$ *for some* $\epsilon > 0$. *Assume that the function* $f(u)$ *is holomorphic on* D_R^γ *and continuous on* $\bar{D}_R^\gamma$, *the closure of* D_R^γ. *Let* $\sum_{k=0}^{\infty} a_k P_k(u)$ *be its Fourier series. Then*

$$|a_n| \le \frac{c(R)}{R^n} \cdot \max_{u\in\gamma_R} |f(u)| \; \text{for } n = 0, 1, \dots \, ,$$

where $c(R) := \dfrac{2R}{R-1}\left(\int_{-1}^{1} w(t)\,dt\right)^{\frac{1}{2}}$.

Proof. Suppose first that $n \geq 1$. By Lemma 3 we have

$$a_n = \int_{-1}^{1} f(t)P_n(t)w(t)\,dt = \int_{-1}^{1} \big(f(t) - q_{n-1}(t)\big)P_n(t)w(t)\,dt,$$

where $q_{n-1}(t)$ is any polynomial of degree $n-1$. It follows that

$$|a_n| \leq E_{n-1}(f)\int_{-1}^{1} |P_n(t)|w(t)\,dt \leq E_{n-1}(f)\left(\int_{-1}^{1} w(t)\,dt\right)^{\frac{1}{2}},$$

where, in the notation of Lorentz [5],

$$E_{n-1}(f) := \inf_{q_{n-1}} \max_{-1\leq t\leq 1} |f(t) - q_{n-1}(t)|.$$

Further, it is proved in [5, inequality (6), p. 78] that

$$E_{n-1}(f) \leq \frac{2R}{R-1}\cdot\frac{1}{R^n}\cdot\max_{u\in\gamma_R}|f(u)|.$$

Combining the above inequalities we obtain the desired result for $n \geq 1$. Finally, the case $n = 0$ of the Cauchy-type inequality is easily seen to be true since, for $P_0 := P_0(t)$, we have

$$|P_0|\left(\int_{-1}^{1} w(t)\,dt\right)^{\frac{1}{2}} = 1. \qquad \square$$

3. Proofs of Theorems 1, 2 and 3. In the proofs of Theorems 1, 2 and 3, u and z are related by $u = \frac{1}{2}(z + z^{-1})$, $z = u + \sqrt{u^2-1}$ with $|z| > 1$, the sign of the square root being chosen so that $|u + \sqrt{u^2-1}| > 1$.

Proof of Theorems 1 *and* 2. We prove these two theorems together.

Sufficiency. We assume that

$$\begin{cases} \lim_{n\to\infty} b_{nk} =: b_k \text{ for } k = 0, 1, \dots\ ; \\ M(p) := \sup_{n\geq 0, k\geq 0} |b_{nk}|\left(\dfrac{p}{R}\right)^k < \infty \quad \text{for} \quad 1 < p < P. \end{cases}$$

Let $\mathbf{a} \in \mathbf{A}_R$, or $\mathbf{a} \in \mathcal{E}_R$. For $1 < p < P$ choose r so that $1 < r < R$ and $\dfrac{p}{r} < \dfrac{P}{R}$. Now choose p_1 so that $p < p_1 < P$ and $\dfrac{p}{r} = \dfrac{p_1}{R}$. Suppose $u \in D_p^\gamma$. Then $u = \frac{1}{2}(z + z^{-1})$ with $1 \leq |z| < p$, and therefore, by Lemma 2,

$$\begin{aligned} |b_{nk}a_kP_k(u)| &\leq K(\epsilon)|b_{nk}||a_k|(1+k)^{2+2/\epsilon}p^k = K(\epsilon)|b_{nk}|\left(\frac{p}{r}\right)^k |a_k|(1+k)^{2+2/\epsilon}r^k \\ &= K(\epsilon)|b_{nk}|\left(\frac{p_1}{R}\right)^k |a_k|(1+k)^{2+2/\epsilon}r^k \leq K(\epsilon)M(p_1)|a_k|(1+k)^{2+2/\epsilon}r^k < \infty. \end{aligned}$$

Further, by (i) (of either Theorem 1 or Theorem 2),

$$\lim_{n\to\infty} b_{nk}a_kP_k(u) = b_ka_kP_k(u) \ .$$

Since $\sum_{k=0}^{\infty} |a_k|(1+k)^{2+2/\epsilon}r^k < \infty$, and since p can be chosen arbitrarily close to P in $(1,P)$, it follows, by the Weierstrass M-test, that $g_n(u)$ exists for $n = 0,1,\dots$, and

$$\lim_{n\to\infty} g_n(u) = \lim_{n\to\infty} \sum_{k=0}^{\infty} b_{nk}a_kP_k(u) = \sum_{k=0}^{\infty} b_ka_kP_k(u)$$

on D_P^γ, and that the sequence (g_n) is uniformly convergent on compact subsets of D_P^γ. This completes the proof of the sufficiency of conditions (i) and (ii) both for Theorem 1 and Theorem 2.

