SHARP EXTENSIONS OF BERNSTEIN'S INEQUALITY TO RATIONAL SPACES

Peter Borwein and Tamás Erdélyi

Department of Mathematics
and Statistics
Simon Fraser University
Burnaby, B.C.
Canada V5A 1S6

Abstract. Sharp extensions of some classical polynomial inequalities of Bernstein are established for rational function spaces on the unit circle, on $K := \mathbb{R} \pmod{2\pi}$, on $[-1, 1]$ and on $\mathbb{R}$. The key result is the establishment of the inequality

$$|f'(z_0)| \leq \max\left\{ \sum_{\substack{j=1 \\ |a_j|>1}}^{n} \frac{|a_j|^2 - 1}{|a_j - z_0|^2}, \quad \sum_{\substack{j=1 \\ |a_j|<1}}^{n} \frac{1 - |a_j|^2}{|a_j - z_0|^2} \right\} \|f\|_{\partial D}$$

for every rational function $f = p_n/q_n$, where p_n is a polynomial of degree at most n with complex coefficients and

$$q_n(z) = \prod_{j=1}^{n} (z - a_j)$$

with $|a_j| \neq 1$ for each j, and for every $z_0 \in \partial D$, where $\partial D := \{z \in \mathbb{C} : |z| = 1\}$. The above inequality is sharp at every $z_0 \in \partial D$.

1. Introduction, Notation.

We denote by $\mathcal{P}_n^r$ and $\mathcal{P}_n^c$ the sets of all algebraic polynomials of degree at most n with real or complex coefficients, respectively. The sets of all trigonometric polynomials of degree at most n with real or complex coefficients, respectively, are denoted by $\mathcal{T}_n^r$ and $\mathcal{T}_n^c$. We will use the notation

$$\|f\|_A = \sup_{z \in A} |f(z)|$$

for continuous functions f defined on A. Let

$$D := \{z \in \mathbb{C} : |z| \leq 1\},$$
$$\partial D := \{z \in \mathbb{C} : |z| = 1\}$$

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and

$$K := \mathbb{R}\ (\mathrm{mod} 2\pi).$$

The classical inequalities of Bernstein [1] state that

$$\begin{aligned} |p'(z_0)| &\leq n\|p\|_{\partial D}, & p \in \mathcal{P}_n^c, \quad z_0 \in \partial D, \\ |t'(\theta_0)| &\leq n\|t\|_K, & t \in \mathcal{T}_n^c, \quad \theta_0 \in K, \\ |p'(x_0)| &\leq \frac{n}{\sqrt{1-x_0^2}}\|p\|_{[-1,1]}, & p \in \mathcal{P}_n^c, \quad x_0 \in (-1,1). \end{aligned}$$

Proofs of the above inequalities may be found in almost every book on approximation theory, see [4], [5], [6] or [8] for instance. An extensive study of Markov- and Bernstein-type inequalities is presented in [3] and [7].

In this paper we study the rational function spaces:

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n; \partial D) := \left\{ \frac{p_n(z)}{\prod_{j=1}^n (z - a_j)} : p_n \in \mathcal{P}_n^c \right\}$$

on ∂D with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \partial D$;

$$\mathcal{T}_n^c(a_1, a_2, \dots, a_{2n}; K) := \left\{ \frac{t_n(\theta)}{\prod_{j=1}^{2n} \sin((\theta - a_j)/2)} : t_n \in \mathcal{T}_n^c \right\}$$

on K with $\{a_1, a_2, \dots, a_{2n}\} \subset \mathbb{C} \setminus \mathbb{R}$;

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n; [-1,1]) := \left\{ \frac{p_n(x)}{\prod_{j=1}^n (x - a_j)} : p_n \in \mathcal{P}_n^c \right\}$$

on $[-1,1]$ with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus [-1,1]$;

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n; \mathbb{R}) := \left\{ \frac{p_n(x)}{\prod_{j=1}^n (x - a_j)} : p_n \in \mathcal{P}_n^c \right\}$$

on $\mathbb{R}$ with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$, and

$$\mathcal{P}_n^r(a_1, a_2, \dots, a_n; \mathbb{R}) := \left\{ \frac{p_n(x)}{\prod_{j=1}^n |x - a_j|} : p_n \in \mathcal{P}_n^r \right\}$$

on $\mathbb{R}$ with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$.

