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CONSTRUCTIVE APPROXIMATION

Incomplete Rational Approximation in the Complex Plane

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Abstract. We consider rational approximations of the form

$$\left\{(1+z)^{\alpha n+1}\frac{p_{cn}(z)}{q_n(z)}\right\}$$

in certain natural regions in the complex plane where p_{cn} and q_n are polynomials of degree cn and n, respectively. In particular we construct natural maximal regions (as a function of α and c) where the collection of such rational functions is dense in the analytical functions. So from this point of view we have rather complete analog theorems to the results concerning incomplete polynomials on an interval.

The analysis depends on an examination of the zeros and poles of the Padé approximants to $(1+z)^{\alpha n+1}$. This is effected by an asymptotic analysis of certain integrals. In this sense it mirrors the well-known results of Saff and Varga on the zeros and poles of the Padé approximant to exp. Results that, in large measure, we recover as a limiting case.

In order to make the asymptotic analysis as painless as possible we prove a fairly general result on the behavior, in n, of integrals of the form

$$\int_0^1 [t(1-t)f_z(t)]^n\,dt,$$

where $f_z(t)$ is analytic in z and a polynomial in t. From this we can and do analyze automatically (by computer) the limit curves and regions that we need.

1. Introduction

In his remarkable paper of 1924, Szegö [11] considered the zeros of the partial sums $s_n(z) := \sum_{k=0}^{n} z^k/k!$ of the MacLaurin expansion for e^z. Szegö [11] established that $\hat{z}$ is a limit point of zeros of the sequence of normalized partial sums, $\{s_n(nz)\}_{n=0}^{\infty}$, if and only if

$$\hat{z} \in \{z : |ze^{1-z}| = 1,\ |z| \le 1\}. \tag{1.1}$$

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Moreover, Szegö [11] showed that $\hat{z}$ is a nontrivial limit point of zeros of the normalized remainder $\{e^{nz} - s_n(nz)\}_{n=1}^{\infty}$ if and only if

$$\hat{z} \in \{z: |ze^{1-z}| = 1, |z| \geq 1\}. \tag{1.2}$$

Saff and Varga [10] established sharp generalizations of Szegö's results to the asymptotic distribution of zeros and poles of more general sequences of the Padé approximants to e^z.

In this paper we consider the Padé approximants to $(1 + z)^{\alpha n+1}$, and locate the limit points of the zeros and poles of the Padé approximants. The Padé approximation to e^z is then a limiting case of the Padé approximations to $(1 + z)^{\alpha n+1}$ (see Section 6).

The approach is to obtain some general theorems (Theorems 2.4 and 2.5) concerning the zeros of the limit function of the integrals

$$\int_0^1 [t(1-t)f_z(t)]^n \, dt, \tag{1.3}$$

where $f_z(t)$ is a polynomial in t and analytic in z. These theorems can be applied to many other cases. As a consequence of these theorems, we not only determine the limit points of the zeros and poles of the Padé approximants to $(1 + z)^{\alpha n+1}$, but also obtain, for example,

Theorem. *The set of functions* $\{(1 + z)^{\alpha n} r_n(z)/s_n(z)\colon r_n(z), s_n(z) \in \pi_n\}_{n=1}^{\infty}$ *is dense in* $A(K)$, *the analytic functions on* K, *where* K *is an arbitrary compact subset of* R_3 *and not in any region strictly containing* R_3 *(where* R_3 *is as defined in Section* 3).

This can be thought of as a generalization to the complex plane of the now numerous results on denseness of incomplete polynomials on an interval.

In Section 2 we give the explicit formulas for the Padé approximants to $(1 + z)^{\alpha n+1}$ in the form of (1.3), and then prove our main theorems. The distribution of the limit points of the zeros and poles of the Padé approximants to $(1 + z)^{\alpha n+1}$ is established in Section 3.

One advantage of our method is that this procedure can be carried out automatically on a computer. So after some theoretical results are obtained the messy algebra of determining the limit curves and regions is entirely automatic (using Maple or some other symbolic algebra package, see Section 3).

We consider incomplete rationals and incomplete polynomials in Sections 4 and 5, respectively. In our last section, Section 6, we discuss the Padé approximation to e^z as a limiting case of the Padé approximation to $(1 + z)^{\alpha n+1}$.

2. The Main Theorems

In this section we discuss the Padé approximation to $(1 + z)^{\alpha n+1}$ at 0, and obtain the corresponding p, q, and error term explicitly in the following integral form:

$$\int_0^1 [t(1-t)f_z(t)]^n \, dt, \tag{2.1}$$

where $f_z(t)$ is a polynomial in t and analytic in z on some compact set K. In the second half of this section we establish a general theorem concerning the limit function of the above integral form (2.1) as $n \to \infty$. From this limit function, we determine the limit points of the zeros of the integral forms (2.1) as n goes to infinity.

Theorem 2.1. *For the* (m, n) *Padé approximation to* $(1 + z)^{\alpha n+1}$ *at* $0, \alpha > 0$, *we have*

(a) $$(1+z)^{\alpha n+1} - \frac{p_m(z)}{q_n(z)} = \frac{z^{m+n+1}\int_0^1 (1-t)^m t^n (1+tz)^{\alpha n-m}\,dt}{q_n(z)},$$

(b) $$p_m(z) = \int_0^1 (t-1)^n t^{\alpha n-m}(1+z-t)^m\,dt,$$

and

(c) $$q_n(z) = \int_0^1 (1-t)^m t^{\alpha n-m}(t(z+1)-1)^n\,dt.$$

Proof. We write

$$(1+z)^{\alpha n+1} - \frac{p_m(z)}{q_n(z)} = \frac{z^{m+n+1}e_{\alpha n-m}(z)}{q_n(z)}.$$

Then

$$[(1+z)^{\alpha n+1}q_n(z) - p_m(z)]^{(m+1)} = z^n T(z),$$

where $T(z)$ is a polynomial in z and $(1+z)^\alpha$. Also,

$$[(1+z)^{\alpha n+1}q_n(z) - p_m(z)]^{(m+1)} = (1+z)^{\alpha n-m}S(z),$$

where $S(z)$ is polynomial of degree n in z.

So we deduce that

(2.2) $$[(1+z)^{\alpha n+1}q_n(z) - p_m(z)]^{(m+1)} = Cz^n(1+z)^{\alpha n-m},$$

which implies

$$e_{\alpha n-m}(z) = \frac{(1+z)^{\alpha n+1}q_n(z) - p_m(z)}{z^{m+n+1}}$$

$$= C^*\int_0^1 (1-t)^m t^n (1+tz)^{\alpha n-m}\,dt.$$

In the last equality we used the following fact: if

$$G(z) = \frac{z^{m+1}}{m!}\int_0^1 (1-t)^m f(tz)\,dt,$$

then, given suitable smoothness of f,

$$G^{(m+1)}(z) = f(z).$$

On the other hand, from (2.2) we have

$$[(1+z)^{\alpha n+1}q_n(z)]^{(m+1)} = Cz^n(1+z)^{\alpha n-m}. \tag{2.3}$$

So integrating from -1 gives back the correct initial terms. Let $y = 1 + z$. From (2.3) we obtain

$$[y^{\alpha n+1}q_n(y-1)]^{(m+1)} = C(y-1)^n y^{\alpha n-m}.$$

Therefore,

$$y^{\alpha n+1}q_n(y-1) = C^* y^{m+1}\int_0^1 (1-t)^m (yt)^{\alpha n-m}(yt-1)^n\,dt$$

or

$$q_n(y-1) = C^*\int_0^1 (1-t)^m t^{\alpha n-m}(yt-1)^n\,dt,$$

which implies (c). (There is one free normalization constant.)

