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On Monochrome Lines and Hyperplanes

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1. INTRODUCTION

If two finite sets of points in the real projective plane are not all collinear, then there exists a line through two points of one of the sets that does not intersect the other set. Such a line is called monochrome. This attractive result is Motzkin's theorem ([1] or [2]). More generally, Shannon has shown that n finite sets of points whose union spans real n-dimensional space must also define a monochrome line [3]. We shall consider an n-dimensional variant of Motzkin's theorem. More precisely, we shall prove the following:

THEOREM. *If R and B are two finite sets whose union spans* E^n *(Euclidean n space), then either there exists a monochrome R line (a line through two points of R that does not intersect B) or there exists a monochrome B hyperplane (a hyperplane spanned by points of B that does not intersect R).*

Both Motzkin's Theorem in E^2 and Shannon's result in E^n are immediate corollaries of this theorem. Our proof is self-contained and unlike the proofs of the above results we shall proceed directly rather than considering the equivalent dual formulation of the problem. We shall, however, offer a detailed proof of the above theorem in three dimensions only. This is conceptually much easier and as the referee points out, the results extend in a straightforward way to higher dimensions. We shall discuss the necessary modifications later.

We note that two colours in three dimensions is insufficient to guarantee the existence of a monochrome plane. See Fig. 1.

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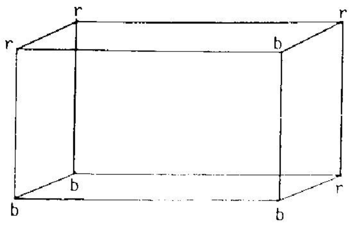

FIGURE 1

2. PRELIMINARIES

We shall denote points in R by $r, r_0, r_1, \ldots$, points in B by $b, b_0, b_1, \ldots$, by interior we mean relative interior. We shall denote tetrahedra by (p_1, p_2, p_3, p_4), triangles by $\Delta(p_1, p_2, p_3)$, and segments by $S(p_1, p_2)$. The plane through p_1, p_2, and p_3 is denoted by $\pi(p_1, p_2, p_3)$ and the line through p_1 and p_2 by $L(p_1, p_2)$. When we write $p \in \Delta(p_1, p_2, p_3)$ etc., we mean p lies in the closed set defined by $\Delta(p_1, p_2, p_3)$.

We require the following lemma:

LEMMA. *If R and B are two finite sets whose union spans* E^3 *and if there are no monochrome R lines, then there exists either:*

(a) *a monochrome B triangle (a triangle with no R points on its boundary or in its interior), or*

(b) *a monochrome B segment and a monochrome R segment which are non-coplanar.*

Proof. We shall first prove that every plane defined by three B points contains a monochrome B segment. Suppose there exists a plane containing three noncollinear B points and no monochrome B segments. We restrict our attention to this plane. This plane must contain at least three R points (one

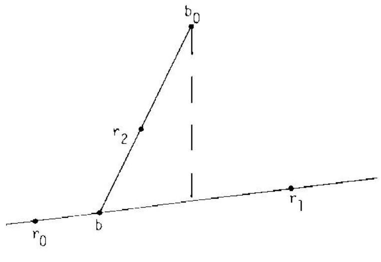

FIGURE 2

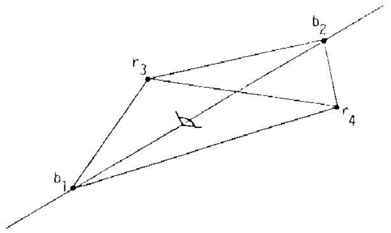

FIGURE 3

on each edge of the triangle defined by the three B points). Rotate this plane so that no two points lie on the same vertical line and so that there exists a vertical line passing through a B point and the interior of an R segment. Now consider the minimum vertical distance from a B point b_0 to the interior of an R segment $S(r_0, r_1)$. See Fig. 2. By assumption there are no monochrome R lines. So there must exist a B point b on $L(r_0, r_1)$. Since $S(b_0, b)$ cannot be monochrome it must contain r_2. Point b_0, however, is now "too close" to either $S(r_0, r_2)$ or $S(r_1, r_2)$. Thus, every B plane contains a monochrome B segment. (We note that the above is obvious if we assume Motzkin's result.) Suppose that $S(b_1, b_2)$ is monochrome.