Necessity. Let $a_k := \dfrac{1}{R^k(k+1)^2}$. Then $\mathbf{a} \in \mathbf{A}_R$ and $\mathbf{a} \in \mathcal{E}_R$. Under the hypotheses of either Theorem 1 or Theorem 2 the series

$$g_n(u) := \sum_{k=0}^{\infty} b_{nk}a_kP_k(u)$$

is convergent on D_P^γ and the sequence (g_n) is uniformly convergent on compact subsets of D_P^γ . Therefore, by the Weierstrass double-series theorem, (g_n) converges to a holomorphic function on D_P^γ. By Lemma 3, we get, for the above sequence $\mathbf{a}$, that

$$b_{nk}a_k = \int_{-1}^{1} g_n(t)P_k(t)\,dt \quad \text{for} \quad n = 0,1,\dots \ .$$

Since $g_n(t)$ converges uniformly on $[-1,1]$ to $g(t)$ say, we get that

$$\lim_{n\to\infty} b_{nk}a_k = \int_{-1}^{1} g(t)P_k(t)\,dt =: d_k.$$

Hence, for $k = 0,1,\dots$,

$$\lim_{n\to\infty} b_{nk} = b_k,$$

where $b_k = d_kR^k(k+1)^2$. This proves the necessity of condition (i) in both Theorem 1 and Theorem 2.

Suppose now that p and $\tilde{p}$ are fixed with $1 < p < \tilde{p} < P$. Since $\mathbf{a}$ satisfies the hypotheses of both Theorem 1 and Theorem 2, the sequence (g_n) is uniformly convergent on $\bar{D}_{\tilde{p}}^\gamma$. Hence we have, for $u \in \bar{D}_{\tilde{p}}^\gamma$ and $n = 0,1,\dots$, that $|g_n(u)| \le M(\tilde{p},\mathbf{a}) < \infty$, $M(\tilde{p},\mathbf{a})$ being independent of n. By Lemma 4 we get that

$$|b_{nk}a_k\tilde{p}^k| \le c(\tilde{p})M(\tilde{p},\mathbf{a}) \text{ for } n,k = 0,1,\dots \ .$$

Since $a_k := \dfrac{1}{R^k(k+1)^2}$, it follows that

$$|b_{nk}|\left(\frac{\tilde{p}}{R}\right)^k \frac{1}{(k+1)^2} \le c(\tilde{p})M(\tilde{p},\mathbf{a}) \text{ for } n,k=0,1,\dots ,$$

and hence that

$$\sup_{n\ge 0,k\ge 0} |b_{nk}|\left(\frac{p}{R}\right)^k \le c(\tilde{p})M(\tilde{p},\mathbf{a}) \sup_{k\ge 0}\left\{\left(\frac{p}{\tilde{p}}\right)^k (k+1)^2\right\} < \infty.$$

Therefore the condition

$$\sup_{n\ge 0,k\ge 0} |b_{nk}|\left(\frac{p}{R}\right)^k < \infty \quad \text{whenever } 1 < p < P ,$$

is necessary, i.e., condition (ii) is necessary in both Theorem 1 and Theorem 2. □

Proof of Theorem 3.

Sufficiency. We assume that

$$\begin{cases} \lim_{n\to\infty} b_{nk} =: b_k \text{ for } k=0,1,\dots ; \\ M := \sup_{n\ge 0,k\ge 0} |b_{nk}|^{\frac{1}{k+1}} < \infty. \end{cases}$$

Let $\mathbf{a} \in \mathcal{E}$, and let $u \in D_R^\gamma$. Then $u = \frac{1}{2}(z+z^{-1})$ with $1 \le |z| < R < \infty$, and so, by Lemma 2,

$$\begin{aligned} \left|b_{nk}a_kP_k(u)\right| &\le K(\epsilon)|b_{nk}||a_k|(1+k)^{2+2/\epsilon}|z|^k \le K(\epsilon)|b_{nk}||a_k|(1+k)^{2+2/\epsilon}R^k \\ &\le K(\epsilon)M|a_k|(1+k)^{2+2/\epsilon}(MR)^k < \infty. \end{aligned}$$