Now; consider $p_m(z)$, from (a) and (c) we can write

$$p_m(z) = (1+z)^{\alpha n+1}\int_0^1 (1-t)^m t^{\alpha n-m}(t(z+1)-1)^n\,dt$$
$$- z^{m+n+1}\int_0^1 (1-t)^m t^n (1+tz)^{\alpha n-m}\,dt.$$

Let $t = (s(1+z)-1)/z$, we can rewrite

$$z^{m+n+1}\int_0^1 (1-t)^m t^n (1+tz)^{\alpha n-m}\,dt$$
$$= z^{m+n+1}\int_{1/(1+z)}^1 \left(\frac{1+z}{z}\right)^m (1-s)^m \frac{1}{z^n}[s(1+z)-1]^n s^{\alpha n-m}(1+z)^{\alpha n-m}\frac{1+z}{z}\,dt$$
$$= (1+z)^{\alpha n+1}\int_{1/(1+z)}^1 (1-s)^m[s(1+z)-1]^n s^{\alpha n-m}\,ds.$$

Therefore,

$$p_m(z) = (1+z)^{\alpha n+1}\int_0^{1/(1+z)} (1-t)^m t^{\alpha n-m}(t(1+z)-1)^n\,dt$$
$$= (1+z)^{\alpha n+1}\int_0^1 \left(1-\frac{s}{1+z}\right)^m\left(\frac{s}{1+z}\right)^{\alpha n-m}(s-1)^n\frac{ds}{1+z}$$
$$= \int_0^1 (1+z-s)^m s^{\alpha n-m}(s-1)^n\,dt,$$

which is (b). ∎

If we let $m = cn$ and suppose that cn is an integer, we have the following corollary.

Corollary 2.2. *For the* (cn, n) *Padé approximation to* $(1+z)^{\alpha n+1}$ *at* 0, $\alpha > 0$, *we have*

(a) $$(1+z)^{\alpha n+1} - \frac{p_{cn}(z)}{q_n(z)} = \frac{z^{cn+n+1}\int_0^1 [(1-t)^c t(1+tz)^{\alpha-c}]^n\,dt}{q_n(z)},$$

(b) $$p_{cn}(z) = \int_0^1 [(t-1)t^{\alpha-c}(1+z-t)^c]^n\,dt,$$

and

(c) $$q_n(z) = \int_0^1 [(1-t)^c t^{\alpha-c}(t(1+z)-1)]^n\,dt.$$

The relations among $p_m(z)$, $q_n(z)$, and $e_{\alpha n-m}$ are considered in the next corollary for some specific α and c.

Corollary 2.3. *When* $c = 1$, *we have*

(2.4) $$(1+z)^n p_n\left(\frac{-z}{1+z}\right) = q_n(z).$$

If $\alpha - c = 1$, *then*

(2.5) $$(-1)^n e_{(\alpha-c)n}(-(1+z)) = q_n(z).$$

Proof. From Corollary 2.2, we have

$$p_n\left(\frac{-z}{1+z}\right) = \int_0^1 \left[(t-1)t^{\alpha-1}\left(1-\frac{z}{1+z}-t\right)\right]^n dt$$
$$= \left(\frac{1}{1+z}\right)^n \int_0^1 [(1-t)t^{\alpha-1}(t(1+z)-1)]^n\,dt,$$

which implies (2.4).

When $\alpha - c = 1$, from (a) of Corollary 2.2,

$$e_{(\alpha-c)n}(-(1+z)) = \int_0^1 [(1-t)^c t(1-t(1+z))]^n\,dt$$
$$= (-1)^n \int_0^1 [(1-t)^c t(t(1+z)-1)]^n\,dt,$$

which completes the proof of the corollary. ∎

Since $p_{cn}(z)$, $q_n(z)$, and $e_{(a-c)n}(z)$ can all be written in the integral form

$$\int_0^1 [t(1-t)f_z(t)]^n\, dt, \tag{2.6}$$

where $f_z(t)$ is a polynomial in both z and t, it is natural to investigate some properties of this integral form.

Theorem 2.4. *Let*

$$I_n = \int_0^1 [t(1-t)f(t)]^n\, dt = \int_0^1 [Q(t)]^n\, dt,$$

where $Q(t) = t(1-t)f(t)$ *is a polynomial of degree* N *in* t. *Let* $t_1, t_2, \ldots, t_{N-1}$ *be the* $N-1$ *zeros of* $Q'(t)$. *Suppose that*

$$|Q(t_i)| \neq |Q(t_j)|, \qquad i \neq j.$$

Then

$$\lim_{n\to\infty} I_n^{1/n} = arg(Q(t_i))|Q(t_i)| = Q(t_i) \qquad \textit{for some } i.$$

Proof. In the proof we use the method of steepest descent and a saddle-point argument (see [5] and for the complex analysis [1]). First, let us recall the real case (see p. 96, #198, of [6]). Suppose two functions $\varphi(x)$ and $g(x)$ are continuous and positive on the interval $[a, b]$. Then

$$\lim_{n\to\infty} \left\{\int_a^b \varphi(x)[g(x)]^n\, dx\right\}^{1/n} \tag{2.7}$$

exists and is equal to the maximum of $g(x)$ on $[a, b]$. (This is in fact a fairly easy exercise.)

Let us return to the proof of the theorem. Observe that from any point there is a downhill contour (in decreasing magnitude) that terminates at one of the zeros of $Q(t)$. Otherwise the path would end at a point of minimum modulus of $Q(t)$ other than a zero which is impossible. So descent is always possible (and not to $t = \infty$ because $Q(t)$ is a polynomial in this case).

Now suppose that A_i is a piece of arc of constant argument for $Q(t)$ through t_i from γ_i to δ_i. Then by the above observation we can connect γ_i to one zero and δ_i to another to form a contour B_i from one zero of $Q(t)$ to another in a descending fashion. Do this procedure for all the t_i. This forms no closed contours since, if it did, integrating around one of these is nonzero by a steepest descent argument but is zero by Cauchy's theorem.

Thus the contour $B_1, B_2, \ldots, B_{N-1}$ in some order must connect all N zeros of $Q(t)$ with exactly one link.

In particular there is a path from 0 to 1 via some of the saddle points of $Q(t)$ (not necessarily all), say t_{i_k}, $k = 1, 2, \ldots, r$, where $r \leq N-1$. We may suppose that

$$|Q(t_{i_1})| > |Q(t_{i_k})|, \qquad k = 2, \ldots, r. \tag{2.8}$$

Now we can apply the modified steepest descent argument in the real case (see p. 287, #198, of [6]) since, along A_{i_1}, $Q(t)$ has constant argument while, for the remainder of the B_{i_1}, the modulus of $Q(t)$ is less than $|Q(t_{i_1})|$ by the construction of B_{i_1}. Along other B_{i_k}, $k = 2, \ldots, r$, we know that the modulus of $Q(t)$ is less than $|Q(t_{i_1})|$ too by (2.8) and the constructions of B_{i_k}, $k = 2, \ldots, r$. This gives us the desired result. ∎

Theorem 2.4 is a pointwise version for z if $f_z(t)$ is a polynomial in t and analytic in z on some compact set K. From Theorem 2.4 we can prove the following uniform version of Theorem 2.4, which is the result we really need.