The spaces

$$\mathcal{T}_n^r(a_1, a_2, \dots, a_{2n}; K) := \left\{ \frac{t_n(\theta)}{\prod_{j=1}^{2n} |\sin((\theta - a_j)/2)|} : t_n \in \mathcal{T}_n^r \right\}$$

on K with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$ and

$$\mathcal{P}_n^r(a_1, a_2, \dots, a_n; [-1,1]) := \left\{ \frac{p_n(x)}{\prod_{j=1}^{n} |x - a_j|} : p_n \in \mathcal{P}_n^r \right\}$$

on $[-1,1]$ with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus [-1,1]$ have been studied in [2] and [3], and the sharp Bernstein-Szegő type inequalities

$$f'(\theta_0)^2 + \widetilde{B}_n(\theta_0)^2 \, f(\theta_0)^2 \le \widetilde{B}(\theta_0)^2 \|f\|_K^2, \qquad \theta_0 \in K$$

for every $f \in \mathcal{T}_n^r(a_1, a_2, \dots, a_{2n}; K)$ with

$$(a_1, a_2, \dots, a_{2n}) \subset \mathbb{C} \setminus \mathbb{R}, \qquad \operatorname{Im}(a_j) > 0, \quad j = 1, 2, \dots, 2n$$

and

$$(1 - x_0^2) f'(x_0)^2 + B_n(x_0)^2 f(x_0)^2 \le B_n(x_0)^2 \|f\|_{[-1,1]}^2, \qquad x_0 \in (-1,1)$$

for every $f \in \mathcal{P}_n^r(a_1, a_2, \dots, a_n; [-1,1]))$ with

$$\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus [-1,1]$$

have been proved, where

$$\widetilde{B}_n(\theta) := \frac{1}{2} \sum_{j=1}^{2n} \frac{1 - |e^{ia_j}|^2}{|e^{ia_j} - e^{i\theta}|^2}, \qquad \theta \in K,$$

and

$$B_n(x) := \operatorname{Re} \left(\sum_{j=1}^{n} \frac{\sqrt{a_j^2 - 1}}{a_j - x} \right), \qquad x \in [-1,1],$$

with the choice of $\sqrt{a_j^2 - 1}$ determined by

$$\left| a_j - \sqrt{a_j^2 - 1} \right| < 1.$$

These inequalities give sharp upper bound for $|f'(\theta)|$ and $|f'(x_0)|$ only at n points in K and $[-1,1]$, respectively. In this paper we establish Bernstein-type inequalities for the spaces

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n, \partial D) \quad \text{and} \quad \mathcal{T}_n^c(a_1, a_2, \dots, a_{2n}; K)$$

which are sharp at every $z \in \partial D$ and $\theta \in K$, respectively. An essentially sharp Bernstein-type inequality is also established for the space

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n; [-1,1]).$$

A Bernstein-type inequality of Russak [7] is extended to the spaces

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n; \mathbb{R}),$$

and a Bernstein-Szegő type inequality is established for the spaces

$$\mathcal{P}_n^r(a_1, a_2, \dots, a_n; \mathbb{R}).$$

For a polynomial

$$q_n(z) = c \prod_{j=1}^{n} (z - a_j), \qquad 0 \neq c \in \mathbb{C}, \qquad a_j \in \mathbb{C},$$

we define

$$q_n^*(z) = \overline{c} \prod_{j=1}^{n} (1 - \overline{a}_j z) = z^n \overline{q}_n(z^{-1}).$$

It is well-known, and simple to check, that

$$|q_n(z)| = |q_n^*(z)|, \qquad z \in \partial D.$$

We also define the Blaschke products

$$S_n(z) := \prod_{j=1}^{n} \frac{1 - \overline{a}_j z}{z - a_j}$$

associated with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \partial D$, and

$$\widetilde{S}_n(z) := \prod_{j=1}^{n} \frac{z - \overline{a}_j}{z - a_j}$$

associated with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$.