Consider two R points r_3 and r_4 such that the dihedral angle $\langle|r_3, L(b_1, b_2), r_4|$ is the smallest among all the nonzero dihedral angles $\langle|r_i, L(b_1, b_2), r_j|$. Except in cases where the lemma is trivial such a pair of points clearly exist. See Fig. 3.

If $S(r_3, r_4)$ contains no B point, then segments $S(r_3, r_4)$ and $S(b_1, b_2)$ satisfy condition (b).

If $S(r_3, r_4)$ contains a B point b_3, then $\Delta(b_1, b_2, b_3)$ satisfies condition (a). ∎

The n-dimensional form of the lemma asserts the existence of either (a) a monochrome B $(n-1)$-simplex, or (b) a monochrome B $(n-2)$-simplex and a monochrome R segment. To prove this we proceed inductively. By the $(n-1)$-dimensional form of the theorem, there exists a monochrome $(n-2)$-dimensional affine variety $H(b_1, ..., b_{n-1})$. We can finish the argument, as in the last paragraphs of the proof of the lemma, by considering the minimal nonzero dihedral angle $\langle|r_i, H(b_1, ..., b_{n-1}), r_j|$.

3. THE PROOF OF THE THEOREM

We assume that there exist neither monochrome B planes nor monochrome R lines. We rotate the configuration so that:

(1) no vertical line passes through both a point and a line of the configuration, and

(2) at least one vertical line passes through either an R point and a noncoplanar monochrome B triangle or through a monochrome B segment and a noncoplanar monochrome R segment.

The lemma and a dimensionality argument allow us to do this.

We now consider the minimum vertical distance from either:

Condition (1): an R point to a noncoplanar monochrome B triangle or

Condition (2): a monochrome R segment to a noncoplanar monochrome B segment.

Before proceeding with the proof we shall indicate the modifications required for the n-dimensional version. Condition (1) becomes the distance from an R point to a monochrome B $(n-1)$-simplex and condition (2) becomes the distance from a monochrome R segment to a monochrome B $(n-2)$-simplex where, in both cases, we assume that the n points involved are spanning. If we now read B $(n-2)$-simplex for B segment, B $(n-1)$-simplex for B triangle, and B hyperplane for B plane, then the proof generalizes in an obvious fashion.

Case 1. A minimum is obtained (as in condition (1)) between an R point r_0 and a monochrome B triangle $\Delta(b_0, b_0, b_0)$. See Fig. 4. Let $S(r_0, t)$ be the vertical segment from r_0 to $\Delta(b_0, b_0, b_0)$. Let π_1 be the plane through r_0 parallel to π_2, the plane of $\Delta(b_0, b_0, b_0)$.

Since π_2 contains an R point r_1 exterior to $\Delta(b_0, b_0, b_0)$, $S(r_0, r_1)$ must contain a B point b_1. One of the triangles $\Delta(b_0, b_0, b_1)$ intersects $s(r_0, t)$. Thus, there are triangles with 3 B-vertices under π_1 and on or above π_2,