Theorem 2.5. *Let*

$$I_n(z) = \int_0^1 [t(1-t)f_z(t)]^n \, dt = \int_0^1 [Q_z(t)]^n \, dt, \tag{2.9}$$

where $Q_z(t) = t(1-t)f_z(t)$ is a polynomial in t and analytic in z on an open simply connected set U. Suppose

$$|Q_z(t_i(z))| \neq |Q_z(t_j(z))|$$

for any $i \neq j$, and any $z \in U$, where $t_i := t_i(z)$ are the zeros of the polynomial $(d/dt)Q_z(t)$ (which by the above assumption can be given so that each t_i is analytic on U). Then

(a) *$I_n(z)^{1/n}$ converges to a nonzero limit pointwise on U.*
(b) *$|I_n(z)|^{1/n}$ is uniformly bounded on compact subsets of U.*
(c) *$I_n(z)^{1/n}$ converges uniformly to a $Q_z(t_i(z))$ on compact subsets of U, and $Q_z(t_i(z))$ is analytic on U. Moreover, $Q_z(t_i(z)) \neq 0$ for all $z \in U$.*

Proof. (a) This is the content of Theorem 2.4.

(b) This is obvious from the definition of $I_n(z)$.

(c) Denote the open disk centered at z with radius ε as $D(z, \varepsilon)$, and the corresponding closed disk as $\bar{D}(z, \varepsilon)$. Now pick a $z_0 \in U$, then there is an i_0 such that

$$|Q_{z_0}(t_{i_0}(z_0))| > |Q_{z_0}(t_i(z_0))| \quad \text{for} \quad i \neq i_0.$$

Let

$$d = \min_{i \neq i_0} \{|Q_{z_0}(t_{i_0}(z_0))| - |Q_{z_0}(t_i(z_0))|\} > 0,$$

then, since $t_i(z)$ $(i = 1, \ldots, N-1)$ is a continuous function of z and $Q_z(t_i(z))$ is analytic, there is an ε_1 such that

$$\| Q_z(t_{i_0}(z)) \|_{C(\bar{D}(z_0, \varepsilon_1))} \geq V(z_0) - \frac{d}{4}$$

and

$$\| Q_z(t_i(z)) \|_{C(\bar{D}(z_0, \varepsilon_1))} \le V(z_0) - \tfrac{3}{4} d$$

for $i \neq i_0$ where

$$V(z_0) = |Q_{z_0}(t_{i_0}(z_0))|.$$

According to the proof of Theorem 2.4, there is a contour $A_{i_0}(z_0)$ through $t_{i_0}(z_0)$ from $\gamma_{i_0}(z_0)$ to $\delta_{i_0}(z_0)$ on which $Q_{z_0}(t)$ has constant argument. In addition, we choose $\gamma_{i_0}(z_0)$ and $\delta_{i_0}(z_0)$ such that

$$|Q_{z_0}(\gamma_{i_0}(z_0))| \le V(z_0) - \tfrac{3}{4} d$$

and

$$|Q_{z_0}(\delta_{i_0}(z_0))| \le V(z_0) - \tfrac{3}{4} d.$$

Denote the whole contour from 0 to 1 through $t_{i_0}(z_0)$ by $\Gamma(z_0)$ and the length of $\Gamma(z_0)$ by $l(z_0)$. Suppose the length of the part of $A_{i_0}(z_0)$ such that $|Q_{z_0}(t)| \ge V(z_0) - d/2$ is r.

Now, for $z \in D(z_0, \varepsilon_1)$, we construct the contour $\Gamma(z)$ in the same fashion, that is,

$$|Q_z(\gamma_{i_0}(z))| \le V(z_0) - \tfrac{3}{4} d$$

and

$$|Q_z(\delta_{i_0}(z))| \le V(z_0) - \tfrac{3}{4} d.$$

Choose $\varepsilon_2 > 0$ such that

$$l(z) \le l(z_0) + 1 \qquad \text{for} \quad z \in D(z_0, \varepsilon_2),$$

and $\varepsilon_3 > 0$ such that the length of the part of $A_{i_0}(z)$ with $|Q_z(t)| \ge V(z_0) - d/2$ is larger than or equal to $r/2$ for $z \in D(z_0, \varepsilon_3)$.

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, then, for $z \in D(z_0, \varepsilon)$, we have

$$\begin{aligned} |I_n(z)|^{1/n} &= \left| \int_0^1 [Q_z(t)]^n \, dt \right|^{1/n} \\ &= \left| \int_{\Gamma(z)} [Q_z(t)]^n \, dt \right|^{1/n} \\ &= \left| \int_{A_{i_0}(z)} [Q_z(t)]^n \, dt + \int_{\Gamma'(z)} [Q_z(t)]^n \, dt \right|^{1/n}, \end{aligned}$$

where $\Gamma'(z)$ is the part of $\Gamma(z)$ without $A_{i_0}(z)$.

From the choice of ε and the construction of $\Gamma(z)$, we have

$$\begin{aligned} |I_n(z)|^{1/n} &\ge \left| \int_{A_{i_0}(z)} [Q_z(t)]^n \, dt \right|^{1/n} \times \left[1 - \frac{\int_{\Gamma'(z)} |Q_z(t)|^n \, dt}{\left|\int_{A_{i_0}(z)} [Q_z(t)]^n \, dt\right|} \right]^{1/n} \\ &\ge \left[\frac{r}{2} \left(V(z_0) - \frac{d}{2} \right)^n \right]^{1/n} \left[1 - \frac{(l(z_0) + 1)(V(z_0) - \frac{3}{4} d)^n}{(r/2)(V(z_0) - d/2)^n} \right]^{1/n}. \end{aligned}$$

Therefore, there is an $n_0(z_0)$ which depends only on z_0 such that

$$|I_n(z)|^{1/n} > 0 \qquad \text{for} \quad z \in D(z_0, \varepsilon), \quad n \geq n_0(z_0).$$

Now consider any compact subset K of U. For any $z_0 \in K$, by the above argument, there are $\varepsilon(z_0)$ and $n(z_0)$ such that

$$|I_n(z)|^{1/n} > 0 \qquad \text{for} \quad z \in D(z_0, \varepsilon(z_0)), \quad n \geq n_0(z_0).$$

Thus, we can pick up finitely many z in K, say z_i, $i = 1, 2, \ldots, M$, such that

$$K \subset \bigcup_{i=1}^{M} D(z_i, \varepsilon(z_i)).$$

Let $n_0 = \max_{1 \leq i \leq M}\{n(z_i)\}$, then

$$|I_n(z)|^{1/n} > 0 \qquad \text{for} \quad z \in K, \quad n \geq n_0(z_0).$$

That is, $I_n(z)^{1/n}$ is analytic on K for $n \geq n_0(z_0)$ (in the sense that there is a well-defined analytic nth root).

From the above arguments, (a) and (b), and applying Vitali's theorem we know that $I_n(z)^{1/n}$ converges uniformly on compact subsets of U to an analytic function $Q_z(t_i(z))$. Now we can apply the uniqueness theorem, which implies that $I_n(z)^{1/n}$ must converge to the same $Q_z(t_i(z))$ on all compact subsets of U. From Hurwitz's theorem, we see that

$$Q_z(t_i(z)) \neq 0 \qquad \text{for all} \quad z \in U. \qquad ■$$

From Theorems 2.4 and 2.5, we have knowledge of the limit function of $I_n(z)^{1/n}$, $n = 1, 2, \ldots$. In fact, the limit function tells us more.

Corollary 2.6. *Let $I_n(z), f_z(t)$, and $Q_z(t)$ be as in Theorem 2.5. Suppose that, for each z, $Q_z(t)$ is a polynomial of degree N in t, and further that $Q_z(t)$ is analytic in z. Then the limit points of the zeros of $I_n(z)$ can only cluster on the curve*

$$\{z\colon |Q_z(t_i(z))| = |Q_z(t_j(z))|, \textit{for some } i \neq j\}$$

or at points where $Q_z(t_i(z)) = 0$, or at points where $Q_z(t_i(z))$ is not analytic.