2. New Results.

Theorem 1. *Let* $\{a_1, a_2, \ldots, a_n\} \subset \mathbb{C} \setminus \partial D$. *Then*

$$|f'(z_0)| \le \max\left\{ \sum_{\substack{j=1 \\ |a_j|>1}}^{n} \frac{|a_j|^2 - 1}{|a_j - z_0|^2}, \sum_{\substack{j=1 \\ |a_j|<1}}^{n} \frac{1 - |a_j|^2}{|a_j - z_0|^2} \right\} \|f\|_{\partial D}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \ldots, a_n; \partial D)$ *and* $z_0 \in \partial D$. *If the first sum is not less than the second sum for a fixed* $z_0 \in \partial D$, *then equality holds for* $f = c\,S_n^+$, $c \in \mathbb{C}$, *where* S_n^+ *is the Blaschke product associated with those* a_j *for which* $|a_j| > 1$. *If the first sum is not greater than the second sum for a fixed* $z_0 \in \partial D$, *then equality holds for* $f = c\,S_n^-$, $c \in \mathbb{C}$, *where* S_n^- *is the Blaschke product associated with those* a_j *for which* $|a_j| < 1$.

Theorem 2. *Let* $\{a_1, a_2, \ldots, a_{2n}\} \subset \mathbb{C} \setminus \mathbb{R}$. *Then*

$$|f'(\theta_0)| \le \max\left\{ \sum_{\substack{j=1 \\ \mathrm{Im}(a_j)<0}}^{2n} \frac{|e^{ia_j}|^2 - 1}{|e^{ia_j} - e^{i\theta_0}|^2}, \sum_{\substack{j=1 \\ \mathrm{Im}(a_j)>0}}^{2n} \frac{1 - |e^{ia_j}|^2}{|e^{ia_j} - e^{i\theta_0}|^2} \right\} \|f\|_K$$

for every $f \in \mathcal{T}_n^c(a_1, a_2, \ldots, a_{2n}; K)$ *and* $\theta_0 \in K$. *If the first sum is not less than the second sum for a fixed* $\theta_0 \in K$, *then equality holds for* $f(\theta) = cS_{2n}^+(e^{i\theta})$, $c \in \mathbb{C}$. *If the first sum is not greater than the second sum for a fixed* $\theta_0 \in K$, *then equality holds for* $f(\theta) = cS_{2n}^-(e^{i\theta})$, $c \in \mathbb{C}$. S_{2n}^+ *and* S_{2n}^- *associated with* $\{e^{ia_1}, e^{ia_2}, \ldots, e^{ia_{2n}}\}$ *are defined as in Theorem 1.*

Theorem 3. *Let* $\{a_1, a_2, \ldots, a_n\} \subset \mathbb{C} \setminus [-1, 1]$ *and*

$$c_j := a_j - \sqrt{a_j^2 - 1}, \qquad |c_j| < 1$$

with the choice of root in $\sqrt{a_j^2 - 1}$ *determined by* $|c_j| < 1$. *Then*

$$|f'(x_0)| \le \frac{1}{\sqrt{1 - x_0^2}} \max\left\{ \sum_{j=1}^{n} \frac{|c_j|^{-2} - 1}{|c_j^{-1} - z_0|^2}, \quad \sum_{j=1}^{n} \frac{1 - |c_j|^2}{|c_j - z_0|^2} \right\} \|f\|_{[-1,1]}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \ldots, a_n; [-1, 1])$ *and* $x_0 \in (-1, 1)$, *where* z_0 *is defined by*

$$z_0 := x_0 + i\sqrt{1 - x_0^2}, \qquad x_0 \in (-1, 1).$$

Note that

$$B_n(x_0) = \mathrm{Re}\left(\sum_{j=1}^{n} \frac{\sqrt{a_j^2 - 1}}{a_j - x_0} \right) = \sum_{j=1}^{n} \frac{1 - |c_j|^2}{|c_j - z_0|^2}, \qquad x_0 \in (-1, 1).$$

Our next result extends an inequality established by Russak [7] to wider families of rational functions.