¿From (i) we get

$$\lim_{n\to\infty} b_{nk}a_kP_k(u) = b_ka_kP_k(u).$$

Since $\sum_{k=0}^{\infty} |a_k|(1+k)^{2+2/\epsilon}(MR)^k < \infty$, and since R can be arbitrarily large, it follows, by the Weierstrass M-test, that $g_n(u)$ exists for $n=0,1,\dots$, and

$$\lim_{n\to\infty} g_n(u) = \lim_{n\to\infty}\sum_{k=0}^{\infty} b_{nk}a_kP_k(u) = \sum_{k=0}^{\infty} b_ka_kP_k(u)$$

on $\mathbb{C}$, and that the sequence (g_n) is uniformly convergent on compact subsets of $\mathbb{C}$.

Necessity. Let $a_k := k^{-k}$, so that $\mathbf{a} \in \mathcal{E}$. Then, by hypothesis, the series

$$g_n(u) := \sum_{k=0}^{\infty} b_{nk}a_kP_k(u)$$

is convergent on $\mathbb{C}$, and the sequence (g_n) is uniformly convergent on compact subsets of $\mathbb{C}$. By the Weierstrass double-series theorem, (g_n) converges to an entire function on $\mathbb{C}$. By Lemma 3 we have

$$b_{nk}a_k = \int_{-1}^{1} g_n(t)P_k(t)\,dt \quad \text{for } n = 0, 1, \dots .$$

Since $g_n(t)$ is uniformly convergent on $[-1, 1]$ to $g(t)$ say, we get, for $k = 0, 1, \dots$, that

$$\lim_{n\to\infty} b_{nk}a_k = \int_{-1}^{1} g(t)P_k(t)\,dt =: d_k,$$

and hence that

$$\lim_{n\to\infty} b_{nk} = b_k,$$

where $b_k = d_k k^k$ for $k = 0, 1, 2, \dots$. Thus condition (i) is necessary.

Suppose now that $\mathbf{a}$ is an arbitrary sequence in $\mathcal{E}$, and that $R > 1$. Since the sequence (g_n) is uniformly convergent on $\bar{D}_R^\gamma$, we have, for $u \in \bar{D}_R^\gamma$ and $n = 0, 1, \dots$, that $|g_n(u)| \le M(R, \mathbf{a}) < \infty$. ¿From Lemma 4 we get that

$$|b_{nk}a_k| \le c(R)M(R, \mathbf{a})R^{-k} \text{ for } n, k = 0, 1, \dots . \tag{1}$$

Hence $\sum_{k=0}^{\infty} b_{nk}a_k$ is convergent whenever $\mathbf{a} \in \mathcal{E}$, and we have, by Lemma 1, that

$$M_n := \sup_{k\ge 0} |b_{nk}|^{\frac{1}{k+1}} < \infty \text{ for } n = 0, 1, \dots .$$

Assume now that

$$\sup_{n\ge 0}\sup_{k\ge 0} |b_{nk}|^{\frac{1}{k+1}} = \sup_{n\ge 0} M_n = \infty.$$

This implies that there exists a strictly increasing sequence of positive integers (n_j) such that $M_{n_j} \to \infty$. This in turn implies that there exists a sequence of non-negative integers (k_j) such that

$$|b_{n_j,k_j}|^{\frac{1}{k_j+1}} > \frac{1}{2}M_{n_j} \to \infty \text{ as } j \to \infty. \tag{2}$$

We show now that the sequence (k_j) is not bounded. Assume that it is bounded. Then there is a positive integer k^* such that $0 \le k_j \le k^*$. Since $\lim_{n\to\infty} b_{nk} = b_k$ for $k = 0, 1, \dots , k^*$, it follows that the set of numbers $(b_{nk})_{n\ge 0, 0\le k\le k^*}$ is bounded, and hence that the set of numbers $\left(|b_{nk}|^{\frac{1}{k+1}}\right)_{n\ge 0, 0\le k\le k^*}$ is bounded. But this contradicts (2). Therefore the sequence (k_j) is not bounded. We can suppose (by considering a subsequence if necessary) that the sequence is strictly increasing. Choose

$$a_k := \begin{cases} \left(\dfrac{1}{|b_{n_j,k}|}\right)^{\frac{k+1}{2}} & \text{if } k = k_j, \\ 0 & \text{otherwise.} \end{cases}$$

