Reprinted from JOURNAL OF APPROXIMATION THEORY
Vol. 35, No. 2, June 1982
Printed in Belgium

Approximations with Negative Roots and Poles

PETER B. BORWEIN

Mathematics Department, Dalhousie University, Halifax, Nova Scotia B3H 4H8, Canada

Communicated by E. W. Cheney

Received April 13, 1981

We ask when best uniform rational or polynomial approximations on $[0, 1]$ have negative roots or poles. We show that the best $(n+k, n)$th rational approximation to a Stieltjes transform has only negative poles. We use this to show that the best $(n+k, n)$th rational approximation is better than the best $(2n+k-1)$th polynomial approximation to such functions. We also construct a class of entire functions whose best polynomial approximations have negative roots. We show that for this class the best $(n+m+1)$th polynomial approximation behaves better than the best (n, m)th rational approximation.

Let π_n denote the collection of polynomials of degree at most n with real coefficients ($\pi_{-1} \equiv 0$). If f is continuous on $[a, b]$ we set

$$E_n(f: [a, b]) = \min_{p_n \in \pi_n} \|f - p_n\|_{[a,b]}$$

and

$$R_{n,m}(f: [a, b]) = \min_{p_n \in \pi_n, q_m \in \pi_m} \|f - p_n/q_m\|_{[a,b]},$$

where $\|\cdot\|_{[a,b]}$ denotes the supremum norm on $[a, b]$. When we talk about best approximations it will be in this norm.

We prove the following

THEOREM 1. *Let $r_k(x) \in \pi_k$, let α be non-decreasing and let*

$$f(x) = \int_0^\infty \frac{1}{x+t}\, d\alpha(t) + r_k(x). \tag{1}$$

Suppose f is defined (as a convergent Stieltjes integral) fr; $x \geqslant c \geqslant 0$. Suppose that $p_{n+k} \in \pi_{n+k} (k \geqslant -1)$, $q_n \in \pi_n$ and suppose that

$$p_{n+k}(x) - q_n(x) f(x)$$

0021-9045/82/060132-10\$02.00/0

has $2n+k+1$ *zeroes on* $[a,b]$, $a \geqslant c$. *Then* q_n *has all negative roots and*

$$\frac{p_{n+k}(x)}{q_n(x)} = s_k(x) + \sum_{i=1}^{n} \frac{\gamma_i}{x+\delta_i}, \tag{2}$$

where $\gamma_i, \delta_i > 0$ *for all* i *and where* $s_k \in \pi_k$.

A function of the form $\int_0^\infty 1/(x+t)\,d\alpha(t)$ is called a Stieltjes transform (of α). In the context of this paper Stieltjes transforms will always be of non-decreasing α. We have the following characterization of Stieltjes transforms.

COROLLARY 1. *Fix* $k \geqslant -1$. *The following two conditions are equivalent for a non-rational* f.

(A) f *can be represented as*

$$f(x) = \int_0^\infty \frac{1}{x+t}\,d\alpha(t) + r_k(x),$$

where r_k *is a polynomial of degree* $\leqslant k$, α *is a non-decreasing function and the above Stieltjes integral converges for* $x \geqslant a \geqslant 0$.

(B) f *is continuous on* $[a,b]$, *for some* $b > a \geqslant 0$ *and for all* n *the best* $(n+k, n)$ *rationl approximation to* f *on* $[a,b]$ *is of the form*

$$s_k^n(x) + \sum_{i=1}^{n} \frac{\gamma_i^n}{x+\delta_i^n},$$

where $\gamma_i^n, \delta_i^n > 0$ *and* s_k^n *is a polynomial of degree* $\leqslant k$.

Two cases of Theorem 1 are known. The case $k=-1$ is proved by Krein [3, p. 166; or 5, p. 96]. The case where p_{n+k}/q_n is the Padé approximant (that is, all the interpolation points are the same) is proved by Baker (see [1]).

We require the following lemmas.