Theorem 4. *Let* $\{a_1, a_2, \ldots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$. *Then*

$$|f'(x_0)| \leq \max \left\{ \sum_{\substack{j=1 \\ \operatorname{Im}(a_j)>0}}^{n} \frac{2\,|\operatorname{Im}(a_j)|}{|x_0 - a_j|^2}, \sum_{\substack{j=1 \\ \operatorname{Im}(a_j)<0}}^{n} \frac{2\,|\operatorname{Im}(a_j)|}{|x_0 - a_j|^2} \right\} \|f\|_{\mathbb{R}}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \ldots, a_n; \mathbb{R})$ *and* $x_0 \in \mathbb{R}$. *If the first sum is not less than the second sum for a fixed* $x_0 \in \mathbb{R}$, *then equality holds for* $f = c\tilde{S}_n^+$, $c \in \mathbb{C}$, *where* $\tilde{S}_n^+$ *is the Blaschke product associated with the poles* a_j *lying in the upper half-plane*

$$H^+ := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}.$$

If the first sum is not greater than the second sum for a fixed $x_0 \in \mathbb{R}$, *then equality holds for* $f = c\tilde{S}_n^-$, $c \in \mathbb{C}$, *where* $\tilde{S}_n^-$ *is the Blaschke product associated with the poles* a_j *lying in the lower half-plane*

$$H^- := \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}.$$

Our last result is a Bernstein-Szegő type inequality for

$$\mathcal{P}_n^r(a_1, a_2, \ldots, a_{2n}; \mathbb{R})$$

which follows from the Bernstein-Szegő type inequality for

$$\mathcal{P}_n^r(a_1, a_2, \ldots, a_n; [-1, 1])$$

mentioned in the introduction.

Theorem 5. *Let*

$$\{a_1, a_2, \ldots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}, \qquad \operatorname{Im}(a_j) > 0, \quad j = 1, 2, \ldots, n.$$

Then

$$f'(x_0)^2 + \widehat{B}_n(x_0)^2 f(x_0)^2 \leq \widehat{B}_n(x_0)^2 \|f\|_{\mathbb{R}}^2, \qquad x_0 \in \mathbb{R},$$

for every $f \in \mathcal{P}_n^r(a_1, a_2, \ldots, a_n; \mathbb{R})$, *where*

$$\widehat{B}_n(x) := \sum_{j=1}^{n} \frac{\operatorname{Im}(a_j)}{|x - a_j|^2}, \qquad x \in \mathbb{R}.$$

We remark that equality holds in Theorem 5 if and only if x_0 is a maximum point of f (that is, $f(x_0) = \pm\|f\|_{\mathbb{R}}$) or f is a "Chebyshev polynomial" for the space $\mathcal{P}_n^r(a_1, a_2, \ldots, a_n; \mathbb{R})$ which can be explicitly expressed by using the results of [2] and [3].

