THE ARC LENGTH OF THE LEMNISCATE $\{|p(z)| = 1\}$

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Abstract. We show that the length of the set

$$\{z \in \mathbb{C} : |\Pi_{i=1}^{n}(z - \alpha_i)| = 1\}$$

is at most $8\pi en$. This gives the correct rate of growth in a long standing open problem of Erdös, Herzog and Piranian and improves the previous bound of $74n^2$ due to Pommerenke.

In 1958 Erdös, Herzog and Piranian [2] raised a number of problems concerned with the lemniscate

$$E_n := E_n(p) := \{z \in \mathbb{C} : |p(z)| = 1\}$$

where p is a monic polynomial of degree n, so

$$p(z) := \Pi_{i=1}^{n}(z - \alpha_l) \qquad \alpha_i \in \mathbb{C}.$$

One in particular, Problem 12, conjectures that the maximum length of E_n is achieved for $p(z) := z^n - 1$. (Which is of length $2n + 0(1)$.) The best partial to date is due to Pommerenke [7] who shows that the maximum length is at most $74n^2$. This problem has been re-posed by Erdös several times, including recently at a Budapest meeting honouring his 80th birthday. (See also [3].) It now carries with it a cash prize from Erdös of $250.

This note derives an upper bound of $8\pi en$, which gives the correct rate of growth.

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Theorem. *Let $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$. Then the length of*

$$E_n := \{z \in \mathbb{C} : |\Pi_{i=1}^n (z - \alpha_i)| = 1\}$$

is at most $8\pi en \, (\leq 69n)$.

The proof relies on two classical theorems. One due to Cartan and one due to Poincaré.

Cartan's Lemma. ([1, p174]) *If $p(z) := \prod_{i=1}^n (z - \alpha_i)$ then the inequality*

$$|p(z)| > 1$$

holds outside at most n circular discs, the sum of whose radii is at most $2e$.

Poincaré's Formula. *[8, 9] Let Γ be a rectifiable curve contained in $\mathbb{S}$ (the Riemann sphere). Let $v(\Gamma, x)$ denote the number of times that a great circle consisting of points equidistant from the antipodes $\pm x$ intersects Γ. (If this is infinite set $v(\Gamma, x) = 0$.) Then the length of Γ, $L_{\mathbb{S}}(\Gamma)$, is given by*

$$L_{\mathbb{S}}(\Gamma) = \frac{1}{4} \int_{\mathbb{S}} v(\Gamma, x) dx$$

where dx is area measure on $\mathbb{S}$.

We need the following corollary of this result.

Corollary. *Suppose Γ is an algebraic curve in $\mathbb{R}^2$ of degree at most N and D is a disc of radius R. Then the length of $\Gamma \cap D$ is at most $2\pi RN$.*

Proof. By an affine scaling it suffices to prove this for D a disc of radius 1 about the origin. Now any conic intersects Γ in at most $2N$ points by Bezout's theorem. It follows that the projection of Γ in the Riemann sphere is intersected by any great circle in $\mathbb{S}$ in at most $2N$ points. Thus the length of the projection of Γ in S is at most $2\pi N$ by Poincaré formula. Since the projection back to the unit disc doesn't increase arclength the result is proved. □

We can now prove the theorem.

Proof of Theorem. Fix $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. By Cartan's Lemma there exist circles $D_1, \dots, D_m$ with radius $r_1, \dots, r_m$ so that

$$E_n \subset \bigcup_{i=1}^m D_i$$

and

$$\sum_{i=1}^m r_i \leq 2e.$$

Observe that E_n is an algebraic curve in $\mathbb{R}^2$ of degree at most $2n$ in x and y where $z = x + iy$. So by the Corollary each disc D_i contains a portion of E_n of length at most $4\pi r_i n$. On summing over i we deduce that the length of E_n is at most $8\pi en$. □

The constant 2 in the Corollary can be removed with some effort, so a sharpening to $4\pi en$ is possible. The constant e in Cartan's Lemma is probably unnecessary, but this is open. Even with these improvements we would only get a bound of $4\pi n$ which still isn't sharp. Indeed it seems likely that this type of method is too blunt to yield an exact result.

There are a number of interesting related results. See for example Pommerenke [4,5,6,7]. In Pommerenke [6] it is shown that if the roots in the Theorem are all real then the length is at most 4π.

In Pommerenke [5] it is shown that if the set E_n is connected then the length is at least 2π, with equality only for z^n. When E_n is connected one can find a disc of radius 2 that contains it [5]. So in this case the length of E_n is at most $4\pi n$.

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