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Arbitrarily Slow Rational Approximations on the Positive Real Line

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INTRODUCTION

This paper will exhibit positive, non-decreasing, infinitely differentiable functions f with the property that the best rational approximations of degree n in the supremum norm to $1/f$ on $[0, \infty)$ tend to zero arbitrarily slowly. Furthermore, such f can be chosen to have very general growth characteristics at infinity.

In particular, this demonstrates that the following two conjectures of Erdös and Reddy [1] are false.

1. Let $f(x)$ be any nonvanishing, infinitely differentiable and monotonic function tending to $+\infty$. Then for infinitely many n

$$\inf_{p \in P_n} \| 1/f(x) - 1/p(x) \|_{[0,\infty)} \leqslant 1/\log n,$$

where P_n denotes the set of polynomials of degree at most n.

2. Let $f(x)$ be any nonvanishing, infinitely differentiable and monotonic function tending to $+\infty$. Then, there exist polynomials of the form

$$Q(x) = \sum_{i=0}^{k} a_i x^{n_i}$$

with $n_0 = 0$, $n_0 < n_1 < n_2 < \cdots$, $\sum_{i=0}^{\infty} 1/n_i = \infty$, for which, for infinitely many k,

$$\| 1/f(x) - 1/Q(x) \|_{[0,\infty)} \leqslant 1/\log \log n_k .$$

THE CONSTRUCTION

We shall make use of the following Lemma due to Gončar [2]. Let R_n denote the set of rational functions which are the quotients of two polynomials each of degree at most n.

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LEMMA. *If g is a continuous function on $[a-1, a+1]$, $g \equiv 0$ on $[a-1, a]$ and g is nondecreasing on $[a, a+1]$, then*

$$\inf_{r \in R_n} \| g - r \|_{[a-1,a+1]} \geqslant \sup_{0<h<1} \frac{g(a+h)}{1 + \exp(\pi^2 n / \ln 1/h)}.$$

THEOREM. *Let α_n be any sequence of positive numbers tending to zero monotonically. Let S_n be any sequence of positive numbers with $S_{n+1} \geqslant S_n + 1$. Then there exists an f satisfying:*

(1) *f is infinitely differentiable and nondecreasing on $[0, \infty)$.*

(2) *$f(2k) = S_k$ for $k = 1, 2, \ldots$.*

(3) *$\inf_{r \in R_n} \| 1/f(x) - r(x) \|_{[0,\infty)} \geqslant \alpha_n$ for all sufficiently large n.*

Proof. (a) Let δ_n be any sequence of positive numbers with $1 \leqslant \delta_n$. Let $h(n) = e^{-\delta_n}$. Define f on $[0, \infty)$ by:

$$f(x) = S_1, \qquad x \in [0, 2]$$
$$f(x) = S_{k+1}, \qquad x \in [2k + h(k), 2k + 2], \quad k = 1, 2, \ldots$$
$$f(x) = Q_k(x), \qquad x \in [2k, 2k + h(k)], \quad k = 1, 2, \ldots$$

where Q_k is any increasing, infinitely deffrentiable function on $[2k, 2k + h(k)]$ which satisfies $Q_k(2k) = S_k$, $Q_k(2k + h(k)) = S_{k+1}$ and for $n \geqslant 1$, $Q_k^{(n)}(2k) = Q_k^{(n)}(2k + h(k)) = 0$.

Parts (1) and (2) now follow from the construction. We show that, for suitably chosen δ_n, (3) holds.

(b) The Lemma applied to $f - S_k$ on $[2k-1, 2k+1]$ with $h = h(k)$ yields

$$\inf_{r \in R_n} \| f - r \|_{[0,2k+2]} \geqslant \inf_{r \in R_n} \| f - r \|_{[2k-1,2k+1]}$$
$$\geqslant \frac{f(2k + h(k)) - S_k}{1 + e^{\pi^2 n/\delta_k}} \geqslant \frac{1}{1 + e^{\pi^2 n/\delta_k}}.$$

(c) If $\delta_k \geqslant n$ then $\inf_{r \in R_n} \| 1/f - 1/r \|_{[0,2k+2]} \geqslant T(k)$, where $T(k) = 1/3(1 + e^{\pi^2})(S_{k+1})^2$.

Suppose on the contrary that there exists $r \in R_n$ with $\| 1/f - 1/r \|_{[0,2k+2]} < T(k)$ (*). Then $\| r \|_{[0,2k+2]} - \| r \|_{[0,2k+2]} \| f \|_{[0,2k+2]} T(k) \leqslant \| f \|_{[0,2k+2]}$ and so

$$\| r \|_{[0,2k+2]} \leqslant \frac{\| f \|_{[0,2k+2]}}{1 - \| f \|_{[0,2k+2]} T(k)} \leqslant 2 \| f \|_{[0,2k+2]},$$

since $\|f\|_{[0,2k+2]} = S_{k+1}$. Thus, using (b) with $\delta_k \geqslant n$, we have

$$\| 1/f - 1/r \|_{[0,2k+2]} \geqslant \frac{\|f - r\|_{[0,2k+2]}}{\|f\|_{[0,2k+2]} \, \|r\|_{[0,2k+2]}} \geqslant \frac{1}{(1 + e^{\pi^2})} \cdot \frac{1}{2(S_{k+1})^2} > T(k),$$

which contradicts (*) and proves (c).

(d) Let $H_k = \{i: T(k) \geqslant \alpha_i > T(k+1)\}$. Pick $\delta_k = \max H_k$ $(= 1$ if H_k is empty). Then, for sufficiently large n, $n \in H_k$ for some k and by (c)

$$\inf_{r \in R_n} \| 1/f - r \|_{[0,\infty)} \geqslant T(k) \geqslant \alpha_n .$$

Remarks. (1) A similar theorem is easily proved for strictly monotone $f(x)$ by considering $f(x) + x$.

(2) Freud, *et al.* [3] have shown that $e^{-x^{-1/2}}$ can be approximated on $[0, \infty)$ by reciprocals of polynomials of degree n with an error of order $(\log n)/n$.

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References

1. P. Erdös and A. R. Reddy, Rational approximation, *Advances in Math.* **21** (1976), 78–109.
2. A. A. Gončar, Estimates of the growth of rational functions and some of their applications, *Mat. Sb.* **72**, No. 114 (1967), 489–503; *Math. USSR-Sb.* **1** (1967), 445–456.
3. G. Freud, D. J. Newman, and A. R. Reddy, Rational approximation to $e^{-|x|}$ on the whole real line, *Quart. J. Math. Oxford Ser.* **28** (1977), 117–123.

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