Note that Bernstein's classical inequalities are contained in Theorem 1, 2, and 3 as limiting cases, by taking

$$\{a_1^{(k)}, a_2^{(k)}, \dots, a_n^{(k)}\} \subset \mathbb{C} \setminus D$$

in Theorems 1 and 3 so that $\lim_{k\to\infty} |a_j^{(k)}| = \infty$ for each $j = 1, 2, \dots, n$, and by taking

$$\{a_1^{(k)}, a_2^{(k)}, \dots, a_{2n}^{(k)}\} \subset \mathbb{C} \setminus \mathbb{R}$$

in Theorem 2 so that $a_{n+j}^{(k)} = \overline{a}_j^{(k)}$ and $\lim_{k\to\infty} |\mathrm{Im}(a_j^{(k)})| = \infty$ for each $j = 1, 2, \dots, n$. Further results can be obtained as limiting cases by fixing $a_1, a_2, \dots, a_m$, $1 \le m \le n$, in Theorems 1 and 3, and by taking

$$\{a_1, a_2, \dots, a_m, a_{m+1}^{(k)}, a_{m+2}^{(k)}, \dots, a_n^{(k)}\} \subset \mathbb{C} \setminus D$$

so that $\lim_{k\to\infty} |a_j^{(k)}| = \infty$ for each $j = m+1, m+2, \dots, n$. One may also fix the poles $a_1, a_2, \dots, a_m$, $a_{n+1}, a_{n+2}, \dots, a_{n+m}$, $1 \le m \le n$, in Theorem 2 and take

$$\{a_1, \dots, a_m, a_{m+1}^{(k)}, \dots, a_n^{(k)}, a_{n+1}, \dots, a_{n+m}, a_{n+m+1}^{(k)}, \dots, a_{2n}^{(k)}\} \subset \mathbb{C} \setminus \mathbb{R}$$

so that $a_{n+j}^{(k)} = \overline{a}_j^{(k)}$ and $\lim_{k\to\infty} |\mathrm{Im}(a_j^{(k)})| = \infty$ for each $j = m+1, m+2, \dots, n$. Several interesting corollaries of the above three theorems can be obtained. We formulate only one of these.

Corollary 6. *Suppose* $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C}$ *and*

$$1 < R \le |a_j|, \qquad j = 1, 2, \dots, n.$$

Then

$$|f'(z_0)| \le \frac{R+1}{R-1} n \|f\|_{\partial D}, \qquad z_0 \in \partial D,$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \dots, a_n; \partial D)$. *For a fixed* $z_0 \in \partial D$ *equality holds if and only if*

$$a_1 = a_2 = \cdots = a_n = Rz_0$$

and $f = cS_n$, $c \in \mathbb{C}$, *where* S_n *is the Blaschke product associated with the poles* a_j, $j = 1, 2, \dots, n$.

3. Proofs.

To prove Theorem 1 we need the following result (see [9, p. 38], for instance).

Interpolation Theorem. Let V be an $n+1$ dimensional subspace over $\mathbb{C}$ of $C(Q)$, the linear space of complex-valued continuous functions defined on a compact Hausdorff space Q, and let $L \not\equiv 0$ be a linear functional on V. Then there exist distinct points $x_1, x_2, \dots, x_r$ in Q, where $1 \leq r \leq 2n+1$, and nonzero real numbers $c_1, c_2, \dots, c_r$ so that

$$L(f) = \sum_{j=1}^{r} c_j f(x_j), \qquad f \in V$$

and

$$\|L\| := \max_{0 \neq f \in V} \frac{|L(f)|}{\|f\|_Q} = \sum_{j=1}^{r} |c_j|.$$

Proof of Theorem 1. For the reason of symmetry it is sufficient to prove the theorem when $z = 1$. Without loss of generality we may assume that

$$\operatorname{Re}\left(\sum_{j=1}^{n} \frac{1}{1-a_j}\right) \neq \frac{n}{2} \tag{1}$$

the other cases follow from this by a limiting argument. Let $Q := \partial D$ (with the usual metric topology),

$$V := \mathcal{P}_n^c(a_1, a_2, \dots, a_n; \partial D)$$

and

$$L(f) := f'(1), \qquad f \in V.$$

We show in this situation that $n+1 \leq r$ in the Interpolation Theorem. Suppose to the contrary that $r \leq n$. By the Interpolation Theorem there are r distinct points $x_1, x_2, \dots, x_r$ on ∂D so that