LEMMA 1. *Consider, for* m *a positive integer,*

$$f(x) = \sum_{i=0}^{\infty} \frac{a_i}{(x+\gamma_i)^m},$$

where $\gamma_{i+1} \geqslant \gamma_i > 0$ *and each* a_i *is real. Then the number of zeroes of* f *on* $[0,\infty)$ *is no greater than the number of sign changes in the sequence* $\{a_0, a_1, a_2, \ldots\}$ (a_i *terms that vanish are ignored and zeroes are counted according to their multiplicities*).

LEMMA. 2. *Suppose that α is non-decreasing. Consider, for a positive integer m and a polynomial g,*

$$f(x)=\int_0^\infty \frac{g(t)}{(x+t)^m}\,d\alpha(t).$$

If f has k possibly multiple non-negative zeroes then g has at least k distinct positive zeroes.

Both lemmas are immediate consequences of results in [3] or [4]. The basic point in Lemma 1 is that

$$(-1)^m m! \sum_{i=1}^{n} \frac{a_i}{(x+\gamma_i)^{m+1}} = \int_0^\infty (-1)^m \left(\sum_{i=1}^{n} a_i e^{-\gamma_i t}\right) e^{-xt}\,dt.$$

Lemma 1 now follows from the variation diminishing properties of the Laplace transform and Descartes rule of signs. Lemma 2 can be deduced from Lemma 1 by approximating α by step functions.

We note that Lemma 1 implies that

$$\left\{\frac{1}{(x+\alpha_1)^k},\dots,\frac{1}{(x+\alpha_n)^k}\right\}, \qquad \alpha_j \neq \alpha_i > 0,$$

is a Tchebycheff system on any positive interval.

Proof of Theorem 1. Let $q_n(x)=\sum_{k=0}^{n} a_k x^k$. Let $h=n+k+1$. Consider

$$(q_n(x)\cdot f(x))^{(h)} = \sum_{m=0}^{h} \binom{h}{m} q_n^{(m)}(x)\, f^{(h-m)}(x).$$

We note that $(q_n(x)\cdot f(x))^{(h)}$ has n zeroes on $[a,b]$. We also note that we do not need to assume that the zeroes of $p_{n+k}-q_n f$ are distinct. Since $q_n^{(m)}\equiv 0$ for $m>n$ we have, for $x\geqslant c$,

$$\begin{aligned}
(q_n(x)\cdot f(x))^{(h)} &= \sum_{m=0}^{n} \binom{h}{m} q_n^{(m)}(x)\, f^{(h-m)}(x)\\
&= \sum_{m=0}^{n} \binom{n}{m} \left(\sum_{k=m}^{n} \frac{a_k x^{k-m} k!}{(k-m)!}\right)\left(\int_0^\infty \frac{(h-m)!(-1)^{h-m}}{(x+t)^{h+1-m}}\,d\alpha(t)\right)\\
&= \sum_{k=0}^{n}\sum_{m=0}^{k} (-1)^h h!\, a_k x^k \int_0^\infty \frac{k!(-x)^{-m}}{(k-m)!\, m!(x+t)^{h+1-m}}\,d\alpha(t)\\
&= \sum_{k=0}^{n} (-1)^h h!\, a_k x^k \int_0^\infty \left(\sum_{m=0}^{k}\binom{k}{m}\left(\frac{-x}{x+t}\right)^{-m}\right)\frac{d\alpha(t)}{(x+t)^{h+1}}
\end{aligned}$$

$$= \sum_{k=0}^{n} (-1)^h h!\, a_k x^k \int_0^\infty (-1)^k \left(\frac{t}{x}\right)^k \frac{d\alpha(t)}{(x+t)^{h+1}}$$

$$= (-1)^h h! \int_0^\infty \left(\sum_{k=0}^{n} a_k(-t)^k\right) \frac{d\alpha(t)}{(x+t)^{h+1}}$$

$$= (-1)^h h! \int_0^\infty \frac{q_n(-t)}{(x+t)^{h+1}}\, d\alpha(t). \tag{3}$$

We observe, by Lemma 2, that

$$\int_0^\infty \frac{q_n(-t)}{(x+t)^{h+1}}\, d\alpha(t)$$

can have n non-negative zeroes only if $q_n(-t)$ has n positive zeroes. It follows that q_n has only negative roots and that these roots are distinct. Thus,

$$\frac{p_{n+k}(x)}{q_n(x)} = s_k(x) + \sum_{i=1}^{n} \frac{b_i}{x+\delta_i},$$

where $s_k \in \pi_k$, $\delta_i > 0$. It remains to show that $b_i > 0$. If, for $x \geqslant c$,

$$f(x) - r_k(x) = \int_0^\infty \frac{1}{x+t}\, d\alpha(t)$$

then there exists m, β_i, $\eta_i \geqslant 0$ so that

$$\left\| \sum_{i=1}^{m} \frac{\beta_i}{x+\eta_i} - f(x) + r_k(x) \right\|_{[c,\infty)} < \varepsilon.$$