We then have, by (2), that

$$|a_{k_j}|^{\frac{1}{k_j+1}} = \frac{1}{\sqrt{|b_{n_j,k_j}|}} < \left(\frac{1}{\frac{1}{2}M_{n_j}}\right)^{\frac{k_j+1}{2}} \to 0 \text{ as } j \to \infty.$$

Therefore $\mathbf{a} \in \mathcal{E}$, but

$$|b_{n_j,k_j}|a_{k_j} = \sqrt{|b_{n_j,k_j}|} \to \infty \text{ as } j \to \infty,$$

which contradicts (1). Thus the condition

$$\sup_{n\ge 0, k\ge 0} |b_{nk}|^{\frac{1}{k+1}} < \infty$$

is necessary, i.e., condition (ii) is necessary. □

4. Additional Theorems. In this section we prove some theorems showing that the ellipse of convergence D_P^γ specified in Theorem 2 cannot be enlarged when the matrix $\mathbf{B}$ satisfies conditions (i) and (ii) of that theorem together with certain other conditions. Analogous theorems concerning matrix transformations of power series appear in [1].

Theorem 4. *Suppose that P and R are finite numbers greater than 1, and that $\mathbf{B} \equiv (b_{nk})$ is a triangular infinite matrix (i.e., $b_{nk} = 0$ for $k > n$) satisfying*

$$M(p) := \sup_{n\ge 0,\, k\ge 0} |b_{nk}| \left(\frac{p}{R}\right)^k < \infty \quad \text{for} \quad 1 < p < P.$$

Then, for each $\mathbf{a} \in \mathbf{A}_R$ and each $R_1 \ge P$,

$$\limsup_{n\to\infty} \max_{u\in\gamma_{R_1}} \left|\sum_{k=0}^{n} b_{nk}a_kP_k(u)\right|^{\frac{1}{n}} \le \frac{R_1}{P}.$$

Proof. Choose $R_1 \ge P > 1$, and suppose $\mathbf{a} \in \mathbf{A}_R$. Let $\frac{1}{P} < \lambda < 1$, and take $p := \lambda P > 1$. Then $1 < p < P$. Since $\limsup |a_k|^{\frac{1}{k+1}} = \frac{1}{R}$, there is a positive constant $c(\lambda)$ such that

$$|a_k| \le \frac{c(\lambda)}{(\lambda R)^k} \quad \text{for } k \ge 0.$$

By Lemma 2, for $u \in \gamma_{R_1}$ we have $|P_k(u)| \le K(\epsilon)(1+k)^{2+2/\epsilon}{R_1}^k$ and hence

$$\begin{aligned}\left|\sum_{k=0}^{n} b_{nk}a_kP_k(u)\right| &\le K(\epsilon)\sum_{k=0}^{n}|b_{nk}|\left(\frac{p}{R}\right)^k|a_k|R^k\left(\frac{R_1}{p}\right)^k(1+k)^{2+2/\epsilon}\\ &\le K(\epsilon)M(p)c(\lambda)\sum_{k=0}^{n}\left(\frac{R}{\lambda R}\right)^k\left(\frac{R_1}{\lambda P}\right)^k(1+k)^{2+2/\epsilon}\\ &\le K(\epsilon)M(p)c(\lambda)(1+n)^{2+2/\epsilon}\sum_{k=0}^{n}\left(\frac{R_1}{\lambda^2 P}\right)^k.\end{aligned}$$

Since $\frac{R_1}{\lambda^2 P} > \frac{R_1}{P} \geq 1$, it follows that

$$\limsup_{n\to\infty} \max_{u\in\gamma_{R_1}} \left|\sum_{k=0}^{n} b_{nk} a_k P_k(u)\right|^{\frac{1}{n}} \leq \lim_{n\to\infty} \left(\sum_{k=0}^{n} \left(\frac{R_1}{\lambda^2 P}\right)^k\right)^{\frac{1}{n}} = \frac{R_1}{\lambda^2 P}.$$

Letting $\lambda \nearrow 1$ we get

$$\limsup_{n\to\infty} \max_{u\in\gamma_{R_1}} \left|\sum_{k=0}^{n} b_{nk} a_k P_k(u)\right|^{\frac{1}{n}} \leq \frac{R_1}{P}\,. \qquad \square$$

Remark. Assume that a triangular matrix $\mathbf{B}$ satisfies

$$M(p) := \sup_{n\geq 0,\, k\geq 0} |b_{nk}| \left(\frac{p}{R}\right)^k < \infty \text{ for } 1 < p < P.$$