$$\frac{p_n'(1)q_n(1) - q_n'(1)p_n(1)}{q_n(1)^2} = \sum_{j=1}^{r} c_j \frac{p_n(x_j)}{q_n(x_j)}, \qquad p_n \in \mathcal{P}_n^c, \tag{2}$$

where

$$q_n(z) := \prod_{j=1}^{n} (z - a_j). \tag{3}$$

We claim that $x_j \neq 1$ for each $j = 1, 2, \dots, r$. Indeed, if there is an index j so that $x_j = 1$, then the Interpolation Theorem implies that

$$p_n(z) := (z+1)^{n-r} \prod_{j=1}^{r} (z - x_j) \in \mathcal{P}_n^c$$

has a zero at 1 with multiplicity at least 2, a contradiction. Applying (2) to the above p_n, we obtain

$$p_n'(1)q_n(1) - q_n'(1)p_n(1) = 0,$$

and since $p_n(1) \neq 0$ and $q_n(1) \neq 0$, this is equivalent to

$$\frac{q_n'(1)}{q_n(1)} = \frac{p_n'(1)}{p_n(1)}$$

or in terms of the zeros of p_n and q_n

$$\sum_{j=1}^{n} \frac{1}{1-a_j} = \frac{n-r}{2} + \sum_{j=1}^{r} \frac{1}{1-x_j}. \tag{4}$$

Since $x_j \in \partial D$ and $x_j \neq 1$, $j = 1,2,\dots,r$, we have

$$\operatorname{Re}\left(\frac{1}{1-x_j}\right) = \frac{1}{2}, \qquad j = 1,2,\dots,r. \tag{5}$$

It follows from (4) and (5) that

$$\operatorname{Re}\left(\sum_{j=1}^{n} \frac{1}{1-a_j}\right) = \frac{n}{2}$$

which contradicts assumption (1). So $n+1 \leq r$, indeed.

A simple compactness argument shows that there is a function $\tilde{f} \in V$ so that $\|\tilde{f}\|_{\partial D} = 1$ and $|L(\tilde{f})| = \|L\|$. The interpolation Theorem implies

$$|\tilde{f}(x_j)| = 1, \qquad j = 1,2,\dots,r.$$

Hence, if

$$\tilde{f} = \frac{\tilde{p}_n}{q_n}, \quad \tilde{p}_n \in \mathcal{P}_n^c, \quad q_n(z) = \prod_{j=1}^{n}(z-a_j),$$

then

$$h(z) = |\tilde{p}_n(z)|^2 - |q_n(z)|^2 \leq 0, \qquad z \in \partial D \tag{6}$$

and

$$h(x_j) = 0, \qquad j = 1,2,\dots,r. \tag{7}$$

Note that $t(\theta) := h(e^{i\theta}) \in \mathcal{T}_n^r$ vanishes at each θ_j, where $\theta_j \in [0,2\pi)$ is defined by $x_j = e^{i\theta_j}$, $j = 1,2,\dots,r$. Because of (6), each of these zeros is of even multiplicity. Hence, $n+1 \leq r$ implies that $t \in \mathcal{T}_n$ has at least $2n+2$ zeros with multiplicities, therefore $t(\theta) \equiv 0$. From this we can deduce that $h(z) = 0$ for every $z \in \partial D$, so

$$|\tilde{p}_n(z)| = |q_n(z)|, \qquad z \in \partial D. \tag{8}$$

We have

$$z^{-n}\tilde{p}_n(z)\tilde{p}_n^*(z) = |\tilde{p}_n(z)|^2 = |q_n(z)|^2 = z^{-n}q_n(z)q_n^*(z), \qquad z \in \partial D,$$

so by the Unicity Theorem of analytic functions

$$\tilde{p}_n \tilde{p}_n^* = q_n q_n^*.$$

¿From this it follows that there is a constant $0 \neq c \in \mathbb{C}$ so that

$$\tilde{f}(z) = \frac{\tilde{p}_n(z)}{q_n(z)} = c \prod_{j=1}^{m} \frac{z - 1/\overline{\alpha}_j}{z - \alpha_j}$$

with some $m \leq n$ and

$$\alpha_j := a_{k_j}, \quad j = 1, 2, \dots, m, \quad 1 \leq k_1 < k_2 < \cdots < k_m \leq n.$$