Proof. Let U be an open and connected set which is disjoint from the curves and points stated in this corollary. Suppose S is a compact subset of U where $I_n(z)^{1/n}$ is analytic, which will be whenever the nth root is well defined and nonzero. Then by Theorems 2.4 and 2.5

$$I_n(z)^{1/n} \to Q_z(t_i) \neq 0 \qquad \text{as} \quad n \to \infty,$$

pointwise on S. Therefore, applying Vitali's theorem, $I_n(z)^{1/n}$ converges uniformly on any such compact subset of U to the nonzero analytic limit $Q_z(t_i(z))$. ■

From now on we call the curve

$$\{z: |Q_z(t_i(z))| = |Q_z(t_j(z))|, \text{ for some } i \neq j\}$$

the critical curve of $I_n(z)$.

3. Padé Approximation to $(1 + z)^{\alpha n+1}$

In this section we apply the results of Theorems 2.4 and 2.5, and their corollary to the Padé approximation to $(1 + z)^{\alpha n+1}$ at 0, and analyze the limiting location of the zeros of $p_{cn}(z)$, $q_n(z)$, *and* $e_{(\alpha-c)n}(z)$. In fact, all the procedures we discuss in this section can be executed automatically by computer. (This we did using Maple.)

First let us note the following facts. For $c = 1$, from Corollary 2.3, if we have knowledge of the distribution of the zeros of $q_n(z)$, then we know the distribution of the zeros of $p_n(z)$. Similarly, when $\alpha - c = 1$, we know the distribution of the zeros of $e_{(\alpha-c)n}(z)$ from that of $q_n(z)$. Since $I_n(z)^{1/n}$ converges uniformly on any compact subset $S \subset U$ to the nonzero analytic limit $Q_z(t_i(z))$, where $I_n(z)^{1/n}$ is analytic, in order to see which root of $(d/dt)(Q_z(t))\, I_n(z)^{1/n}$ goes to, it is sufficient, by analytic continuation, to check which root it will approach on a segment A of the real axis provided that $A \subset U$.

It is amusing to observe that the critical curves for $p_{cn}(z)$, $q_n(z)$, and $e_{(\alpha-c)n}(z)$ are all the same, essentially since we can write

$$p_{cn}(z) = (1 + t)^{\alpha n+1} \int_0^{1/(1+z)} [(1 - t)^c t^{\alpha-c}(t(1 + z) - 1)]^n \, dt, \tag{3.1}$$

$$q_n(z) = \int_0^1 [(1 - t)^c t^{\alpha-c}(t(1 + z) - 1)]^n \, dt, \tag{3.2}$$

and

$$e_{(\alpha-c)n}(z) = \frac{(1 + z)^{\alpha n+1}}{z^{cn+n+1}} \int_{1/(1+z)}^1 [(1 - t)^c t^{\alpha-c}(t(1 + z) - 1)]^n \, dt \tag{3.3}$$

from the proof of Theorem 2.1. Notice that

$$g_z(0) = g_z(1) = g_z\left(\frac{1}{1+z}\right) = 0,$$

where $g_z(t) = (1 - t)^c t^{\alpha-c}(t(1 + z) - 1)$. However, $p_{cn}(z)$, $q_n(z)$, and $e_{(\alpha-c)}(z)$ may pick up different branches of that critical curve as we will see later.

To illustrate the procedures, we consider the case $c = 1$. In this case we have

$$p_n(z) = \int_0^1 [(t - 1)t^{\alpha-1}(1 - t + z)]^n \, dt, \tag{3.4}$$

$$q_n(z) = \int_0^1 [(1 - t)t^{\alpha-1}(t(1 + z) - 1)]^n \, dt, \tag{3.5}$$

and

$$e_{(\alpha-1)n}(z) = \int_0^1 [(1-t)t(1+tz)^{\alpha-1}]^n \, dt. \tag{3.6}$$

Let $Q_z(t) = (1-t)t^{\alpha-1}(t(1+z)-1)$, then

$$Q_z(0) = Q_z(1) = Q_z\left(\frac{1}{1+z}\right) = 0$$

and

$$\frac{d}{dt} Q_z(t)|_{t=t_{1,2}(z)} = 0,$$

where

$$t_{1,2}(z) = \frac{\alpha(z+2) \pm \mu}{2(z+1)(1+\alpha)}, \tag{3.7}$$

$$\mu = (\alpha^2 z^2 + 4z + 4)^{1/2}.$$

Therefore, from Corollary 2.6 and the above observations, the critical curve for $p_n(z)$, $q_n(z)$, and $e_{(\alpha-1)n}(z)$ is

$$\{z: |Q_z(t_1(z))| = |Q_z(t_2(z))|\}, \tag{3.8}$$

which is

$$\left\{z: \left|\frac{\alpha z + 2z + 2 + \mu}{\alpha z + 2z + 2 - \mu}\right| \left|\frac{\alpha z - 2 - \mu}{\alpha z - 2 + \mu}\right| \left|\frac{\alpha z + 2\alpha - \mu}{\alpha z + 2\alpha + \mu}\right|^{\alpha-1} = 1\right\}, \tag{3.9}$$

where

$$\mu = (\alpha^2 z^2 + 4z + 4)^{1/2}.$$

From (3.9), it can be seen that the critical curve is always symmetric about the real axis for any α. The critical curves for $\alpha = 2$, $\alpha = 3$, $\alpha = 5$, and $\alpha = 8$ are shown in Figs. 1, 2, 3, and 4, respectively. In Figs. 5 and 6 we plot the zeros of $p_n(z)$ and $q_n(z)$ for $\alpha = 2$, $n = 20$ and $\alpha = 3$, $n = 10$, respectively. (Since the zeros are symmetric in the real axis we only plot the portion in $\{\text{Im}(z) \geq 0\}$.) We also plot the zeros of $e_{(\alpha-1)n}(z)$ for $\alpha = 3$, $n = 15$ in Fig. 7. These pictures indicate that the zeros of $p_n(z)$, $q_n(z)$, and $e_{(\alpha-1)n}(z)$, $n = 1, 2, \ldots$, are dense on the three different branches of the critical curve (3.8). Indeed, we can prove this fact.

Now we restrict our attention to the case $\alpha = 2$, $c = 1$. Then we have

$$q_n(z) = \int_0^1 [(1-t)t(t(1+z)-1)]^n \, dt \tag{3.10}$$

and the critical curve (3.9) is replaced by

$$\left\{z: \left|\frac{2z+1+\nu}{2z+1-\nu}\right| \left|\frac{z-1-\nu}{z-1+\nu}\right| \left|\frac{z+2-\nu}{z+2+\nu}\right| = 1\right\}, \tag{3.11}$$

where

$$\nu = (1 + z + z^2)^{1/2}.$$

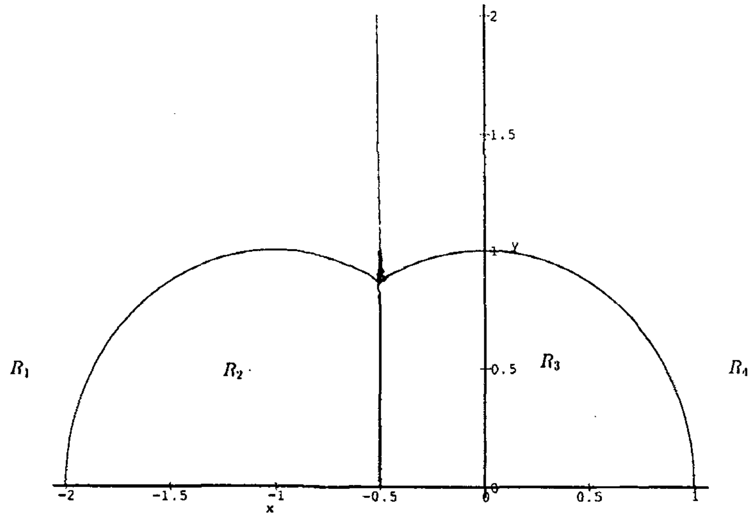

Fig. 1. Critical curve for $c = 1$, $\alpha = 2$.