Then

$$|b_{nn}|^{\frac{1}{n}} \frac{p}{R} \leq M(p)^{\frac{1}{n}} \to 1 \text{ as } n \to \infty,$$

and hence

$$\limsup_{n\to\infty} |b_{nn}|^{\frac{1}{n}} \leq \frac{R}{p} \text{ for each } p \in (1, P).$$

Letting $p \nearrow P$ we get

$$\limsup_{n\to\infty} |b_{nn}|^{\frac{1}{n}} \leq \frac{R}{P}.$$

This suggests that it is not inappropriate to impose the condition

$$\lim_{n\to\infty} |b_{nn}|^{\frac{1}{n}} = \frac{R}{P},$$

as we do in the following theorem.

Theorem 5. *Let* $\mathbf{B}$ *be a triangular matrix. Suppose that*

$$\lim_{n\to\infty} |b_{nn}|^{\frac{1}{n}} = \frac{R}{P},$$

where P *and* R *are finite numbers greater than* 1 *. Then for each* $\mathbf{a} \in \mathbf{A}_R$ *and each* $R_1 \geq P$ *we have*

$$\limsup_{n\to\infty} \max_{u\in\gamma_{R_1}} \left|\sum_{k=0}^{n} b_{nk} a_k P_k(u)\right|^{\frac{1}{n}} \geq \frac{R_1}{P}.$$

Proof. Assume that the conclusion of the theorem is not true. Then there is an $\mathbf{a}^* \in \mathbf{A}_R$ and an $R_1 \geq P > 1$ such that

$$\limsup_{n\to\infty} \max_{u\in\gamma_{R_1}} \left|\sum_{k=0}^{n} b_{nk} a_k^* P_k(u)\right|^{\frac{1}{n}} < \frac{R_1}{P}.$$

Therefore there exists a number $\tilde{R}$ such that $1 < \tilde{R} < R_1$ and, for all n sufficiently large,

$$\max_{u\in\gamma_{R_1}} \left|\sum_{k=0}^{n} b_{nk}a_k^* P_k(u)\right|^{\frac{1}{n}} \le \frac{\tilde{R}}{P}, \text{ and hence } \max_{u\in\gamma_{R_1}} \left|\sum_{k=0}^{n} b_{nk}a_k^* P_k(u)\right| \le \left(\frac{\tilde{R}}{P}\right)^n .$$

Applying Lemma 4 to the function $g_n(u) := \sum_{k=0}^{n} b_{nk}a_k^* P_k(u)$ we get in particular that, for all large n,

$$|b_{nn}||a_n^*|R_1^n \le c(R_1)\left(\frac{\tilde{R}}{P}\right)^n, \text{ and therefore } |b_{nn}|^{\frac{1}{n}}|a_n^*|^{\frac{1}{n}}R_1 \le c(R_1)^{\frac{1}{n}}\frac{\tilde{R}}{P}.$$

¿From the last inequality we get that

$$\frac{\tilde{R}}{P} \ge \limsup_{n\to\infty}\left(|b_{nn}|^{\frac{1}{n}}|a_n^*|^{\frac{1}{n}}R_1\right) = R_1 \lim_{n\to\infty}|b_{nn}|^{\frac{1}{n}} \cdot \limsup_{n\to\infty}|a_n^*|^{\frac{1}{n}} = \frac{R_1}{P}.$$

But this is a contradiction since $1 < \tilde{R} < R_1$. Hence the conclusion of the theorem must hold. □

The next two theorems are analogues of Theorems 6 and 7 (concerning matrix transformations of power series) in [1], which in turn generalize results about regular and non-regular Nörlund matrices due respectively to Luh [6] and K. Stadtmüller [9, Theorems 6 and 7]. The first of these new theorems, which follows immediately from Theorems 4 and 5, shows, inter alia, that the sequence (g_n) specified in Theorem 2 cannot converge uniformly in the interior of any ellipse γ_{P_1} with $P_1 > P$ when $\mathbf{B}$ is a triangular matrix satisfying condition (ii) of Theorem 2 together with the diagonal condition of Theorem 5.