A straightforward calculation gives

$$|\tilde{f}'(1)| = \left| \frac{\tilde{f}'(1)}{\tilde{f}(1)} \right| = \left| \sum_{j=1}^{m} \left(\frac{1}{1 - 1/\overline{\alpha}_j} - \frac{1}{1 - \alpha_j} \right) \right|$$
$$= \left| \sum_{j=1}^{m} \frac{|\alpha_j|^2 - 1}{|\alpha_j - 1|^2} \right| \leq \max \left\{ \sum_{\substack{j=1 \\ |a_j| > 1}} \frac{|a_j|^2 - 1}{|a_j - 1|^2}, \sum_{\substack{j=1 \\ |a_j| < 1}} \frac{1 - |a_j|^2}{|a_j - 1|^2} \right\}$$

which finishes the proof. □

Proof of Theorem 2. Observe that if

$$h_n(\theta) := \prod_{j=1}^{2n} \sin((\theta - a_j)/2) \in \mathcal{T}_n^c$$

and $t_n \in \mathcal{T}_n^c$, then there are $p_{2n} \in \mathcal{P}_{2n}^c$ and $q_{2n} \in \mathcal{P}_{2n}^c$ so that

$$\frac{t_n(\theta)}{h_n(\theta)} = \frac{p_{2n}(e^{i\theta})e^{-in\theta}}{q_{2n}(e^{i\theta})e^{-in\theta}} = \frac{p_{2n}(e^{i\theta})}{q_{2n}(e^{i\theta})},$$

where

$$q_{2n}(z) = c \prod_{j=1}^{2n} (z - e^{ia_j})$$

with some $0 \neq c \in \mathbb{C}$. Therefore the theorem follows from Theorem 1. □

Proof of Theorem 3. The result follows from Theorem 1 by the substitution

$$x = \frac{1}{2}(z + z^{-1}).$$

□

Proof of Theorem 4. The function

$$x = i\frac{z+1}{z-1}$$

maps $\partial D \setminus \{1\} = \{z \in \mathbb{C} : |z| = 1, \quad z \neq 1\}$ onto the real line. A straightforward calculation shows that the inequality of the theorem follows from Theorem 1 by the above substitution. □

Proof of Theorem 5. By Corollary 3.3. of [2] we have

$$(1 - y_0^2)g'(y_0)^2 + B_n(y_0)^2 g(y_0)^2 \leq B_n(y_0)^2 \|g\|_{[-1,1]}^2 \tag{9}$$

for every $g \in \mathcal{P}_n^r(b_1, b_2, \dots, b_n; [-1,1])$ and $y_0 \in [-1,1]$, where

$$\{b_1, b_2, \dots, b_n\} \subset \mathbb{C} \setminus [-1,1]$$

and

$$B_n(y_0) := \operatorname{Re}\left(\sum_{j=1}^{n} \frac{\sqrt{b_j^2 - 1}}{b_j - y_0}\right), \qquad y_0 \in [-1,1],$$

with the choice of root in $\sqrt{b_j^2 - 1}$ determined by

$$|b_j - \sqrt{b_j^2 - 1}| < 1.$$

Let $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}, \quad x_0 \in \mathbb{R}$, and

$$f \in \mathcal{P}_n(a_1, a_2, \dots, a_n; \mathbb{R})$$

be fixed. Let $a \in \mathbb{R}$ be chosen so that $|x_0| < a$, let $y_0 := x_0/a \in (-1,1)$, $b_j := a_j/a, \ \ j = 1, 2, \dots, n$, and