If we consider $(k+1)$st derivatives we see that, for an appropriate ε,

$$\sum_{i=1}^{n} \frac{b_i}{(x+\delta_i)^{k+2}} - \sum_{i=1}^{m} \frac{\beta_i}{(x+\eta_i)^{k+2}}$$

has $2n$ zeroes on $[c, \infty)$ and we deduce from Lemma 1 that each $b_i > 0$. ∎

Proof of Corollary 1. That condition A implies condition B follows from Theorem 1 and the alternation criteria for best rational approximations (see [2, p. 158]). Calculation (3) of the proof of Theorem 1 allows us to deduce that if f is not a rational function then the best $(n+k, n)$ rational approximation to f is nondegenerate (see [6, pp. 163–165]).

The proof that B implies A is a consequence of results in Widder (see

[10, p. 364]). We observe that since π_k is finite dimensional we have, passing to subsequences if necessary,

$$\lim_{n\to\infty} s_k^n(x) = r_k(x).$$

Also,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\gamma_i^n}{x+\delta_i^n} = \int_0^\infty \frac{1}{x+t}\, d\alpha(t).$$

Each rational function of the form

$$\sum_{i=1}^{n} \frac{\gamma_i^n}{x+\delta_i^n}$$

is a Stieltjes transform of an increasing (step) function β_n. It is now possible, via Helly's theorem, to write α as a limit of the β_n on $[a, \infty)$. ∎

One can deduce from Corollary 1 and Lemma 1 that the poles of the best $(n+k, n)$th rational approximation to a Stieltjes transform interlace with the poles of the best $(n+k+1, n+1)$th approximation.

THEOREM 2. *Suppose f is continuous on $[a, b], a \geqslant 0$ and suppose the best $(n+k, n)$ rational approximation to f on $[a, b]$ is of the form*

$$s_k(x) + \sum_{i=1}^{n} \frac{\gamma_i}{1+\delta_i x},$$

where $s_k \in \pi_k$ and $\gamma_i, \delta_i > 0$. Then

$$R_{n+k,n}(f\colon [a, b]) \leqslant P_{2n+k-1}(f\colon [a, b]).$$

Proof. Let

$$r(x) = s_k(x) + \sum_{i=1}^{n} \frac{\gamma_i}{1+\delta_i x}$$

be the best $(n+k, n)$th rational approximation to f on $[a, b]$ and let $p(x)$ be the best $(2n+k-1)$th polynomial approximation to f on $[a, b]$. We assume that

$$P_{2n+k-1}(f\colon [a, b]) < R_{n+k,n}(f\colon [a, b])$$

and derive a contradiction. Under the above assumption, appealing to the usual alternation criteria for best approximation, we deduce that $r(x)-p(x)$ has $2n+k+1$ zeroes on $[a, b]$ and hence that

$$(r(x)-p(x))^{(2n+k)}=(r(x))^{(2n+k)}$$

has a zero on $[a, b]$. This is impossible since

$$(r(x))^{2n+k}=(-1)^{2n+k}(2n+k)!\sum_{i=1}^{n}\frac{(\delta_i)^{2n+k}\gamma_i}{(1+\delta_i x)^{2n+k+1}}$$

is never zero on $[a, b]$. ∎

The above theorem can be applied, by Corollary 1, to Stieltjes transforms. We note that $\log(x+1)/x$ and $x^{-\delta}$, $0<\delta<1$, are Stieltjes transforms [10, p. 346].