Theorem 6. *Suppose that P and R are finite numbers greater than 1, and that $\mathbf{B}$ is a triangular matrix satisfying*

$$M(p) := \sup_{n\ge 0,\, k\ge 0} |b_{nk}|\left(\frac{p}{R}\right)^k < \infty \quad \text{for } 1 < p < P, \text{ and } \lim_{n\to\infty}|b_{nn}|^{\frac{1}{n}} = \frac{R}{P}.$$

Then, for each $\mathbf{a} \in \mathbf{A}_R$ and each $R_1 \ge P$,

$$\limsup_{n\to\infty} \max_{u\in\gamma_{R_1}} \left|\sum_{k=0}^{n} b_{nk}a_k P_k(u)\right|^{\frac{1}{n}} = \frac{R_1}{P}.$$

The next theorem shows that the ellipse γ_{R_1} in the conclusion of Theorem 6 can be replaced by any arc of that ellipse (provided condition (i) of Theorem 2 is also satisfied when $R_1 = P$).

Theorem 7. *Suppose that P and R are finite numbers greater than 1, and that $\mathbf{B}$ is a triangular matrix such that*

$$M(p) := \sup_{n\geq 0,\, k\geq 0} |b_{nk}|\Big(\frac{p}{R}\Big)^k < \infty \text{ for } 1 < p < P, \text{ and } \lim_{n\to\infty} |b_{nn}|^{\frac{1}{n}} = \frac{R}{P}.$$

(i) *Then, for each $\mathbf{a} \in \mathbf{A}_R$ and each $R_1 > P$,*

$$\limsup_{n\to\infty} \max_{u\in\Gamma} \left|\sum_{k=0}^{n} b_{nk} a_k P_k(u)\right|^{\frac{1}{n}} = \frac{R_1}{P} ,$$

where Γ is any closed non-trivial arc of γ_{R_1}.

(ii) *If, in addition,*

$$\lim_{n\to\infty} b_{nk} =: b_k \text{ for } k = 0, 1, \dots , \text{ where } b_k \neq 0 \text{ for } k > k^*,$$

then, for each $\mathbf{a} \in \mathbf{A}_R$,

$$\limsup_{n\to\infty} \max_{u\in\Gamma} \left|\sum_{k=0}^{n} b_{nk} a_k P_k(u)\right|^{\frac{1}{n}} = 1 ,$$

where Γ is any closed non-trivial arc of γ_P.

Proof of (i). By Theorem 6 we know that

$$\limsup_{n\to\infty} \max_{u\in\Gamma} \left|\sum_{k=0}^{n} b_{nk} a_k P_k(u)\right|^{\frac{1}{n}} \leq \frac{R_1}{P}.$$

Hence it is enough to prove that, for every $\mathbf{a} \in \mathbf{A}_R$,

$$\limsup_{n\to\infty} \max_{u\in\Gamma} \left|\sum_{k=0}^{n} b_{nk} a_k P_k(u)\right|^{\frac{1}{n}} \geq \frac{R_1}{P}, \tag{3}$$

which we now proceed to do. Assume that (3) is not true. Then there exists a sequence $\mathbf{a}^* \in \mathbf{A}_R$ and a number $\tilde{R}$ such that $P < \tilde{R} < R_1$ and

$$\limsup_{n\to\infty} \max_{u\in\Gamma} |g_n(u, \mathbf{a}^*)|^{\frac{1}{n}} \leq \frac{\tilde{R}}{P}.$$

Hence given $\epsilon > 0$ we have, for $z := u + \sqrt{u^2 - 1}$ and all sufficiently large n,

$$\max_{u\in\Gamma} \left|\frac{g_n(u, \mathbf{a}^*)}{z^n}\right| \leq \left(\frac{\tilde{R}}{P} \cdot \frac{1}{R_1}\right)^n 2^{\epsilon n} = \left(\frac{\tilde{R}}{R_1}\right)^n \left(\frac{2^\epsilon}{P}\right)^n .$$

Further, from Theorem 6 we get that, for all large n,

$$\max_{u\in\gamma_P}\left|\frac{g_n(u,\mathbf{a}^*)}{z^n}\right| \le \left(\frac{2^\epsilon}{P}\right)^n$$

and

$$\max_{u\in\gamma_{R_1}}\left|\frac{g_n(u,\mathbf{a}^*)}{z^n}\right| \le \left(\frac{2^\epsilon}{P}\right)^n.$$

Let $P < r < R_1$. Since the function $z = u+\sqrt{u^2-1}$ is holomorphic and different from zero on $\mathbb{C}\backslash[-1,1]$, we have, by Nevanlinna's N-constants theorem (see [3, Theorem 18.3.3]), that there exist positive constants $\theta_1, \theta_2, \theta_3$ (depending on r but not on ϵ) such that $\theta_1+\theta_2+\theta_3 = 1$ and

$$\max_{u\in\gamma_r}\left|\frac{g_n(u,\mathbf{a}^*)}{z^n}\right| \le \left(\frac{\tilde{R}}{R_1}\frac{2^\epsilon}{P}\right)^{n\theta_1}\left(\frac{2^\epsilon}{P}\right)^{n\theta_2}\left(\frac{2^\epsilon}{P}\right)^{n\theta_3} = \left(\frac{\tilde{R}}{R_1}\right)^{n\theta_1}\left(\frac{2^\epsilon}{P}\right)^n$$

for all sufficiently large n. Hence, choosing $\epsilon > 0$ so small that $\left(\frac{\tilde{R}}{R_1}\right)^{\theta_1} 2^\epsilon < 1$, we get

$$\limsup_{n\to\infty}\max_{u\in\gamma_r}|g_n(u,\mathbf{a}^*)|^{\frac{1}{n}} \le \left(\frac{\tilde{R}}{R_1}\right)^{\theta_1} 2^\epsilon\frac{r}{P} < \frac{r}{P}.$$

Since $r > P$, the last inequality contradicts the conclusion of Theorem 5. Hence (3) must hold when $R_1 > P$.

Proof of (ii). By Theorem 6 we know in this case that

$$\limsup_{n\to\infty}\max_{u\in\Gamma}\left|\sum_{k=0}^n b_{nk}a_kP_k(u)\right|^{\frac{1}{n}} \le 1.$$

Hence it is enough to prove that, for every $\mathbf{a}\in\mathbf{A}_R$,

$$\limsup_{n\to\infty}\max_{u\in\Gamma}\left|\sum_{k=0}^n b_{nk}a_kP_k(u)\right|^{\frac{1}{n}} \ge 1, \tag{4}$$

Suppose (4) is not true. Then for some $\mathbf{a}^*\in\mathbf{A}_R$ we have

$$\limsup_{n\to\infty}\max_{u\in\Gamma}\left|\sum_{k=0}^n b_{nk}a_k^*P_k(u)\right|^{\frac{1}{n}} < 1.$$

Write

$$g_n(u, \mathbf{a}^*) := \sum_{k=0}^{n} b_{nk} a_k^* P_k(u) \ .$$

It follows that there exists a positive number $q < \dfrac{R_1}{P} = 1$, such that, for all n sufficiently large,

$$\sup_{u \in \Gamma} |g_n(u, \mathbf{a}^*)| < q^n.$$

Given $\alpha > 0$ we get from Theorem 6 that, for all n sufficiently large,

$$\max_{u \in \gamma_P} |g_n(u, \mathbf{a}^*)| \le 2^{\alpha n}.$$

By Nevanlinna's N-constants theorem, there exists a positive number $\theta < 1$ (independent of α) such that, for all large n,

$$\max_{-1 \le u \le 1} |g_n(u, \mathbf{a}^*)| \le \left(q^\theta 2^{(1-\theta)\alpha}\right)^n.$$

Since we can choose $\alpha > 0$ so small that $q^\theta 2^{(1-\theta)\alpha} < 1$, it follows that

$$\max_{-1 \le u \le 1} |g_n(u, \mathbf{a}^*)| \to 0 \text{ as } n \to \infty.$$

By Lemma 3 we have

$$b_{nk} a_k = \int_{-1}^{1} g_n(t, \mathbf{a}^*) P_k(t) dt \quad \text{for } n = 0, 1, \dots \ .$$

Since $g_n(t, \mathbf{a}^*)$ tends uniformly to 0 on $[-1, 1]$ as $n \to \infty$, it follows that

$$0 = \lim_{n \to \infty} b_{nk} a_k^* = b_k a_k^* \text{ for } k = 0, 1, \dots \ .$$

Since $\mathbf{a}^* \in \mathbf{A}_R$ we have that $a_k^* \ne 0$ for some $k > k^*$. Hence $b_k = 0$ for such a k. But this contradicts the assumption that $b_k \ne 0$ for $k > k^*$. Therefore (4) must hold. □

5. Chebyshev Polynomials. In this section we restrict (P_n) to be the orthonormal sequence on $[0, 1]$ of Chebyshev polynomials of the first or second kind, the corresponding weight functions of which are respectively $w(x) = \frac{\pi}{2}(1 - x^2)^{-\frac{1}{2}}$ and $w(x) = \frac{\pi}{2}(1 - x^2)^{\frac{1}{2}}$. The special properties of these Chebyshev polynomials that makes them amenable to the proof of Theorem 8 (below) are the familiar identities

$$2P_n\left(\frac{1}{2}(z + z^{-1})\right) = z^n + z^{-n} \tag{5}$$

when P_n is of the first kind, and

$$(z - z^{-1})P_n\left(\frac{1}{2}(z + z^{-1})\right) = z^{n+1} - z^{-n-1} \tag{6}$$

when P_n is of the second kind.