To analyze which root $q_n(z)$ will pick up on the four regions bounded by (3.11) and the branch lines where v changes its branches (see Fig. 1), it is sufficient to consider the real segments contained in these four regions R_1, R_2, R_3, and R_4. We specify the four regions by R_1 contains $-\infty$, R_2 contains -1, R_3 contains 0, and R_4 contains ∞. From (3.10) and (3.7), we know that

$$Q_z(t) = (1 - t)t(t(1 + z) - 1) \tag{3.12}$$

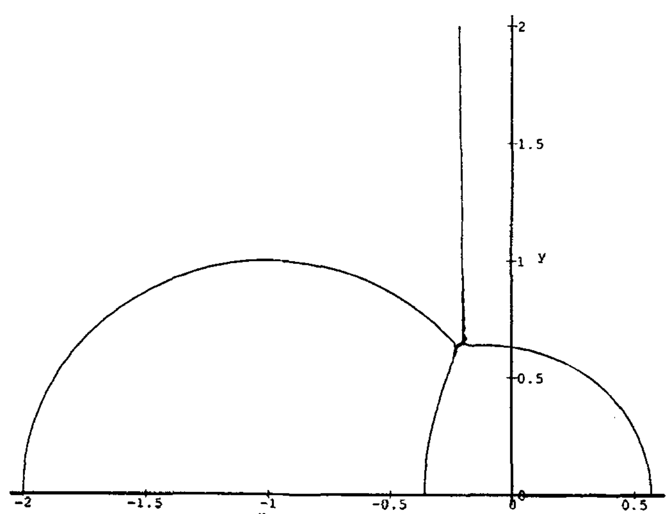

Fig. 2. Critical curve for $c = 1$, $\alpha = 3$.

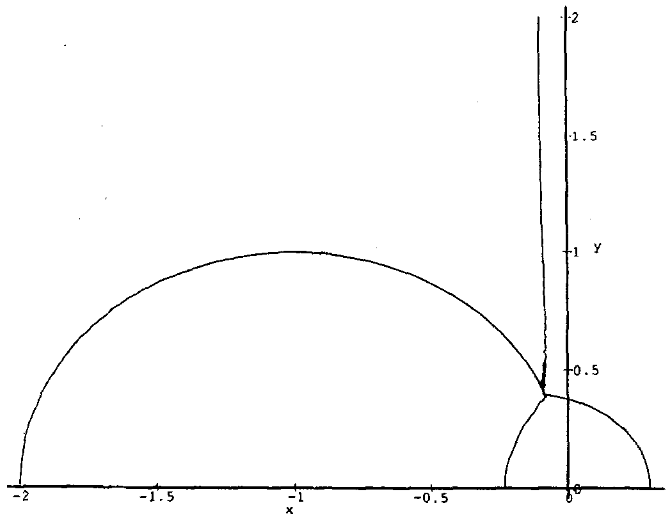

Fig. 3. Critical curve for $c = 1$, $\alpha = 5$.

and

$$t_{1,2}(z) = \frac{z + 2 \pm \nu}{3(1 + z)}. \tag{3.13}$$

Let $A_1 = \{x\colon x \text{ is real}, -5 \le x \le -3\} \subset R_1$, then

$$t_1(x) = \frac{x + 2 + \nu}{3(1 + x)} \notin [0, 1] \qquad \text{for} \quad x \in A_1$$

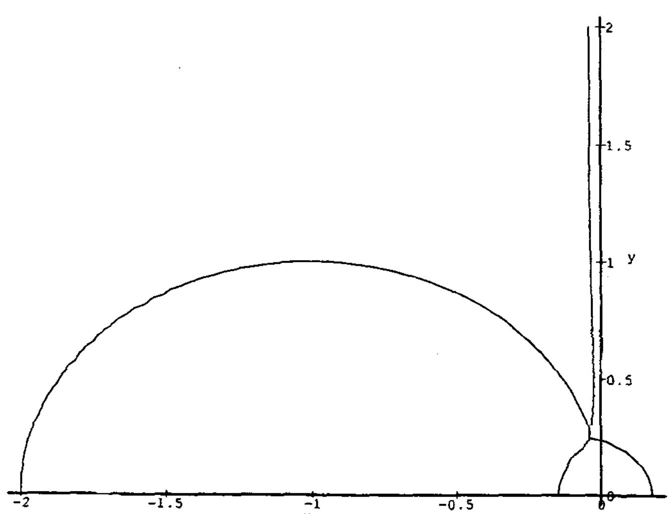

Fig. 4. Critical curve for $c = 1$, $\alpha = 8$.

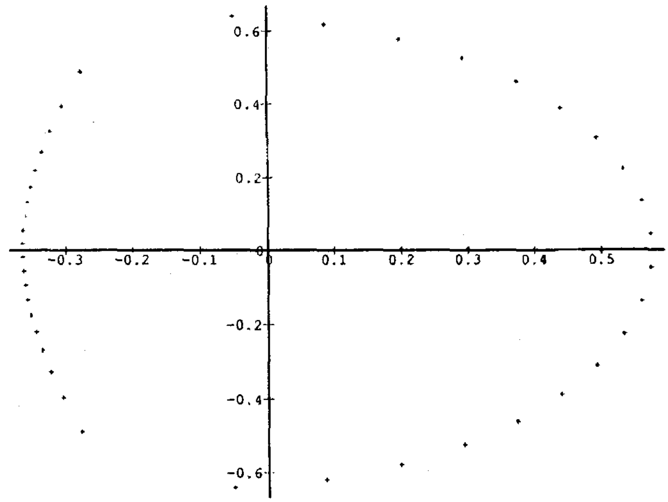

Fig. 5. Zeros of $p_n(z)$ and $q_n(z)$, $\alpha = 3$, $n = 20$.

and

$$t_2(x) = \frac{x + 2 - v}{3(1 + x)} \in [0, 1] \qquad \text{for} \quad x \in A_1.$$

Then by a saddle-point argument (see p. 287, #198, of [6])

$$\{q_n(x)\}^{1/n} \to Q_x(t_2(x))$$

pointwise on A_1. Now applying the argument we used in the proof of Theorem 2.5, we obtain that

$$\{q_n(z)\}^{1/n} \to Q_z(t_2(z)) \tag{3.14}$$

uniformly on compact subsets of R_1.

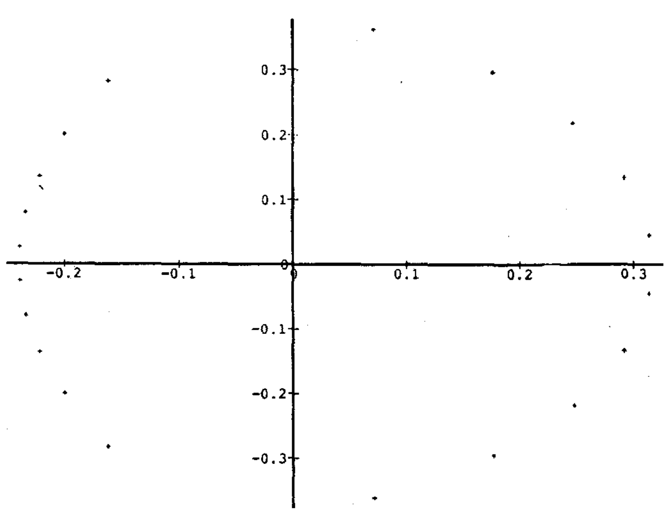

Fig. 6. Zeros of $p_n(z)$ and $q_n(z)$, $\alpha = 5$, $n = 10$.