Polynomial Approximations with Real Roots

Let Γ be the set of entire functions defined by

$$\Gamma=\left\{\sum_{n=0}^{\infty} a_n z^n \,|\, 0 \leqslant 4\cdot a_{n+1} \leqslant (a_n)^2 \text{ and } a_0<\tfrac{1}{3}\right\}$$

and

$$\Gamma^*=\left\{\sum_{n=0}^{\infty} a_n z^n \in \Gamma \,|\, a_0 \leqslant 5/16, a_1 \leqslant a_0^2/10 \text{ and } a_2 \leqslant a_1^2/5\right\}.$$

THEOREM 3. (a) *If $f\in\Gamma$, then for all n the* nth *partial sum of f has only negative roots.*

(b) *If $f\in\Gamma^*$ then every best uniform polynomial approximation to f on $[-1, 1]$ has only negative roots.*

Polya and Szegö [8, p. 66] show that

$$f(z)=\sum_{n=0}^{\infty}\frac{z^n}{a^{n^2}}, \qquad a>2,$$

has the property that all its partial sums have negative roots. We use an analogous argument for Theorem 3.

Proof. To prove (a) we suppose that

$$f(z)=\sum_{n=0}^{\infty} a_n z^n, \qquad 0<4a_{n+1}\leqslant (a_n)^2, \qquad a_0<\tfrac{1}{3}$$

and consider

$$S_N(z) = \sum_{n=0}^{N} a_n z^n.$$

We evaluate

$$\frac{S_N(-1/a_n)}{a_n(-1/a_n)^n}.$$

For $j > 0$

$$\left| \frac{a_{n+j}(-1/a_n)^{n+j}}{a_n(-1/a_n)^n} \right| \leqslant \frac{a_{n+j}}{(a_n)^{j+1}} \leqslant \frac{1}{4^j}.$$

For $j = -n$

$$\left| \frac{a_0}{a_n(-1/a_n)^n} \right| \leqslant a_0 < \tfrac{1}{3}.$$

For $-n < j \leqslant -1$,

$$\left| \frac{a_{n+j}(-1/a_n)^{n+j}}{a_n(-1/a_n)^n} \right| \leqslant a_{n+j} \leqslant \frac{1}{4^{n+j}}.$$

Therefore,

$$\frac{S_N(-1/a_n)}{a_n(-1/a_n)^n} \geqslant 1 - \frac{1}{3} - 2 \sum_{k=1}^{n} \frac{1}{4^j} > 0.$$

Thus, S_N changes sign between $-1/a_n$ and $-1/a_{n+1}$ and, hence, has real negative roots.

We now prove part (b). We need the following inequality (see [9, p. 226]): If $p_n \in \pi_n$ then

$$|p_n^{(k)}(0)| \leqslant n^k \, \|p_n\|_{[-1,1]}.$$

Suppose $p_n = \sum_{h=0}^{n} b_h x^h$ is the best nth degree polynomial approximation to f on $[-1, 1]$. Then

$$\|p_n - s_n\|_{[-1,1]} \leqslant 2 \sum_{h=n+1}^{\infty} a_h \leqslant 4a_{n+1}$$

and

$$|p_n^{(k)}(0) - s_n^{(k)}(0)| \leqslant 4n^k a_{n+1}.$$

Thus,

$$|b_k - a_k| \leqslant (4n^k/k!)\, a_{n+1}.$$

Since

$$a_{n+1} \leqslant \frac{(a_k)^{2(n+1-k)}}{4^{(n+1-k)}} \qquad \text{and} \qquad a_k \leqslant \left(\frac{1}{4}\right)^{2k}$$

we have, for all $0 < k \leqslant n$,

$$|b_k - a_k| \leqslant \frac{n^k a_k}{k!\, 4^{2k(2n+1-k-1)}} \leqslant \frac{a_k}{8}.$$

It follows that, for $0 < k \leqslant n$,

$$0 \leqslant b_{k+1} \leqslant (9/8)\, a_{k+1} \leqslant (9/32)\, a_k^2 \leqslant (18/49)\, b_k^2.$$

If we consider $q_n(x) = p_n(2x)$ we see that q_n satisfies the conditions of part (a) provided $a_0 \leqslant 5/16$, $a_1 \leqslant a_0^2/10$ and $288a_2 \leqslant 49a_1^2$. ∎

Contained in the Polya class is the set of functions which are uniform limits of polynomials with negative roots. These functions are all of the form

$$f(x) = \gamma e^{\alpha x} \prod_{i=1}^{\infty} (1 + \delta_i x), \qquad \alpha, \delta_i \geqslant 0.$$

Since neither all the partial sums nor all the best polynomial approximations to $e^{\alpha x}$ on $[0, 1]$ have all negative roots it is apparent that Γ is a proper subclass of this class (see [7]). The next theorem can be applied to the class Γ^*.