The said theorem deals with the possibility of pointwise convergence of the sequence $(g_n(u))$ specified in Theorem 2 outside the convergence ellipse γ_P. It's analogue for power series is Theorem 8 in [1], which generalizes results due to Lejá [4] and Stadtmüller [9, Theorem 8] about regular and non-regular Nörlund matrices respectively.

Theorem 8. *Suppose that P and R are finite numbers greater than 1, and that $\mathbf{B}$ is a triangular matrix such that*

(i) $\lim_{n\to\infty} b_{nk} =: b_k$ *for* $k = 0, 1, \dots$ *where* $b_k \neq 0$ *for* $k > k^*$;

(ii) $M(p) := \sup_{n\geq 0,\, k\geq 0} |b_{nk}| \left(\frac{p}{R}\right)^k < \infty$ *for* $1 < p < P$; $\lim_{n\to\infty} |b_{nn}|^{\frac{1}{n}} = \frac{R}{P}$, *and*

(iii) $|b_{nk}| \leq c(\tilde{R})|b_{nn}| \left(\frac{P}{\tilde{R}}\right)^{n-k}$ *for* $1 < \tilde{R} < R$ *and* $0 \leq k \leq n$.

Suppose that $\mathbf{a} \in \mathbf{A}_R$ *and that* $\limsup_{n\to\infty} |a_n|R^n > 0$. *Let*

$$g_n(u) := \sum_{k=0}^{n} b_{nk} a_k P_k(u),$$

where (P_k) *is the orthonormal sequence on* $[-1, 1]$ *of Chebyshev polynomials of the first or second kind, and let* $P_1 > P$. *Then* $\limsup_{n\to\infty} |g_n(u)|^{\frac{1}{n}} \leq 1$ *for at most a finite number of points* u *outside the ellipse* γ_{P_1} *and hence, in particular, the sequence* (g_n) *can converge at most at a finite number of points* u *outside the ellipse* γ_{P_1}.

Proof. Assume that u is a point outside the ellipse γ_{P_1} for which

$$\limsup_{n\to\infty} |g_n(u)|^{\frac{1}{n}} \leq 1. \tag{7}$$

Let $z := u + \sqrt{u^2 - 1}$, so that $|z| > P_1$; and let

$$\tilde{g}_n(z) := \sum_{k=0}^{n} b_{nk} a_k z^k.$$

Then, by (5),

$$2g_n(u) = 2\sum_{k=0}^{n} b_{nk} a_k P_k(u) = \tilde{g}_n(z) + \tilde{g}_n(z^{-1})$$

when the Chebyshev polynomials P_k are of the first kind; and, by (6),

$$(z - z^{-1})g_n(u) = z\tilde{g}_n(z) - z^{-1}\tilde{g}_n(z^{-1})$$

when the Chebyshev polynomials P_k are of the second kind.

Since $|z^{-1}| < P_1^{-1} < P$ it follows from Theorem 2 in [1] that $\tilde{g}_n(z^{-1})$ tends to a finite limit as $n \to \infty$, and therefore from (7) that, in either case,

$$\limsup_{n\to\infty} |\tilde{g}_n(z)|^{\frac{1}{n}} \leq 1. \tag{8}$$

Theorem 8 in [1] tells us that inequality (8) can hold for at most a finite number of points z satisfying $|z| > P_1$, and thus (7) can hold for at most finitely many points u outside the ellipse γ_{P_1}. □

Remarks. A Nörlund matrix $\mathbf{N}_B$ for which

$$\lim_{n\to\infty} \frac{B_{n-1}}{B_n} = b \text{ with } |b| = \frac{R}{P}$$

satisfies all the conditions on the matrix in Theorem 8. In this case, however, the condition $\limsup |a_n| R^n > 0$ can be omitted since the corresponding version of the theorem for power series has recently been proved by K. Stadmüller and Gross-Erdman [10, Remark 3.7].

An open and challenging question is whether Theorem 8 holds for other orthogonal polynomials.

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