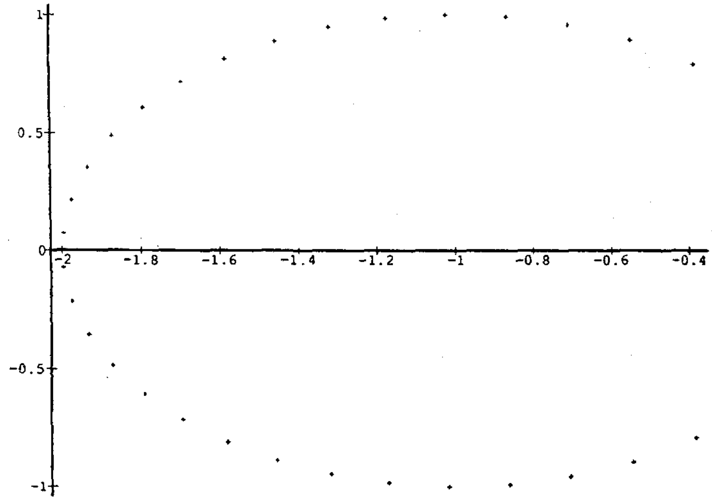

Fig. 7. Zeros of $e_{(\alpha-1)}(z)$, $\alpha = 3$, $n = 15$.

Let $A_2 = \{x: x \text{ is real}, -\frac{3}{2} \le x \le -\frac{11}{10}\} \subset R_2$, then

$$t_1(x) \notin [0, 1] \qquad \text{for} \quad x \in A_2$$

and

$$t_2(x) \in [0, 1] \qquad \text{for} \quad x \in A_2.$$

Therefore, $\{q_n(z)\}^{1/n}$ converges to $Q_z(t_2(z))$ uniformly on compact subsets of R_2.

Set $A_3 = \{x: x \text{ is real}, 0 \le x \le \frac{1}{2}\} \subset R_3$, then, for $x \in A_3$, we have $t_1(x) \in [0, 1]$ and $t_2(x) \in [0, 1]$. However,

$$|Q_x(t_1(x))| < |Q_x(t_2(x))|. \tag{3.15}$$

Thus, $\{q_n(z)\}^{1/n}$ converges to $Q_z(t_2(z))$ uniformly on compact subsets of R_3.

Set $A_4 = \{x: x \text{ is real}, 2 \le x \le 4\} \subset R_4$, then $t_1(x), t_2(x) \in [0, 1]$, for $x \in A_4$, but

$$|Q_x(t_1(x))| > |Q_x(t_2(x))|. \tag{3.16}$$

Therefore, $\{q_n(z)\}^{1/n}$ converges to $Q_z(t_1(z))$ uniformly on compact subsets of R_4.

From the above consideration, the uniqueness theorem, and Montel's theorem (see [1]) we can prove that the limit points of the zeros of $\{q_n(z)\}_{n=1}^{\infty}$ are dense on the branch B_3, which is the boundary between R_3 and R_4.

Therefore, we have proved

Theorem 3.1. *For $\alpha > 1$, $\{q_n(z)\}^{1/n}$ converges to $Q_z(t_2(z))$ uniformly on any compact subset of R_1, R_2, and R_3, and to $Q_z(t_1(z))$ uniformly on any compact subset of R_4. Moreover, the limit points of the zeros of $\{q_n(z)\}_{n=1}^{\infty}$ are dense on the branch B_3, which is the boundary between R_3 and R_4.*

Similarly, we can consider $p_n(z)$ and $e_{(\alpha-1)n}(z)$. The analogs for $p_n(z)$ and $e_{(\alpha-1)n}(z)$ are summarized in Theorems 3.2 and 3.3.

Theorem 3.2. *For $\alpha > 1$, $\{p_n(z)\}^{1/n}$ converges to $(1+z)^\alpha Q_z(t_1(z))$ uniformly on any compact subset of R_1 and R_2, and to $(1+z)^\alpha Q_z(t_2(z))$ uniformly on any compact subset of R_3 and R_4. Moreover, the limit points of the zeros of $\{p_n(z)\}_{n=1}^\infty$ are dense on the branch B_2, which is the boundary between R_2 and R_3.*

Theorem 3.3. *For $\alpha > 1$, $\{e_{(\alpha-1)n}\}^{1/n}$ converges to $(1+z)^\alpha Q_z(t_2(z))/z^2$ uniformly on any compact subset of R_1 and to $(1+z)^\alpha Q_z(t_1(z))/z^2$ uniformly on any compact subset of R_2, R_3, and R_4. Moreover, the limit points of the zeros of $\{e_n(z)\}_{n=1}^\infty$ are dense on the branch B_1, which is the boundary between R_1 and R_2.*

4. Incomplete Rationals

We have established the results on the zeros and poles of Padé approximants to $(1+z)^{\alpha n+1}$, and on the zeros of the Padé remainder in Section 3. In addition, we know that $\{p_n(z)\}^{1/n}$, $\{q_n(z)\}^{1/n}$, and $\{e_{(\alpha-1)n}(z)\}^{1/n}$ converge to some analytic functions on R_1, R_2, R_3, and R_4, respectively. In this section we apply these results to analyze the limit functions of $(1+z)^{\alpha n+1}q_n(z)/p_n(z)$ on R_1, R_2, R_3, and R_4. Then we prove that the collection of functions of the form $\{(1+z)^{\alpha n}r_n(z)/s_n(z)\}_{n=1}^\infty$ is dense on R_3 where $r_n(z)$ and $s_n(z)$ belong to π_n.

First we prove the following theorem.

Theorem 4.1. *Let $p_n(z)$, $q_n(z)$, and $e_{(\alpha-1)n}(z)$ be as in Corollary 2.2 in the case $c = 1$. Then we have that $(1+z)^{\alpha n+1}q_n(z)/p_n(z)$ converges*

(a) *to ∞ uniformly on any compact subset of R_1 and R_4,*
(b) *to 0 uniformly on any compact subset of R_2, and*
(c) *to 1 uniformly on any compact subset of R_3.*

Remark. Observe that 1 cannot be approximated on any region strictly larger than R_3 by Rouché's theorem, so R_3 is a natural maximal region of denseness.

Proof. (a) we consider R_1 first (similarly for R_4). Let K_1 be a compact subset of R_1. Then from (3.1), (3.2), (3.14), and Theorems 3.1 and 3.2, we have

$$\{p_n(z)\}^{1/n} \to (1+z)^\alpha Q_z(t_1(z))$$

and

$$\{q_n(z)\}^{1/n} \to Q_z(t_2(z))$$

uniformly on K_1.

Therefore,

$$\left|(1+z)^{\alpha n+1}\frac{q_n(z)}{p_n(z)}\right|^{1/n} \to \frac{|Q_z(t_2(z))|}{|Q_z(t_1(z))|} > 1+\varepsilon$$

by the definition of critical curve (see (3.8)) and nature of R_1, where $\varepsilon \in (0, 1)$

depends only on $K_1 \subset R_1$. Thus we conclude that $(1+z)^{\alpha n+1} q_n(z)/p_n(z)$ converges to ∞ uniformly on K_1.