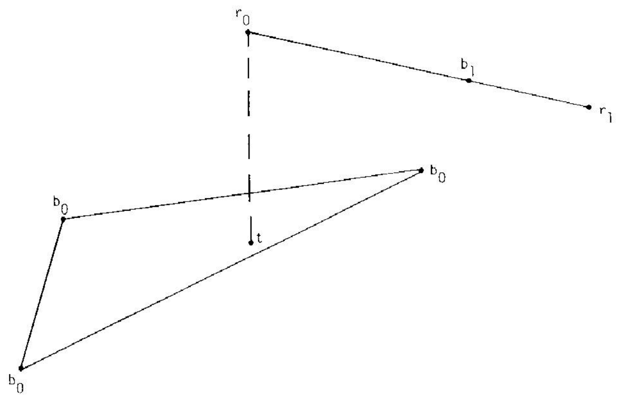

FIGURE 4

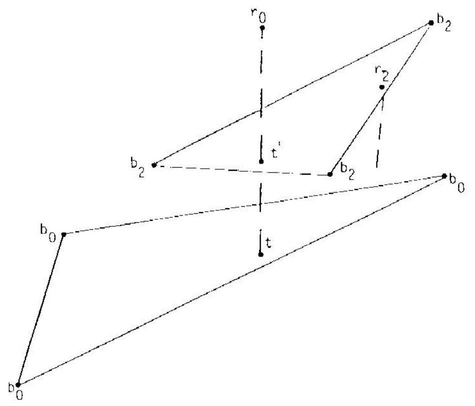

FIGURE 5

which intersect $S(r_0, t)$ above t. Consider the $\Delta(b_2, b_2, b_2)$ in this set which has the highest intersection t', with $S(r_0, t)$. See Fig. 5.

The minimality of $S(r_0, t)$ implies that $\Delta(b_2, b_2, b_2)$ must contain an R point r_2, where r_2 lies above π_2 but does not lie vertically above $\Delta(b_0, b_0, b_0)$. Hence, $S(r_0, r_2)$ passes vertically over an edge of this triangle and must, therefore, contain a B point b_3. This leads to the contradiction that one of the triangles $\Delta(b_3, b_2, b_2)$ intersects the segment $S(r_0, t')$.

Case 2. A minimum is obtained (as in condition (2)) between a monochrome R segment $S(r_0, r_0^*)$ and a monochrome B segment $S(b_0, b_0)$. See Fig. 6. Since $L(r_0, r_0^*)$ is not monochrome there exists $b_1 \in L(r_0, r_0^*) - S(r_0, r_0^*)$. Thus, there exists $r_1 \in \Delta(b_1, b_0, b_0)$ or condition (1) is violated with respect to r_0 and $\Delta(b_0, b_0, b_1)$. In order that $S(r_1, r_0^*)$ and $S(b_0, b_0)$ do not violate condition (2) we must place b_2 on $S(r_1, r_0^*)$.

We have now guaranteed the existence of a segment $S(r, r)$ having R endpoints, containing B points in the interior of tetrahedron (b_1, b_0, b_0, r_0^*)

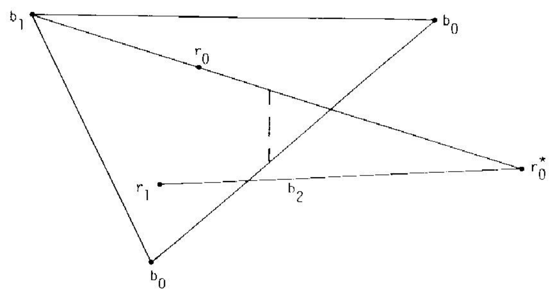

FIGURE 6

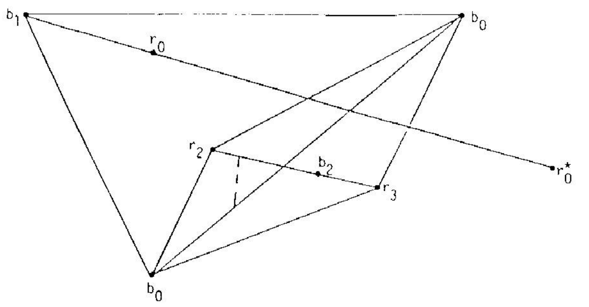

FIGURE 7

and passing vertically over $S(b_0, b_0)$. Consider the class of all such segements and suppose that $S(r_2, r_3)$ is one subtending the smallest nonzero dihedral angle $\langle [r_2, L(b_0, b_0), r_3]$ at the line $L(b_0, b_0)$. See Fig. 7.

This segment must contain a B point $b_2$and $\Delta(b_2, b_1, b_0)$ must be monochrome. But either r_0 is vertically over $\Delta(b_2, b_0, b_0)$ or $s(r_0, r_0^*)$ passes vertically over one of the monochrome B segments $s(b_0, b_2)$. Either case violates the minimality of the vertical distance between segments $S(r_0, r_0^*)$ and $S(b_0, b_0)$.

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