$$g(x) := f(ax) \in \mathcal{P}_n^r(b_1, b_2, \dots, b_n; [-1,1]).$$

Applying (9) with the above g and y_0, we obtain

$$(1 - y_0)^2 a^2 f'(x_0)^2 + B_n(y_0)^2 f(x_0)^2 \leq B_n(y_0)^2 \|f\|_{[-a,a]}^2$$

so

$$\frac{a^2 - x_0^2}{a^2} f'(x_0)^2 + (a^{-1}B_n(y_0))^2 f(x_0)^2 \leq (a^{-1}B_n(y_0))^2 \|f\|_{\mathbb{R}}^2 \tag{10}$$

where

$$\lim_{a\to+\infty} a^{-1}B_n(y_0) = \lim_{a\to+\infty} \operatorname{Re}\left(\sum_{j=1}^{n} \frac{\sqrt{b_j^2-1}}{a(b_j-y_0)}\right) \tag{11}$$

$$= \lim_{a\to+\infty} \operatorname{Re}\left(\sum_{j=1}^{n} \frac{\sqrt{(a_j/a)^2-1}}{a_j-x_0}\right)$$

$$= \lim_{a\to+\infty} \operatorname{Re}\left(\sum_{j=1}^{n} \frac{\sqrt{(a_j/a)^2-1}-a_j/a}{a_j-x_0}\right)$$

$$= \operatorname{Re}\left(\sum_{j=1}^{n} \frac{i \operatorname{sign}\left(\operatorname{Im}\left(\sqrt{a_j^2-1}-a_j\right)\right)(\overline{a}_j-x_0)}{|\overline{a}_j-x_0|^2}\right)$$

$$= \sum_{j=1}^{n} \frac{\operatorname{Im}(a_j)}{|a_j-x_0|^2} = \widehat{B}_n(x_0)$$

(note that the map $a \to \sqrt{(a_j/a)^2-1}-a_j/a$ is a continuous map on $(0,\infty)$ taking only nonreal values, and

$$\operatorname{Im}\left(\sqrt{a_j^2-1}-a_j\right) < 0$$

follows from $\left|a_j-\sqrt{a_j^2-1}\right| < 1$ and $\operatorname{Im}(a_j) > 0$.) Therefore, taking the limit in (10) when $a \to +\infty$, we obtain the theorem by (11).

Proof of Corollary 6. The inequality follows from Theorem 1 since $R \le |a_j|$ and $|z_0| = 1$ imply

$$\frac{|a_j|^2-1}{|a_j-z_0|^2} \le \frac{R+1}{R-1}, \qquad j = 1, 2, \dots, n.$$

Now assume that $\tilde{f} \ne 0$ satisfies

$$|\tilde{f}'(z_0)| = \frac{R+1}{R-1}\,n, \qquad \|\tilde{f}\|_{\partial D} = 1,$$

for some $z_0 \in \partial D$. Then we obtain from Theorem 1 that

$$\frac{|a_j|^2-1}{|a_j-z_0|^2} = \frac{R+1}{R-1}, \qquad j = 1, 2, \dots, n,$$

therefore

$$a_j = Rz_0, \qquad j = 1, 2, \dots, n.$$

Now observe that $1 < R \le |a_j|, \quad j = 1, 2, \dots, n$, implies

$$\operatorname{Re}\left(\sum_{j=1}^{n} \frac{1}{1-a_j}\right) < \sum_{j=1}^{n} \frac{1}{2} = \frac{n}{2},$$

so the proof of Theorem 1 yields that $\tilde{f} = cS_n$, $|c| = 1$, where S_n is the Blaschke product associated with $\{a_1, a_2, \dots, a_n\}$.

On the other hand, if $z_0 \in \partial D$, $a_1 = a_2 = \cdots = a_n = Rz_0$, S_n is the Blaschke product associated with $\{a_1, a_2, \dots, a_n\}$ and $f = cS_n$, $c \in \mathbb{C}$, then

$$|f'(z_0)| = \frac{R+1}{R-1}\, n\, \|f\|_{\partial D}$$

and the proof is finished. □

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