Now let K_4 be a compact subset of R_4, then

$$\{p_n(z)\}^{1/n} \to (1+z)^\alpha Q_z(t_2(z))$$

and

$$\{q_n(z)\}^{1/n} \to Q_z(t_1(z))$$

uniformly on K_4, which implies

$$\left|(1+z)^{\alpha n+1} \frac{q_n(z)}{p_n(z)}\right|^{1/n} \to \frac{|Q_z(t_1(z))|}{|Q_z(t_2(z))|} > 1+\varepsilon \qquad \text{on } K_4.$$

The above inequality comes from the definition of R_4 and the compactness of K_4. Therefore, we complete the proof of (a).

(b) Let K_2 be a compact subset of R_2, then

$$\{p_n(z)\}^{1/n} \to (1+z)^\alpha Q_z(t_1(z))$$

and

$$\{q_n(z)\}^{1/n} \to Q_z(t_2(z))$$

uniformly on K_2, thus we have

$$\left|(1+z)^{\alpha n+1} \frac{q_n(z)}{p_n(z)}\right|^{1/n} \to \frac{|Q_z(t_1(z))|}{|Q_z(t_2(z))|} < 1-\varepsilon \qquad \text{on } K_2$$

(see (3.15)). Therefore, $(1+z)^{\alpha+1} q_n(z)/p_n(z)$ converges to 0 uniformly on K_2.

(c) Let K_3 be a compact subset of R_3. Since

$$(1+z)^{\alpha n+1} - \frac{p_n(z)}{q_n(z)} = \frac{z^{\alpha n+1} e_{(\alpha-1)n}(z)}{q_n(z)}, \tag{4.1}$$

from (3.1), (3.2), Theorems 3.2, and 3.3, we have

$$(1+z)^{\alpha n+1} \frac{q_n(z)}{p_n(z)} - 1 = z^{2n+1} \frac{e_{(\alpha-1)n}(z)}{p_n(z)}. \tag{4.2}$$

However,

$$\{p_n(z)\}^{1/n} \to (1+z)^\alpha Q_z(t_2(z))$$

and

$$\{e_{(\alpha-1)n}(z)\}^{1/n} \to (1+z)^\alpha \frac{Q_z(t_1(z))}{z^2}$$

uniformly on K_3. Thus, from (3.15), we obtain

$$\left|(1+z)^{\alpha n-1}\frac{q_n(z)}{p_n(z)}-1\right|^{1/n}=\left|z^{\alpha n+1}\frac{e_{(\alpha-1)n}(z)}{p_n(z)}\right|^{1/n}\to\frac{|Q_z(t_1(z))|}{|Q_z(t_z(z))|}<1-\varepsilon \qquad \text{on } K_3.$$

Therefore, we obtain the desired results. ■

Theorem 4.2. *$\{(1+z)^{\alpha n}r_n(z)/s_n(z): r_n(z), s_n(z)\in\pi_n\}_{n=1}^{\infty}$ is dense in $A(K)$ where K is an arbitrary compact subset of R_3.*

Proof. Note first that

$$T=\left\{f(z)=(1+z)^{\alpha n}\frac{r_n(z)}{s_n(z)}: r_n(z), s_n(z)\in\pi_n,\ n\in\mathbf{N}\right\}$$

is closed under addition, provided that we have the same degree and same denominator, and is also closed under multiplication.

Therefore, if $(1+z)^{\alpha n}r_n(z)/s_n(z)$ can approximate 1 and z with the same $s_n(z)$, they can approximate the linear form $az+b$. From the above observation we see that we can approximate any polynomial $p(z)$ since it can be written as

$$p(z)=\prod(a_kz+b_k).$$

Notice that the collection of all polynomials is dense in $A(K)$, thus

$$\{(1+z)^{\alpha n}r_n(z)/s_n(z): r_n(z), s_n(z)\in\pi_n\}_{n=1}^{\infty}$$

is dense in $A(K)$ provided that $(1+z)^{\alpha n}r_n(z)/s_n(z)$ can approximate 1 and z with the same denominator.

Let K be an arbitrary compact subset of R_3. We choose a rational number $\delta>0$ small enough such that K is a subset of R_3' corresponding to $\alpha'=\alpha(1+\delta)$. Note that from (3.8) or (3.9) we know that the critical curve is a continuous function of α and $R_3'\subset R_3$.

From Theorem 4.1, we have $p_n(z)$ and $q_n(z)$ for $\alpha'=\alpha(1+\delta)$ such that $(1+z)^{\alpha(1+\delta)n+1}q_n(z)/p_n(z)$ converges uniformly to 1 on K. Now we choose $p_{[\delta n]}(z)$, $q_{[\delta n]}(z)$, and $\bar{q}_{[\delta n]}(z)\in\pi_{[\delta n]}$ such that

$$\frac{q_{[\delta n]}(z)}{p_{[\delta n]}(z)}\to 1+z,\qquad \frac{\bar{q}_{[\delta n]}(z)}{p_{[\delta n]}(z)}\to z(1+z)$$

uniformly on K. We have

$$(1+z)^{\alpha(1+\delta)n+1}\frac{q_n(z)}{p_n(z)}\frac{q_{[\delta n]}(z)}{p_{[\delta n]}(z)}\to 1+z \tag{4.3}$$

and

$$(1+z)^{\alpha(1+\delta)n+1}\frac{q_n(z)}{p_n(z)}\frac{\bar{q}_{[\delta n]}(z)}{p_{[\delta n]}(z)}\to z(1+z) \tag{4.4}$$

uniformly on K. That is

(4.5)
$$(1+z)^{\alpha(1+\delta)n}\frac{q^*_{n+[\delta n]}(z)}{p^*_{n+[\delta n]}(z)} \to 1$$

and

(4.6)
$$(1+z)^{\alpha(1+\delta)n}\frac{\bar{q}^*_{n+[\delta n]}(z)}{p^*_{n+[\delta n]}(z)} \to z$$

uniformly on K where $p^*_{n+[\delta n]}(z) = p_n(z)p_{[\delta n]}(z)$, $q^*_{n+[\delta n]}(z) = q_n(z)q_{[\delta n]}(z)$, and $\bar{q}^*_{n+[\delta n]}(z) = q_n(z)\bar{q}_{[\delta n]}(z)$.

From (4.5) and (4.6) we know that $(1+z)^{\alpha(1+\delta)n}(a\bar{q}^*_{n+[\delta n]}(z) + bq^*_{n+[\delta n]}(z))/p^*_{n+[\delta n]}(z)$ converges to $az+b$ uniformly on K, which completes the proof of the theorem. ■

5. Incomplete Polynomials

If we let $c = 0$, then instead of incomplete rationals we have incomplete polynomials. (For a discussion of approximation by incomplete polynomials, applications, and the relations among Padé approximants, incomplete polynomials, and orthogonal polynomials, see [3] and [7] and the references therein.)

From Corollary 2.2 and (3.3) we have

(5.1)
$$p_0(z) = \int_0^1 [(t-1)t^\alpha]^n\, dt,$$

(5.2)
$$q_n(z) = \int_0^1 [t^\alpha(t(1+z)-1)]^n\, dt,$$

and

(5.3)
$$e_{\alpha n}(z) = \frac{(1+z)^{\alpha n+1}}{z^{n+1}}\int_{1/(1+z)}^1 [t^\alpha(t(1+z)-1)]^n\, dt.$$

Let $R_z(t) = t^\alpha(t(1+z)-1)$, then

(5.4)
$$R_z(0) = R_z\left(\frac{1}{1+z}\right) = 0.$$

Since we do not have the factor $(1-t)$ in $R_z(t)$, we cannot apply Theorems 2.4 and 2.5 to $q_n(z)$ and $e_{\alpha n}(z)$ directly. However, since $R_z(t)$ is a polynomial in both t and z, and has exactly one nontrivial critical point $t^* = \alpha/[(1+\alpha)(1+z)]$, by the argument in Theorem 2.4, there is a contour B from 0 to $1/(1+z)$, and a downhill contour which starts at 1, and terminates at 0 or $1/(1+z)$. Therefore, there are contours that connect 0 and 1 (for $q_n(z)$) or $1/(1+z)$ and 1 (for $e_{\alpha n}(z)$).

From this observation, and modifying the proofs of Theorems 2.4, 2.5, and Corollary 2.6, we have

Theorem 5.1. *Let $q_n(z)$ and $e_{\alpha n}(z)$ be as stated in (5.2) and (5.3). Then $q_n(z)$ and $e_{\alpha n}(z)$ have the same critical curve*

$$\{z: |R_z(t^*)| = |R_z(1)|\}, \tag{5.5}$$

where $R_z(t) = t^\alpha(t(1+z)-1)$ and $t^* = \alpha/[(1+\alpha)(1+z)]$. That is, the limit points of the zeros of $q_n(z)$ or $e_{\alpha n}(z)$ can only cluster on the curve (5.5). (Note $R_z(1) = z$.)

We can write (5.5) explicitly:

$$\left\{z: |z(1+z)^\alpha| = \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\right\}. \tag{5.6}$$

Figure 8 is the critical curve (5.6) when $\alpha = 2$. By almost identical arguments to those in Section 3, the following theorems can be proved. Note that this time we do not have any branch lines.

Theorem 5.2. *$\{q_n(z)\}^{1/n}$ converges to $R_z(1)$ uniformly on any compact subset of R_1 and R_2, and to $R_z(t^*)$ uniformly on any compact subset of R_3. Moreover, the limit points of the zeros of $\{q_n(z)\}_{n=1}^\infty$ are dense on the branch B_2, which is the boundary between R_1 and R_3.*

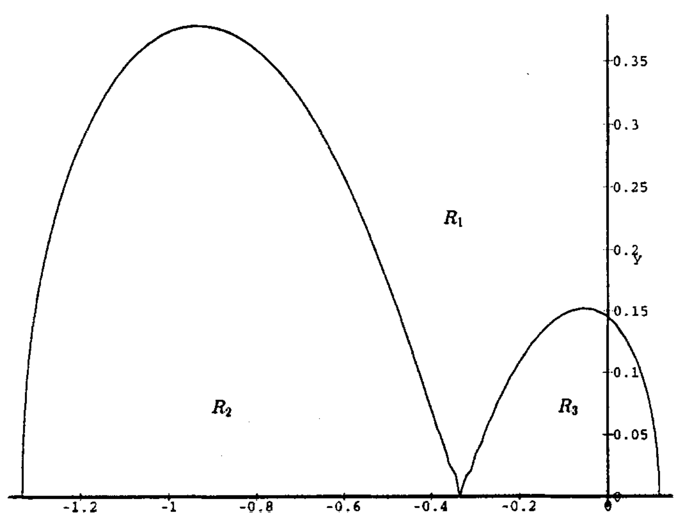

Fig. 8. Critical curve for $c = 0$, $\alpha = 2$.

Theorem 5.3. *$\{e_{\alpha n}(z)\}^{1/n}$ converges to $(1+z)^\alpha R_z(1)/z$ uniformly on any compact subset of R_1 and R_3, and to $(1+z)^\alpha R_z(t^*)/z$ uniformly on any compact subset of R_2. Moreover, the limit points of the zeros of $\{e_{\alpha n}(z)\}_{n=1}^{\infty}$ are dense on the branch B_1, which is the boundary between R_1 and R_2.*

The analog of Theorem 4.2 is the following:

Theorem 5.4. *$\{(1+z)^{\alpha n}p_n(z): p_n(z)\in\pi_n\}_{n=1}^{\infty}$ is dense in $A(K)$ where K is an arbitrary compact subset of R_3.*

6. Padé Approximation to e^z

In this section we consider the Padé approximation to e^z. In a sequence of papers [8]–[10] Saff and Varga examined the Padé approximation to e^z in detail. The purpose of this section is to observe that this is the limiting case of the Padé approximation to $(1+z)^{\alpha n+1}$. We verify this as follows.

From Corollary 2.2 and using the substitution $t = 1 - s/\alpha$, we can write

$$p_{cn}(z) = \int_0^1 [(t-1)t^{\alpha-c}(1+z-t)^c]^n\,dt = (-1)^n\left(\frac{1}{\alpha}\right)^{cn+n+1}\int_0^\alpha\left[s\left(1-\frac{s}{\alpha}\right)^{\alpha-c}(\alpha z+s)^c\right]^n ds. \tag{6.1}$$

Similarly, we have

$$q_n(z) = (-1)^n\left(\frac{1}{\alpha}\right)^{cn+n+1}\int_0^\alpha\left[t^c\left(1-\frac{t}{\alpha}\right)^{\alpha-c}((1+z)t-\alpha z)\right]^n dt. \tag{6.2}$$

Therefore, (a) of Corollary 2.2 can be written as

$$(1+z)^{\alpha n+1} - \frac{\int_0^\alpha [t(1-t/\alpha)^{\alpha-c}(\alpha z+t)^c]^n\,dt}{\int_0^\alpha [t^c(1-t/\alpha)^{\alpha-c}((1+z)t-\alpha z)]^n\,dt} = \frac{(-1)^n(\alpha z)^{cn+n+1}\int_0^1 [(1-t)^c t(1+tz)^{\alpha-c}]^n\,dt}{\int_0^\alpha [t^c(1-t/\alpha)^{\alpha-c}((1+z)t-\alpha z)]^n\,dt}. \tag{6.3}$$

Let $z = y/\alpha$, and next let $\alpha\to\infty$, then from (6.3), we obtain

$$e^{ny} - \frac{\int_0^\infty [te^{-t}(t+y)^c]^n\,dt}{\int_0^\infty [t^c e^{-t}(t-y)]^n\,dt} = \frac{(-1)^n y^{cn+n+1}\int_0^1 [(1-t)^c te^{ty}]^n\,dt}{\int_0^\infty [t^c e^{-t}(t-y)]^n\,dt}, \tag{6.4}$$

which is exactly the Padé approximations to e^z (see [10]).

From (3.8) and (3.9), we have the critical curve for the Padé approximants to $(1+z)^{\alpha n+1}$, which is

$$\left\{z: \left|\frac{\alpha z+2z+2-\mu}{\alpha z+2z+2+\mu}\right| \left|\frac{\alpha z-2+\mu}{\alpha z-2-\mu}\right| \left|\frac{\alpha z+2\alpha+\mu}{\alpha z+2\alpha-\mu}\right|^{\alpha-1}=1\right\}, \tag{6.5}$$

where $\mu=(\alpha^2 z^2+4z+4)^{1/2}$.

Set $\alpha z=y$, and then let $\alpha\to\infty$, from (6.5) we have

$$\left\{y: \left|\frac{2-\sqrt{y^2+4}}{2+\sqrt{y^2+4}}\right| |e^{\sqrt{y^2+4}}|=1\right\}. \tag{6.6}$$

Let $y=2x$, then we can rewrite (6.6) as

$$\left\{x: \left|\frac{xe^{\sqrt{x^2+1}}}{1+\sqrt{x^2+1}}\right|=1\right\},$$

which is exactly the critical curve for the Padé approximates to e^z with $\sigma=1$ in [10]. This limiting argument, however, requires some careful justification.

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