THEOREM 4. *Suppose that f is continuous on $[a, b]$, $a \geqslant 0$ and suppose that the best polynomial approximation of degree n to f has only negative roots. Then*

$$E_n(f\colon [a, b]) \leqslant R_{n-k,k-1}(f\colon [a, b]).$$

Proof. Let $p \in \pi_n$ be the best polynomial approximation to f on $[a, b]$. Let r/s be the best $(n-k, k-1)$th approximation to f on $[a, b]$ where $r \in \pi_{n-k}$, $s \in \pi_{k-1}$. Suppose, for the sake of a contradiction, that $R_{n-k,k-1}(f\colon [a, b]) < E_n(f\colon [a, b])$. Once again, appealing to the alternation characterization of best polynomial approximations, we deduce that

$$r(x) - s(x) \cdot p(x)$$

has $n+1$ zeroes on $[a, b]$. This implies that

$$(r(x) - s(x) \cdot p(x))^{(n+1-k)}$$

has at least k zeroes on $[a, b]$. However, since $p(x)$ has n zeroes on $(-\infty, 0)$,

$$(r(x) - s(x) \cdot p(x))^{(n+1-k)} = (p(x) \cdot s(x))^{(n+1-k)}$$

has $k-1$ zeroes on $(-\infty, 0)$. This yields the contradiction that the polynomial

$$(r(x) - s(x) \cdot p(x))^{(n+1-k)}$$

of degree $2k-2$ has $2k-1$ roots.

Informally, Theorems 2 and 4 say that best rational approximations of total degree n always reduce to polynomial approximations for functions of class Γ and never reduce to polynomial approximations for Stieltjes transforms. We observe that for $x^{1/2}$

$$R_{n,n}(x^{1/2}: [0, 1]) \leqslant e^{-c_1 n^{1/2}}$$

but

$$P_{2n}(x^{1/2}: [0, 1]) \geqslant c_2/n$$

and hence, that $R_{n,n}$ can be very much smaller than P_{2n} for functions satisfying the conditions of Theorem 2. (See [6, pp. 64 and 169].)

REFERENCES

1. G. A. BAKER, JR., The theory and application of the Padé approximant method, *in* "Advances in Theoretical Physics" (K. A. Brueckner, Ed.), Vol. 1, Academic Press, New York, 1965.
2. E. W. CHENEY, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.
3. S. KARLIN AND W. J. STUDDEN, "Tchebycheff Systems: With Applications in Analysis and Statistics," Wiley, New York, 1966.
4. S. KARLIN, "Total Positivity," Vol. 1, Stanford Univ. Press, Stanford, Calif., 1968.
5. M. G. KREIN, The ideas of P. L. Chebysev and A. A. Markov in the theory of limiting values of integrals and their further development, *Usp. Mat. Nauk (N.S.)* **6**(1951), No. 4(44), 3–120; *Amer. Math. Soc. Transl.*, Ser. 2, **12**, 1–122.
6. G. MEINARDUS, "Approximation of Functions: Theory and Numerical Methods," Springer-Verlag, New York/Berlin, 1967.
7. D. J. NEWMAN, Rational approximation to e^x with negative zeros and poles, *J. Approx. Theory* **20** (1977), 173–175.

8. G. POLYA AND G. SZEGÖ, "Problems and Theorems in Analysis," Vol. 11, Springer-Verlag, Berlin/Heidelberg/New York, 1976.
9. A. F. TIMAN, "Theory of Approximation of Functions of a Real Variable," Pergamon, New York, 1963.
10. D. V. WIDDER, "The Laplace Transform," Princeton Univ. Press, Princeton, N.J., 1941.

Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium